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A NOTE ON A PROBLEM OF A RING COVERED BY FIELDS

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In [1] the following problem was given:
Prove that every ring $\mathfrak{R}$ which is the set theoretical union of a finite number of commutative fields $\mathfrak{F}_i$ ($i = 1, 2, \ldots, n$) is a commutative field.

However, the direct sum of two prime fields of characteristic 2 gives us a simple example of a ring which shows that the statement which has to be proved is not true. Of course, additional conditions may be given to make the problem correct.

1. Thus, if supposed that all covering fields $\mathfrak{F}_i$ have the same unit element (see [2]) it follows in an easy way that $\mathfrak{R}$ is a division ring. Now the commutativity law for multiplication may be proved just as it is done in paragraph 2 of this paper.

But the same is true if $\mathfrak{R}$ is supposed to be *infinite*. Realy, if L is a left ideal of $\mathfrak{R}$, $L \neq O$, then for every $x \in L$, $x \neq 0$, $x \in \mathfrak{F}_i$ we have $\mathfrak{F}_i = \mathfrak{F}_i x \subset L$ and thus every left ideal $L \neq O$ of $\mathfrak{R}$ is a set theoretical union of some finite number of the given fields $\mathfrak{F}_i$. Thus both chain conditions for left ideals are satisfied and because no left ideal may be nilpotent it follows that $\mathfrak{R}$ is semisimple. Now $\mathfrak{R}$ may be decomposed into a direct sum of minimal left ideals

$$\mathfrak{R} = X_1 + X_2 + \ldots + X_r$$

If $u = u_1 + u_2 + \ldots + u_r \neq 0$ and $v = v_1 + v_2 + \ldots + v_r \neq 0$ (with $u_j \in X_j$, $v_j \in X_j$ for every $j = 1, 2, \ldots, r$) belong to the same field $\mathfrak{F}_i$ then $u_j \neq 0 \Leftrightarrow v_j \neq 0$ for every $j = 1, 2, \ldots, r$, for otherwise no $w \in \mathfrak{F}_i$ can satisfy $wu = v$ or $wv = u$. Now if some X_j, say X_1, is infinite and if $r \geq 2$ we choose a fixed $x_2 \in X_2$, $x_2 \neq 0$ and we form all sums of type $x_1 + x_2$ where x_1 runs in X_1. Distinct sums must belong to distinct fields according to previous statement what is evidently impossible. We conclude that for an infinite $\mathfrak{R}$ there is $r = 1$. Now for every $a \in \mathfrak{R} = X_1$, $a \neq 0$ we have $\mathfrak{R}a \neq 0$ and consequently $\mathfrak{R}a = \mathfrak{R}$. In a similar way, taking right ideals instead of the left ones we find $a\mathfrak{R} = \mathfrak{R}$ for every $a \in \mathfrak{R}$, $a \neq 0$. Thus $\mathfrak{R}$ is a division ring.

2. The commutativity law for $\mathfrak{R}$ may be proved by using a theorem due to I. KAPLANSKY (see [3] page 185). Denoting by $\mathfrak{C}$ the centre of a division ring $\mathfrak{R}$ and by $\mathfrak{C}^*$ and $\mathfrak{R}^*$ the corresponding multiplicative groups, we may state Kaplansky's theorem in the following way: *If $\mathfrak{R}^*/\mathfrak{C}^*$ is a torsion group then $\mathfrak{R}$ is commutative.*

In our case the division ring $\mathfrak{R}$ is covered by commutative fields $\mathfrak{F}_i$ ($i = 1, 2, \ldots, n$) and consequently the group $\mathfrak{R}^*$ is covered by abelian groups $\mathfrak{F}_i^*$. Now we need only to use the following lemma.

LEMMA. *If a group $\mathfrak{G}$ (with centre $\mathfrak{Z}$) is covered by a finite number of its abelian subgroups $\mathfrak{A}_i (i = 1, 2, \ldots, n)$, then $\mathfrak{G}/\mathfrak{Z}$ is a torsion group.*

Proof: Suppose that the covering system $\mathfrak{A}_i$ ($i = 1, 2, \ldots, n$) is minimal. This may be obtained of course from the original covering system by crossing out some A_is (if possible) not destroying the covering property. Now let $x \in \mathfrak{G}$, $x \notin \mathfrak{Z}$. Denote by $\mathfrak{U}$ the union of all $\mathfrak{A}_i$ with $x \in \mathfrak{A}_i$ and by $\mathfrak{B}$ the union of all $\mathfrak{A}_i$ with $x \notin \mathfrak{A}_i$. $\mathfrak{U}$ and $\mathfrak{B}$ are not empty. Now we can find $y \in \mathfrak{B}$ such that $y \notin \mathfrak{U}$. For every integer k we have $x^k y \notin \mathfrak{U}$ and consequently $x^k y \in \mathfrak{B}$. We may find $k_1 < k_2$ such that $x^{k_1} y$ and $x^{k_2} y$ belong to the same $\mathfrak{A}_j \subset \mathfrak{B}$ and then $x^{k_2 - k_1} = x^{k_2} y \, (x^{k_1} y)^{-1} \in \mathfrak{A}_j$. Thus $x^{k_2 - k_1}$ belongs to all $\mathfrak{A}_i \subset \mathfrak{U}$ and to some new $\mathfrak{A}_j \subset \mathfrak{B}$. Repeating this step starting with $x^{k_2 - k_1}$ instead of x and making it again and again we get finally a positive integer m such that $x^m \in \mathfrak{A}_i$ for all $i = 1, 2, \ldots, n$ and thus $x^m \in \mathfrak{Z}$. Hence $\mathfrak{G}/\mathfrak{Z}$ is a torsion group. We have proved

THEOREM: *If an infinite ring $\mathfrak{R}$ is covered by a finite number of subrings $\mathfrak{F}_i$ each of them being a commutative field then it is a commutative field.*

POZNÁMKA K PROBLÉMU OKRUHU POKRYTÉHO TĚLESY

Souhrn

V práci se dokazuje, že problém daný I. H. HERSTEINEM v [1] je korektní, předpokládáme-li, že daný okruh $\mathfrak{R}$ je nekonečný:

VĚTA: *Je-li nekonečný okruh $\mathfrak{R}$ pokryt konečným počtem podokruhů, z nichž každý je komutativním tělesem, pak $\mathfrak{R}$ je komutativní těleso.*

REFERENCES

[1] I. H. HERSTEIN: American Mathematical Monthly 67 (1960) p. 927, Problem No 4933.

[2] I. H. HERSTEIN: American Mathematical Monthly 68 (1961) p. 299, Correction.

[3] N. JACOBSON: Structure of Rings, American Mathematical Society Colloquium Publications vol. XXXVII, 1956.

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