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Regular Mappings of Groupoids

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1° We shall usually write the binary operation of a groupoid multiplicatively. When using the other symbol of the binary operation, we put this symbol into brackets, e. g.: $G(\cdot)$, $H(o)$ e. t. c.

Let $G$ be a groupoid and $x \in G$. By the symbol $L_x$ ($R_x$) we shall denote the mapping of the set $G$ into $G$ such that for every $y \in G$, $L_x(y) = xy$ ($R_x(y) = yx$).

A groupoid $G$ is called a groupoid with left (right) cancellation, if for every $x \in G$ the mapping $L_x$ ($R_x$) is one — to — one.

A groupoid $G$ is called a groupoid with left (right) division, if for every $x \in G$ the mapping $L_x$ ($R_x$) is onto $G$.

A groupoid $G$ is called a left (right) quasigroup, if for every $x \in G$ the mapping $L_x$ ($R_x$) is a permutation of the set $G$ (permutation is a mapping of a set into itself, which is one — to — one and onto the set).

A groupoid, which is simultaneously with left and right cancellation (division), is called a groupoid with cancellation (division).

A groupoid, which is simultaneously a left and right quasigroup, is called a quasigroup.

2° Definition 1: Let $G$ be a groupoid. A mapping $\lambda (\varphi)$ of the set $G$ into $G$ is called left (right) regular, if there is a mapping $\lambda^*$ ($\varphi^*$) such that for every $x, y \in G$, $\lambda(xy) = \lambda^*(x) \cdot y$ ($\varphi(xy) = x \cdot \varphi^*(y)$).

A mapping $\varphi$ is called central regular, if there is a mapping $\varphi^*$ such that for every $x, y \in G$, $\varphi(x) \cdot y = x \cdot \varphi^*(y)$. By the symbol $A_G$ we shall denote the set of all left regular mappings of the groupoid $G$ and by $A_G^*$ the set of all possible mappings $\lambda^*$ corresponding to the left regular mappings. Similarly introduce the symbols $R_G$, $R_G^*$, $\Phi_G$, $\Phi_G^*$.

Lemma 1: Let $G$ be a groupoid. Then the sets $A_G, A_G^*, R_G, R_G^*, \Phi_G, \Phi_G^*$ are semigroups with unit under the binary operation of composition of mappings.

Proof: We shall prove the Lemma for $A_G, A_G^*$ only. For the other cases the proof is similar.

Let $\lambda_1, \lambda_2 \in A_G$. Let $\lambda_1^*, \lambda_2^* \in A_G^*$ be arbitrary mappings corresponding to the mappings $\lambda_1, \lambda_2$. For every $x, y \in G$, $\lambda_1 \lambda_2(xy) = \lambda_1(\lambda_2^*(x) \cdot y) = \lambda_1^* \lambda_2^*(x) \cdot y$. 

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Hence \( \lambda_1 \lambda_2 \in A_G \) and \( \lambda_1^* \lambda_2^* \in A_G^* \). Evidently \( 1_G \in A_G \), \( 1_G \in A_G^* \).

Lemma 2: Let \( G \) be a groupoid and \( \lambda \in A_G, \varrho \in R_G, \varphi \in \Phi_G \). Then \( \lambda L_x = L_{\lambda^*(x)}, \lambda R_x = R_{\lambda^*(x)}, \varrho R_x = R_{\varphi^*(x)}, R_x \varphi = R_{\varphi^*(x)}, L_{\varphi(x)} = L_{\varphi^*(x)} \) for every \( x \in G \).

Proof: By Definition 1.

Corollary: 1. Let \( G \) be a groupoid with left (right) division. Then all left (right) regular mappings of the groupoid \( G \) are onto \( G \).
2. Let \( G \) be a groupoid with cancellation. Then all central regular mappings of \( G \) are one-to-one.
3. Let \( G \) be a quasigroup. Let \( \lambda \in A_G, \varrho \in R_G, \varphi \in \Phi_G \). Then the mappings \( \lambda^*, \varrho^*, \varphi^* \) are uniquely determined and \( \lambda, \lambda^*, \varrho, \varrho^*, \varphi, \varphi^* \) are permutations.

Lemma 3: Let \( G \) be a groupoid and \( \lambda \in A_G, \varrho \in R_G, \varphi \in \Phi_G \) such that \( \lambda^*, \varrho^*, \varphi^* \) are mappings onto \( G \). Let \( \alpha, \beta, \gamma \) be arbitrary mappings such that \( \alpha \lambda = \beta \varrho = \varphi \gamma = 1_G \). Then also \( \alpha \in A_G, \beta \in R_G, \gamma \in \Phi_G \).

Proof: 1) Since \( \lambda^* \) is a mapping onto, there is a mapping \( \delta \) so that \( \lambda^* \delta = 1_G \). For every \( x, y \in G \) we have \( \lambda(x) = \lambda^*(x) \cdot y \). Hence \( \alpha \lambda(\delta(x)) \cdot y = \delta(x) \cdot y = \alpha(\lambda^* \delta(x)) \cdot y \). Thus \( \alpha \in A_G \). For \( \beta \) similarly.

2) There is a mapping \( \varepsilon \) such that \( \varphi^* \varepsilon = 1_G \). For every \( x, y \in G \), \( \varphi(x) \cdot y = x \cdot \varphi^*(y) \). Hence \( x \cdot \varepsilon(y) = \gamma(x) \cdot y \). Thus \( \gamma \in \Phi_G \).

Theorem 1: Let \( G \) be a quasigroup. Then the semigroups \( A_G, R_G, \Phi_G \) are groups.

Proof: By Corollary and Lemma 1, 3.

Lemma 4: Let \( G \) be a groupoid with right (left) unit \( e \). Let \( \lambda \in A_G, \varrho \in R_G, \varphi \in \Phi_G \). Then the mapping \( \lambda^* (\varrho^*) \) is uniquely determined and \( \lambda = \lambda^* (\varrho = \varrho^*) \).

Proof: Let \( x \in G \). Then \( \lambda(x) = \lambda(xe) = \lambda^*(x) \cdot e = \lambda^*(x) \). Hence \( \lambda = \lambda^* \).

Similarly for \( \varrho \).

Lemma 5: Let \( G \) be a groupoid with unit \( e \). Let \( \lambda \in A_G, \varrho \in R_G, \varphi \in \Phi_G \). Then \( \lambda = L_{\lambda(e)}, \varrho = R_{\varrho(e)}, \varphi = L_{\varphi(e)} \). Let \( x, y \in G \). Then \( \lambda(e)(xy) = \lambda(e)x = \lambda(e)(x) y = \lambda(x) \cdot y = \lambda^*(x) \cdot y \). Hence \( L_{\lambda(e)} = \lambda \). For every \( x, y \in G \), \( \lambda(e)(xy) = \lambda(e)(xy) = \lambda(x) y = = \lambda(x)e \cdot y \).

For \( \varrho \) similarly.

2) Let \( x, y \in G \). We have \( \varphi(x) = \varphi(x) \cdot e = \varphi^*(e) \). Hence \( \varphi^*(e) = e \cdot \varphi^*(e) = \varphi(e) \), hence, \( \varphi = R_{\varphi(e)} \). Further, \( x(\varphi(e) \cdot y) = x(e \cdot \varphi^*(y)) = x \cdot \varphi^*(y) = \varphi(x) \cdot y = = (x \cdot \varphi(e)) \cdot y \).

Theorem 2: Let \( G \) be a groupoid (quasigroup) with unit \( e \).

Put \( A_G = E(x \in G \mid \lambda \in A_G, x = \lambda(e)), B_G = E(x \mid \varrho \in R_G, x = \varrho(e)), C_G = E(x \varphi \in \Phi_G, x = \varphi(e)) \). Then the sets \( A_G, B_G, C_G \) are subsemigroups (subgroups) with unit of the groupoid (quasigroup) \( G \).
Proof: We shall prove the theorem for $A_G$ only.

1) Let $x, y \in A_G$. Then there are $\lambda_1, \lambda_2 \in A_G$ so that $x = \lambda_1(e)$, $y = \lambda_2(e)$. Hence $xy = \lambda_1(e) \cdot \lambda_2(e) = \lambda_1(e \cdot \lambda_2(e)) = \lambda_1 \lambda_2(e)$. But $\lambda_1 \lambda_2 \in A_G$ by Lemma 1. Hence $xy \in A_G$. We have proved that $A_G$ is a subgroupoid of $G$. By Lemma 5, $A_G$ is a semigroup. Evidently $e \in A_G$.

2) Let $G$ be a quasigroup. By 1), $A_G$ is a semigroup with unit $e$.

Let $x \in A_G$. Then there is $\lambda \in A_G$ such that $x = \lambda(e)$. By Theorem 1, $\lambda$ is a permutation and $\lambda^{-1} \in A_G$. Hence $\lambda^{-1}(e) \in A_G$.

But $x \cdot \lambda^{-1}(e) = \lambda(e) \cdot \lambda^{-1}(e) = \lambda(e \cdot \lambda^{-1}(e)) = \lambda \lambda^{-1}(e) = e$.

The element $\lambda^{-1}(e)$ is a right inverse element to $x$. Therefore $A_G$ is a group.

Definition 2: A groupoid $G$ is called transitive if for every $x, y \in G$ there is $\lambda \in A_G$ such that $x = \lambda(x)$. Similarly for $A^*, R, R^*, \Phi, \Phi^*$ — transitivity. A groupoid $G$ is called transitive, if at least one of the cases defined is valid.

Lemma 6: Every group is transitive in all possible cases.

Proof: As a group $G$ is a groupoid with unit, we have $A_G = A_G^*$, $R_G = R_G^*$. Further for all $x \in G$, $L_x \in A_G$, $R_x \in R_G$, $R_x \in R_G^*$, $L_x \in L_G^*$. Hence $G$ is $A, A^*, R, R^*, \Phi, \Phi^*$ — transitive.

Lemma 7: Let $G$ be a $A$ or $R^*$ — transitive groupoid with left unit. Then $G$ is a groupoid with right division. Let $G$ be a $R$ or $\Phi$ — transitive groupoid with right unit. Then $G$ is a groupoid with left division.

Proof: 1) Let $G$ be $A$ — transitive and $e$ be a left unit of $G$.

Let $x, y \in G$. There is $\lambda \in A_G$ such that $x = \lambda(x)$. But $\lambda(x) = \lambda (ex) = \lambda (e) \cdot x = y$.

Hence $R_x$ is a mapping onto $G$. Hence $G$ is with right division.

2) Let $G$ be $R^*$ — transitive and $e$ be a left unit of $G$.

Let $x, y \in G$. There is $\varphi^* \in \Phi_G^*$ such that $\varphi^*(x) = y$. We have, $y = e \cdot \varphi^*(x) = \varphi^*(e) \cdot x$. Hence $G$ is with right division.

Similarly for the other cases.

Theorem 3: A groupoid with unit is transitive if and only if it is a group.


Since $G$ is transitive, we have, by Definition 2, $G = A_G$ or $G = B_G$ or $G = C_G$.

Thus by Theorem 2 $G$ is a semigroup with unit. But every semigroup with unit, which is with left or right division, is a group.

2) Let $G$ be a group. By Lemma 6 $G$ is transitive in all possible cases.

3° Definition 3: Let $G$ be a groupoid. A groupoid $G(\cdot)$ is called a homotope of the groupoid $G$, if there are mappings $\alpha, \beta$ of the set $G$ into $G$ and a permutation $\gamma$ of the set $G$ so that for every $x, y \in G$, $\gamma(x \cdot y) = \alpha(x) \cdot \beta(y)$. We shall write $G(\cdot) = G(\alpha, \beta, \gamma)$. The groupoid $G(\cdot)$ is called a $\mu$ — homotope of the groupoid $G$, if $\alpha, \beta$ are onto $G$. The groupoid $G(\cdot)$ is called an isotope of $G$, if $\alpha, \beta$ are permutations. The groupoid $G(\cdot)$ is called a principal homotope, if $\gamma = 1_G$.

The following Lemma is evident.
Lemma 8: 1) Let $G(\cdot) = G(o)(a,\beta,\gamma)$ and $G(o) = G(\delta,\epsilon,\kappa)$. Then $G(\cdot) = G(\tau o, e_0, x y)$.

2) For every $G$ is $G = G(1 o, 1 o, 1 o)$.

3) A mapping $\gamma : G(\cdot) \to G$ is an isomorphism if and only if $G(\cdot) = G(\tau o, e_0, x y)$.

4) Let $G(\cdot) = G(a,\beta,\gamma)$ and $\delta, \epsilon$ be arbitrary mappings such that $\alpha \delta = \beta \epsilon = 1_G$. Then $G$ is a homotope of $G(\cdot)$ and $G = G(\tau o, e_0, x y)$.

5) Let $G(\cdot) = G(a,\beta,\gamma)$ be an isotope of $G$. Then $G$ is an isotope of $G(\cdot)$ and $G = G(\tau o, e_0, x y)$.

6) Let $G(\cdot) = G(a,\beta,\gamma)$ be an isotope of $G$. Then $G$ is an isotope of $G(\cdot)$ and $G = G(\tau o, e_0, x y)$.

Lemma 9: Every $\mu$-homotope of a groupoid with division is a groupoid with division.

Proof: Let $G$ be a groupoid with division and $G(o)$ be a $\mu$-homotope of $G$. Let $\gamma : G(a,\beta,\gamma)$ be $\mu$-homotope of $G$. Then $G(o)$ is a groupoid with division.

Lemma 10: Let $G$ be a groupoid with cancellation and $G(o)$ be an $\mu$-homotope of $G$. Then $G(o)$ is also a groupoid with cancellation.

Theorem 4: Every groupoid which is an isotope of a quasigroup, is a quasigroup.

Proof: By Lemma 8, 10.

Theorem 5: Let $G(o)$ be an isotope of a groupoid $G$. Then the following isomorphisms are valid: $A_{G(o)} \cong A_G, A^*_{G(o)} \cong A^*_G, R_{G(o)} \cong R_G, R^*_{G(o)} \cong R^*_G, \Phi_{G(o)} \cong \Phi_G$.

Proof: Let $G(o) = G(a,\beta,\gamma)$. Then, by Lemma 8, $G = G(o)(\tau o, e_0, x y)$. By Lemma 11, $\lambda \in A_{G(o)} \iff \lambda \tau o \in A_G, \lambda^* \in A^*_G \iff \lambda \tau o \in A^*_G$. The mappings $A : A_{G(o)} \to A_G, B : A^*_{G(o)} \to A^*_G$ such that $A(\lambda) = \lambda \tau o, B(\lambda^*) = \lambda \tau o$ for all $\lambda \in A_{G(o)}$, $\lambda^* \in A^*_{G(o)}$, are evidently isomorphisms.

Similarly for the other cases.

Theorem 6: Let $G(o)$ be a $\mu$-homotope of $G$. Let $G(o)$ be $\mu$-transitive
Proof: Let $G(o) = G^{(a, b, y)}$. Since the mappings $a, b$ are onto $G$, there are mappings $\delta, \varepsilon$ such that $a\delta = b\varepsilon = 1_G$. Let $G(o)$ be $\Lambda$ — transitive and $x, y \in G$. There is $\lambda \in A_G$ such that $\lambda y^{-1}(x) = y^{-1}(y)$. Hence $y\lambda y^{-1}(x) = y$. By Lemma 11, $\gamma \lambda y^{-1} \in \Lambda_G$. Hence $G$ is $\Lambda$ — transitive.

Similarly for the other cases.

Theorem 7: Let a transitive groupoid $G(o)$ be a $\mu$ — homotope of a groupoid $G$, which has a unit. Then $G$ is a group.

Proof: By Theorem 6 and Theorem 3.

Theorem 8: Let a commutative groupoid $G(o)$ be a $\mu$ — homotope of a group $G$. Then $G$ is an Abelian group.

Proof: Let $G(o) = G^{(a, b, y)}$. Since the mappings $a, b$ are onto $G$, there are mappings $\delta, \varepsilon$ such that $a\delta = b\varepsilon = 1_G$. Let $u \in G$ be such that $\beta(u) = e$. Then $a(x) = a(u) \cdot \beta(x)$. Hence $\beta(x) \cdot \beta(y) = \beta(x) \cdot \beta(y) = \alpha(y) \cdot \beta(x) \cdot \beta(y)$. But $\beta(x)$, hence $\beta(x) \cdot \beta(y) = \beta(y) \cdot \beta(x)$. Thus for every $v, z \in G$ we have $vz = \gamma = \beta(e) \cdot \beta(z) \cdot \beta(e(z)) = zv$.

Lemma 12: Let $G(o) = G^{(a, b, y)}$ and let $G(o)$ be a groupoid with unit $e$. Then the translations $\Lambda a$ and $\Lambda b$ of the groupoid $G$ are mappings onto $G$.

Proof: For every $x \in G$ we have $\gamma(x) = \gamma(xo \circ e) = \alpha(x) \cdot \beta(e)$. Hence $\gamma = \Lambda a \circ \beta$. Similarly $\gamma = \Lambda a \circ \beta$. Thus $\Lambda a \circ \beta$ and $\Lambda b \circ \beta$ are mappings onto $G$.

Lemma 13: Let $G$ be a groupoid and $x, y \in G$. Let $\alpha, \beta$ be arbitrary mappings such that $L_x\beta = R_y\alpha = 1_G$ and $\alpha(xy) = x, \beta(xy) = y$ (the mappings $\alpha, \beta$ exist if and only if the mappings $L_x, R_y$ are onto $G$). Put $G(o) = G^{(a, b, 1_G)}$. Then $G(o)$ is a groupoid with unit $e$, where $e = xy$.

Proof: Let $u \in G$. Then $u \circ e = u \circ (xy) = \alpha(u) \cdot \beta(xy) = \alpha(u) \cdot y = R_y\alpha(u) = u, e \circ u = (xy) \circ u = \alpha(xy) \cdot \beta(u) = L_x\beta(u) = u$.

Definition 4: Let $G$ be a groupoid and $x, y \in G$. We say that two elements $x, y$ satisfy the $\mu$ — condition if:

1) The mappings $L_x, R_y$ are onto $G$.
2) For every $u, v, z \in G$, $R_y(u) = R_y(v)$ implies $R_z(u) = R_z(v)$.
3) For every $u, v, z \in G$, $L_x(u) = L_x(v)$ implies $L_z(u) = L_z(v)$.

Lemma 14: Let $G$ be a groupoid and $x, y \in G$. Then the following conditions are equivalent:

1) The elements $x, y$ satisfy the $\mu$ — condition.
2) There are mappings $\alpha, \beta$ such that $R_y\alpha = L_x\beta = 1_G$ and $uv = \alpha R_y(u) \cdot \beta L_x(v)$ for every $u, v \in G$.
3) There are mappings $\alpha, \beta$ such that $R_y\alpha = L_x\beta = 1_G$. For all possible mappings $\delta, \varepsilon$ such that $R_y\delta = L_x\varepsilon = 1_G$ and for all $u, v \in G, uv = \delta R_y(u) \cdot \varepsilon L_x(v)$.
Proof: 1) Implies 3). Since $R_y, L_x$ are onto $G$, there are mappings $\alpha, \beta$ such that $R_y = L_x = 1_G$. Let $\delta, \epsilon$ be arbitrary mappings such that $R_y \delta = L_x \epsilon = 1_G$. Let $u, v \in G$. Set \( z = \delta R_y(u), \ t = \epsilon L_x(v) \). We have $R_y(z) = R_y \delta R_y(u) = R_y(u)$.

Hence $zt = ut$ (by $\mu$ — condition). Further, $L_x(t) = L_x \epsilon L_x(v) = L_x(v)$. Hence $ut = uv$. Thus $zt = uv$.

Evidently 3) implies 2).

2) implies 1). Since $R_y \alpha = L_x \beta = 1_G$, the mappings $R_y, L_x$ are onto $G$. Let $u, v \in G$ such that $R_y(u) = R_y(v)$. Let $x \in G$ be arbitrary element. Then $R_x(u) = uz = \alpha R_y(u)$. Hence we have proved that:

$$R_y(u) = R_y(v) \text{ implies } R_x(u) = R_x(v).$$

Similarly we can prove the last part of the $\mu$ — condition.

Lemma 15: Let $G$ be a groupoid and $x, y \in G$. Then the following conditions are equivalent:

1. The elements $x, y$ satisfy the $\mu$ — condition.
2. There are mappings $\alpha, \beta$ such that $R_y \alpha = L_x \beta = 1_G$ and $\alpha(xy) = x, \beta(xy) = y$.

Let $\alpha_1, \beta_1$ be arbitrary such mappings. Put $G(o) = G(\alpha_1, \beta_1, 1)$. Then $G(o)$ is a groupoid with unit $xy$ and $G = G(o)(R_y, L_x, 1)$.

Proof: 1) implies 2). The mappings $R_y, L_x$ are onto. Hence there are mappings $\alpha, \beta$ such that $R_y \alpha = L_x \beta = 1_G$ and $\alpha(xy) = x, \beta(xy) = y$. Let $\alpha_1, \beta_1$ be arbitrary such mappings. For every $u, v \in G$ by Lemma 14, we have $uv = \alpha_1 R_y(u) \beta_1 L_x(v)$. Hence $R_y(u) \circ L_x(v) = \alpha_1 R_y(u) \beta_1 L_x(v) = uv$. Thus $G = G(o)(R_y, L_x, 1)$.

2) implies 1). The mappings $R_y, L_x$ are evidently onto $G$. Let be $u, v \in G$ such that $R_y(u) = R_y(v)$. Let $x \in G$ be an arbitrary element. We have $R_x(u) = uz = R_y(u) \circ L_x(z) = R_y(v) \circ L_x(z) = vz = R_x(v)$.

Similarly we can prove the last part of the $\mu$ — condition.

Definition 5: A groupoid $G$ is called $\mu$ — groupoid if there is a groupoid with unit, $G(o)$, such that the groupoid $G$ is a $\mu$ — homotope of the groupoid $G(o)$.

Lemma 16: Let $G$ be a $\mu$ — groupoid. Then there is a groupoid with unit, $G(o)$, such that $G$ is a principal $\mu$ — homotope of $G(o)$.

Proof: By Lemma 8.

Theorem 9: Every groupoid $G$ is a $\mu$ — groupoid if and only if there are two elements $x, y \in G$ such that $x, y$ satisfy the $\mu$ — condition.

Proof: 1) Let $G$ be a $\mu$ — groupoid. By Lemma 16 there is a groupoid $G(o)$, which has a unit $e$, such that $G = G(o)(\delta, \epsilon, 1)$. Moreover, the mappings $\delta, \epsilon$ are onto $G$.

Hence there are mappings $\alpha, \beta$ such that $\delta \alpha = \epsilon \beta = 1_G$. Set $x = \alpha(e), y = \beta(e)$.

For every $u, v \in G$, $R_y(u) = uv = \delta(u) \circ \epsilon(y) = \delta(u) \circ \epsilon \beta(e) = \delta(u) \circ \epsilon = \delta(u), L_x(u) = \delta \alpha(e) \circ \epsilon(u) = \epsilon(u)$. Thus $\delta = R_y, \epsilon = L_x$. Further, $\alpha(xy) = \alpha(\alpha(e) \cdot \beta(e)) = \alpha(\delta \alpha(e) \circ \epsilon \beta(e)) = \alpha(e) = x$. Similarly $\beta(xy) = \epsilon = 1_G$. Finally, $\alpha(u) \cdot \beta(v) = \alpha(\delta \alpha(e) \circ \epsilon \beta(v)) = \alpha(e) = x$.

Now we can use Lemma 15. Therefore $x, y$ satisfy the $\mu$ — condition.

2) Let $x, y \in G$ be two elements satisfying the $\mu$ — condition. By Lemma 15 there is
a groupoid with unit, $G(o)$, such that $G = G(o)^{(R_y, L_x, 1)}$. Since $R_y, L_x$ are mappings onto $G$, $G$ is a $\mu$-groupoid.

**Theorem 10:** Let $G$ be a transitive $\mu$-groupoid. Then $G$ is a principal $\mu$-homotope of a group. Hence $G$ is with division.

**Proof:** There is a groupoid with unit, $G(o)$, and there are mappings $\alpha, \beta$, which are onto $G$, such that $G = G(o)^{(\alpha, \beta, 1)}$. By Theorem 6, $G(o)$ is transitive and hence, by Theorem 3, $G(o)$ is a group. By Lemma 9, $G$ is a groupoid with division.

**Theorem 11:** Let $G$ be a transitive groupoid. Let there be two elements of $G$ which satisfy the $\mu$-condition. Then arbitrary two elements of $G$ satisfy the $\mu$-condition.

**Proof:** The groupoid $G$ is a $\mu$-groupoid. Then, by Theorems 9, 10, there is a group $G(o)$ and there are mappings $\alpha, \beta$ (which are onto $G$) such that $G = G(o)^{(\alpha, \beta, 1)}$. Let $x, y$ be arbitrary elements of $G$. By Theorem 10, $G$ is a groupoid with division, hence $L_x, R_y$ are mappings onto $G$. Let $u, v \in G$ be such that $R_y(u) = R_y(v)$ and $z \in G$ be arbitrary element. We have $R_y(u) = uy = \alpha(u) \circ \beta(y) = vy = \alpha(v) \circ \beta(y)$. Hence $\alpha(u) = \alpha(v)$, and hence, $R_z(u) = uz = \alpha(u) \circ \beta(z) = \alpha(v) \circ \beta(z) = vz = = R_z(v)$. Similarly we can prove the last part of the $\mu$-condition. Thus $x, y$ satisfy the $\mu$-condition.

**Lemma 17:** Let $G$ be a group and $\alpha, \beta, \gamma$ be three mappings of $G$ into $G$ such that for every $x, y \in G$ is $\gamma(xy) = \alpha(x) \cdot \beta(y)$. Then there are elements $a, b, c$ of the group $G$ such that the mappings $L_0\alpha, \alpha R_0, L_0\beta, R_0\beta, L_0\gamma, \gamma R_0$ are endomorphisms of the group $G$.

**Proof:** Let 1 be the unit of $G$. For every $x \in G, \gamma(x) = \alpha(1) \cdot \beta(x), \gamma(x) = \alpha(x) \cdot \beta(1)$. Therefore $\alpha(x) \cdot \beta(1) = \alpha(1) \cdot \beta(x)$. Hence $\alpha(x) = \alpha(1) \cdot \beta(x) \cdot (\beta(1))^{-1}$. Further, for every $x, y \in G, \gamma(xy) = \alpha(x) \cdot \beta(y) = \alpha(1) \cdot \beta(xy) = \alpha(1) \cdot \beta(x) \cdot (\beta(1))^{-1} \cdot \beta(y)$. Hence $\beta(xy) = \beta(x) b \beta(y)$, where $b = (\beta(1))^{-1}$. Thus the mappings $L_0\beta, R_0\beta$ are endomorphisms of $G$.

Similarly, there exist $a \in G$ such that $L_0\alpha, R_0\alpha$ are endomorphisms of $G$.

Now for $\gamma$. We have $\beta(x) = (\alpha(1))^{-1} \cdot \gamma(x)$, $\alpha(x) = \gamma(x) \cdot (\beta(1))^{-1}$ for every $x, y \in G$. Since $\gamma(xy) = \alpha(x) \cdot \beta(y)$, we have $\gamma(xy) = \gamma(x) \cdot (\beta(1))^{-1} \cdot (\alpha(1))^{-1} \cdot \gamma(y) = \gamma(x) \cdot c \cdot \gamma(y)$, where $c = (\beta(1))^{-1} \cdot (\alpha(1))^{-1}$. Thus $L_0\gamma, \gamma R_0$ are endomorphisms of the group $G$.

**4° Definition 6:** A groupoid $G$ is called $B_1$ ($B_2$) - groupoid if $x(yz) = y(xz)$ ($xy \cdot z = xz \cdot y$) for all $x, y, z \in G$.

**Lemma 18:** Let $G$ be a $B_1$ - groupoid. Let $x \in G$ be such that $R_x$ is onto $G$. Then $G$ has a left unit $e$. Moreover, the elements $e, x$ satisfy the $\mu$-condition.

**Proof:** Let $y \in G$. There are $e, z \in G$ such that $zx = y$ and $ex = x$. We have $y = zx = = z(ex) = e(zx) = ey$. Therefore, $e$ is a left unit of $G$. The mappings $L_e, R_x$ are onto $G$. Further, let $u, v$ be elements of $G$ such that $R_x(u) = R_x(v)$. Let $z \in G$. There is $t \in G$ such that $tx = z$. Then $uz = u(tx) = t(ux) = t(vx) = vz$. The last part of the $\mu$-condition (for $e$) is evident (as $L_e = 1_G$).

**Lemma 19:** Every $B_1$ - groupoid with right division is $R$ - transitive.

**Proof:** For all $x, y, z \in G, x \cdot yz = y \cdot xz$. Hence $L_x(yz) = y \cdot L_x(z)$. Thus $L_x \in R_G$. 31
Let $u, v \in G$ be arbitrary elements. There is $z \in G$ such that $zu = L_z(u) = v$. Hence $G$ is $R$-transitive.

**Theorem 12:** Let $G$ be a $B_1$-groupoid. Then the following conditions are equivalent:
1) There exists $x \in G$ such that $R_x$ is onto $G$.
2) There is a commutative semigroup with unit, $G(o)$, and a mapping $\alpha$ which is onto $G$ such that $uv = \alpha(u) o v$ for every $u, v \in G$.

**Proof:** 1) implies 2). By Lemma 18, $G$ has a left unit $e$. Since $R_x$ is onto $G$, there is a mapping $\beta$ such that $R_x \beta = 1_G$ and $\beta(x) = \beta(ex) = e$. Put $G(o) = G(\beta, 1, 1)$. Since $e, x$ satisfy the $\mu$-condition, hence, by Lemma 15, $G(o)$ is a groupoid with unit $x$ and $G = G(o)(R_x, 1, 1)$. Let $u, v, z \in G$. We have $u(vz) = R_x(u) o (R_x(v) o z) = v(uz) = R_x(v) o (R_x(u) o z)$. From this we deduce that $G(o)$ is $B_1$-groupoid.

2) implies 1). This part of the proof is evident.

**Theorem 13:** Let $G$ be a groupoid. Then the following conditions are equivalent:
1) $G$ is a $B_1$-groupoid with right division.
2) $G$ is a $B_1$-groupoid with division and simultaneously a left quasigroup.
3) There is an Abelian group $G(+)$ and a mapping $\alpha$ which is onto $G$ such that $xy = \alpha(x) + y$ for every $x, y \in G$.

**Proof:** 3) implies 2) and 2) implies 1) evidently.

1) implies 3). By Theorem 12, there is a commutative semigroup with unit, $G(+)$, and a mapping $\alpha$ which is onto $G$ such that $G = G(\alpha, 1, 1)$. Therefore, $G$ is a $\mu$-homotope of $G(+)$. Since, by Lemma 19, $G$ is transitive, the semigroup $G(\alpha)$ is, by Theorem 7, a (Abelian) group.

**Theorem 14:** Let $G$ be a $B_1$-groupoid with left cancellation. Let there be $x \in G$ such that $R_x$ is onto $G$ (a permutation). Then the groupoid $G$ can be imbedded in a $B_1$-groupoid $G_1$ which is with division (which is a quasigroup).

**Proof:** By Theorem 12, there is a commutative semigroup $G(o)$ and a mapping $\alpha$ which is onto $G$ such that $G = G(o)(\alpha, 1, 1)$. Let $\beta$ be a mapping such that $\alpha \beta = 1_G$. Then, by Lemma 18, $G(\beta, 1, 1) = G(o)$. Therefore, for every $u, v \in G$ we have,

(1) $u o v = \beta(u) \cdot v$.

Since $G$ is with left cancellation, we get, applying (1), that $G(o)$ is with left cancellation, too. As $G(o)$ is commutative, $G(o)$ is with cancellation. It is well known that every commutative semigroup with cancellation can be imbedded in an Abelian group. Let $G_1(\cdot)$ be any such Abelian group and $\varphi : G(o) \rightarrow G_1(\cdot)$ be a monomorphism. Define the mapping $\kappa$ of $G_1$ into $G_1$ as follows: $\kappa(y) = \varphi R_x \varphi^{-1}(y)$ for $y \in \varphi(G)$, $\kappa(y) = y$ for $y \in G_1, y \not\in \varphi(G)$. The mapping $\kappa$ is, evidently, onto $G_1$. When $R_x$ is moreover one-to-one, then $\kappa$ is a permutation. Put $G_1(\cdot) = G_1(\kappa, 1, 1)$. $G_1(\cdot)$ is a $B_1$-groupoid with division. The mapping $\varphi$ is also monomorphism of $G$ into $G_1(\cdot)$. This completes the proof.
For $B_2$-groupoids we can prove Theorems dual to Theorems 12—14.

**Definition 7:** A groupoid $G$ is called an $A_1$-groupoid ($A_2$-groupoid) if $xy \cdot uv = xu \cdot yv$ ($xy \cdot uv = vy \cdot ux$), for all $x, y, u, v \in G$.

**Lemma 19:** Let $G$ be a groupoid with left (right) division. Let there be $x \in G$ such that the mapping $R_x (L_x)$ is onto $G$ and for $y, u, v \in G$, $yx \cdot uv = yu \cdot xv$. Then the groupoid $G$ is $\Phi (\Phi^*)$-transitive.

**Proof:** For every $y, u, v \in G$ we have $R_u (y). L_x (v) = R_x (y). L_u (v)$. Let $\alpha, \beta$ be any mappings such that $R_x \alpha = L_x \beta = 1_G$. Then $R_u \alpha (y) \cdot v = y \cdot L_u \beta (v)$. Hence $R_u \alpha \in \Phi_G$ for every $u \in G$. From this we see that $G$ is a $\Phi$-transitive groupoid. Similarly for the remaining case.

**Corollary:** Every $A_1$-groupoid with division is $\Phi$ and $\Phi^*$-transitive.

**Theorem 15:** Let $G$ be a groupoid. Then the following conditions are equivalent:

1) $G$ is a $\mu$-groupoid with division and there is $x \in G$ such that for every $u, y, v \in G$, $uy \cdot xv = yu \cdot xv$. 

2) There is a group $G(o)$, its endomorphisms $\varphi, \psi$ which are onto $G(o)$ and $g, h \in G(o)$ such that for every $u, v \in G$, $uv = \varphi(u) \circ g \circ \varphi(v), \varphi(v) \circ h = h \circ \varphi(u)$. 

**Proof:** 1) implies 2). Since $G$, by Lemma 19, is $\Phi$-transitive, there is a group $G(o)$ and mappings $\alpha, \beta$ such that $G = G(o) (\alpha, \beta)$. The mappings $\alpha, \beta$ are onto $G$. For all $y, u, v \in G$ we have $ux \cdot xy = \alpha (\alpha(u) \circ \beta(x)) \circ \beta(\alpha(y) \circ \beta(v)) = uy \cdot xv = \alpha(\alpha(u) \circ \beta(y)) \circ (\beta(\alpha(x) \circ \beta(v)))$. Hence $\alpha(u \circ y) = \alpha_1(u) \circ \beta_1(y)$, $\beta(y \circ v) = \alpha_2(y) \circ \beta_2(v)$, where $\alpha_1, \beta_1$ are convenient mappings. Thus, by Lemma 17, there exist endomorphisms $\varphi, \psi$ of the group $G(o)$ and elements $a, b$ in $G$ such that $\alpha(u) = \varphi(u) \circ a, \beta(u) = b \circ \psi(u)$ for every $u \in G$. Therefore, $uv = \varphi(u) \circ g \circ \psi(v)$, where $g = a \circ b$. Now we can write,

$ux \cdot xv = \varphi^2(u) \circ \varphi(g) \circ \varphi(x) \circ g \circ \varphi(y) \circ \psi(g) \circ \psi(v) =
= uy \cdot xv = \varphi^2(u) \circ \varphi(g) \circ \varphi(y) \circ g \circ \varphi(x) \circ \psi(g) \circ \psi^2(v)$.

From this we get $\varphi(y) \circ g \circ \varphi(x) = \varphi(y) \circ g \circ \varphi^2(y)$. 

Put $y = 1$, where $I$ is the unit of the group $G(o)$. Then $g \circ \varphi(x) = \varphi(x) \circ g = h$. Hence $\varphi(y) \circ h = h \circ \varphi(y)$ for every $y \in G$.

2) implies 1). The groupoid $G$ is, evidently, a $\mu$-homotope of the group $G(o)$. Hence $G$ is a $\mu$-groupoid with division. Put $x = \psi^{-1}(h \circ g^{-1})$. Then $h = \varphi(x) \circ g = \varphi(x) \circ h \circ h^{-1} \circ g = h \circ \varphi(x) \circ h^{-1} \circ g$. Hence $g^{-1} = \varphi(x) \circ h^{-1}$, and hence, $h = g \circ \varphi(x)$.

For every $u, y, v \in G$ we have,

$yx \cdot uv = \varphi^2(y) \circ \varphi(g) \circ \varphi(x) \circ g \circ \varphi(u) \circ \psi(g) \circ \psi^2(v) =
= \varphi^2(y) \circ \varphi(g) \circ h \circ \varphi(u) \circ \psi(g) \circ \psi^2(v) =
= \varphi^2(y) \circ \varphi(g) \circ \varphi^2(u) \circ \psi(g) \circ \psi^2(v) =
= \varphi^2(y) \circ \varphi(g) \circ \varphi(u) \circ g \circ \varphi(x) \circ \psi(g) \circ \psi^2(v) = yu \cdot xv$.

This completes the proof.

**Theorem 16:** Let $G$ be a groupoid. Then the following conditions are equivalent:
1) G is a $\mu$-groupoid with division and there exist elements $x, a, b$ of $G$ such that $ux . vt = uv . xt, au . vb = av . ub$ for all $u, v, t \in G$.
2) G is a $\mu$-groupoid with division and $G$ is an $A_1$-groupoid.
3) There is an Abelian group $G(+)$, its endomorphisms $\varphi, \psi$ which are onto $G$ and $g \in G$ such that $uv = \varphi(u) + \psi(v) + g$ for all $u, v \in G$ and $\varphi \psi = \psi \varphi$.

Proof: 1) implies 3). By Theorem 15, there is a group $G(+)$, its endomorphisms $\varphi, \psi$ which are onto $G$ and $g, h \in G$ such that $uw = \varphi(u) + g + \psi(w)$, $au = \varphi(u) + h$ for all $u, w \in G$.

Put $G(*) = G^L$. For every $u, v \in G$ we have $u \ast v = au \ast vb = av \ast ub = v \ast u$. Thus $G(*)$ is a commutative $\mu$-homotope of $G$. Hence $G(*)$ is a commutative $\mu$-homotope of the group $G(+)$.

Therefore, by Theorem 8, $G(+)$ is an Abelian group. Hence $h + \psi \varphi(u) = \psi \varphi(u) + h = \varphi \varphi(u) + h$, and hence, $\varphi \psi = \psi \varphi$.

3) implies 2) and 2) implies 1) evidently.

Definition 8: Let $G$ be a non-empty set, $n \geq 2$ be a positive integer and $f$ be an $n$-ary operation completely defined on $G$. The algebra $(G, f)$ is called $n$-groupoid. Instead of $(G, f)$ and $f(x_1, \ldots, x_n)$ we shall usually write $G$ and $(x_1, \ldots, x_n)$ only.

Definition 9: Let $G$ be a $n$-groupoid. A mapping $\lambda$ of the set $G$ into $G$ is called $i$-regular, where $1 \leq i \leq n$ if there exists a mapping $\lambda^*$ such that for every $x_1, \ldots, x_n \in G, \lambda(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \lambda^*(x_i), x_{i+1}, \ldots, x_n)$.

Denote by symbol $A_i^G$ the set of all $i$-regular mappings of the $n$-groupoid $G$.

Lemma 20: Let $G$ be a $n$-groupoid. Then for every $i$, $1 \leq i \leq n$, the set $A_i^G$ is a semigroup with unit under the operation of composition of mappings.

Proof: Proof is the same as for Lemma 1.

Definition 10: Let $G$ be a $n$-groupoid. Let $i$ be a positive integer, $1 \leq i \leq n$. An element $e$ of $G$ is called an $i$-unit if for every $x \in G$,

$$(e, \ldots, e, x, e, \ldots, e) = x.$$  

An element $e$ is called a unit if $e$ is a $j$-unit for every $j, 1 \leq j \leq n$.

Lemma 21: Let $G$ be a $n$-groupoid with $i$-unit $e$, $1 \leq i \leq n$. Let $\lambda \in A_i^G$. Then $\lambda = \lambda^*$.

Proof: For every $x \in G$ we have $\lambda(x) = \lambda^* (e, \ldots, e, x, e, \ldots, e) = (e, \ldots, e, \lambda^*(x), e, \ldots, e) = \lambda^*(x)$. Thus $\lambda = \lambda^*$.

Definition 11: Let $G$ be a $n$-groupoid and $a$ be an element of $G$. We say that $a$ satisfies the $\nu$-condition if for every $j, 1 \leq j \leq n$, and for every $x_1, \ldots, x_n \in G$,

$$(x_1, \ldots, x_{j-1}, a, x_j, \ldots, x_n) = (x_1, \ldots, x_{j-1}, a, x_j, a, x_{j+1}, \ldots, x_n)$$

Lemma 22: Let $G$ be a $n$-groupoid with $i$-unit $e$, $1 \leq i \leq n$. Let $e$ satisfy the $\nu$-condition. Then $e$ is a unit of $G$.

Proof: This Lemma follows directly from Definition 11.
Definition 12: Let G be a n-groupoid and i be a positive integer, \(1 \leq i \leq n\). The n-groupoid G is called \(A^i\)-transitive if for every \(x, y \in G\) there is \( \lambda \in A^i \) such that \( \lambda(x) = y \).

Definition 13: Let G be a n-groupoid with \(i, j\)-unit \(e\), where \(1 \leq i, j \leq n\), \(i \neq j\). Define the binary operation \(f_{i,j}\) on G as follows:

For every \(x, y \in G\),
\[
f_{i,j}(x, y) = (e, \ldots, e, x, e, \ldots, e, y, e, \ldots, e)
\]
if \(i < j\)

and \(f_{i,j}(x, y) = (e, \ldots, e, y, e, \ldots, e, x, e, \ldots, e)\) if \(j < i\).

Just defined groupoid \((G, f_{i,j})\) we shall denote by symbol \(G(o)^{i,j}\).

Theorem 17: Let G be a \(A^i\)-transitive n-groupoid with \(i, j\)-unit \(e\), where \(1 \leq i, j \leq n\), \(i \neq j\). Then \(G(o)^{i,j}\) is a group.

Proof: Suppose \(i < j\). The element \(e\) is a unit of the groupoid \(G(o)^{i,j}\). Indeed, for every \(x \in G\) we have
\[
x \circ e = (e, \ldots, e, x, e, \ldots, e) = (e, \ldots, e, x, e, \ldots, e) = e \circ x.
\]
Further, let \(\lambda \in A^i\). For every \(x, y \in G\) we have,
\[
\lambda(x \circ y) = \lambda(e, \ldots, e, x, e, \ldots, e, y, e, \ldots, e) = (e, \ldots, e, \lambda(x), e, \ldots, e, y, e, \ldots, e) = \lambda(x) \circ y.
\]
Hence \(\lambda\) is a left regular mapping of \(G(o)^{i,j}\). Since G is \(A^i\)-transitive, \(G(o)^{i,j}\) is \(A\)-transitive. Therefore, by Theorem 3, \(G(o)^{i,j}\) is a group.

If \(j < i\) the proof is similar.

Definition 13: Let G be a n-groupoid and \(\sigma\) be a permutation of elements \(1, 2, \ldots, n\). The n-groupoid G is called a \(\sigma\)-n-groupoid if there exists a group \(G(o)\) such that for every \(x_1, \ldots, x_n \in G\),
\[
(x_1, x_2, \ldots, x_n) = x_{\sigma(1)} \circ x_{\sigma(2)} \circ \ldots \circ x_{\sigma(n)}.
\]

Lemma 23: Let G be a \(\sigma\)-n-groupoid. Then G is \(A^{\sigma(1)}\)–transitive and \(A^{\sigma(n)}\)–transitive.

Proof: There exists a group \(G(o)\) such that for every \(x_1, \ldots, x_n \in G\),
\[
(x_1, x_2, \ldots, x_n) = x_{\sigma(1)} \circ x_{\sigma(2)} \circ \ldots \circ x_{\sigma(n)}.
\]

Let \(u \in G\). The translation \(R_u\) of the group \(G(o)\) is a \(\sigma(n)\)–regular mapping of the n-groupoid G. Indeed,
\[
R_u (x_1, \ldots, x_n) = x_{\sigma(1)} \circ x_{\sigma(2)} \circ \ldots \circ x_{\sigma(n)} \circ u = x_{\sigma(1)} \circ \ldots \circ x_{\sigma(n-1)} \circ (x_{\sigma(n)} \circ u) = (x_1, \ldots, x_{\sigma(n)-1}, R_u (x_{\sigma(n)}), x_{\sigma(n)+1}, \ldots, x_n).
\]
Since the group \(G(o)\) is a groupoid with division, G is \(A^{\sigma(n)}\)–transitive. Similarly, G is \(A^{\sigma(1)}\)–transitive.

Lemma 24: Let G be a \(A^i\)-transitive n-groupoid with \(i\)–unit \(e\), \(1 \leq i \leq n\). Let \(e\) satisfy the \(\sigma\)-condition. Then G is a \(\sigma\)-n-groupoid for \(\sigma = (i, i+1, \ldots, n, i-1, i-2, \ldots, 1)\).

Proof: There is \(j, 1 \leq j \leq n\), such that \(i \neq j\). Suppose \(i < j\).
By Lemma 22, the element $e$ is a unit of $G$. Therefore, by Theorem 17, the groupoid $G(o)^{t, f}$ is a group.

Let $x_1, \ldots, x_n \in G$ be arbitrary elements. Since $G$ is $A^t$-transitive, there are mappings $\lambda_1, \ldots, \lambda_n \in A^t_o$ such that $x_1 = \lambda_1(e)$, $x_2 = \lambda_2(e)$, $\ldots$, $x_n = \lambda_n(e)$. Since $e$ satisfies the $\nu$-condition and $A^t$ are $i$-regular, we have $(x_1, \ldots, x_n) = (\lambda_1(e), \ldots, \lambda_n(e)) = (\lambda_2(e), \ldots, \lambda_{i-1}(e), e, \lambda_{i+1}(e), \ldots, \lambda_n(e)) = \ldots = (\lambda_n(e), \ldots, \lambda_{i-1}(e), \lambda_i(e), \ldots, e) = (\lambda_{i+1}(e), \ldots, \lambda_n(e), e, \lambda_{i-1}(e), \ldots, e) = \ldots = (e, \ldots, e, \lambda_{i+1}(e), \ldots, \lambda_n(e), e, \lambda_{i-1}(e), \ldots, e).

Conversely, $(x_1, \ldots, x_n) = (\lambda_1(e), \ldots, \lambda_n(e)) = (\lambda_{i+1}(e), \ldots, \lambda_n(e), e, \lambda_{i-1}(e), \ldots, e) = \ldots = (\lambda_n(e), \ldots, \lambda_{i-1}(e), \lambda_i(e), \ldots, e) = \ldots = (e, \ldots, e, \lambda_{i+1}(e), \ldots, \lambda_n(e), e, \lambda_{i-1}(e), \ldots, e).

Thus $G$ is a $\sigma$-groupoid for $\sigma = (1, \ldots, n, i, i-1, \ldots, 1)$.

If $j < i$ the proof is similar.

**Theorem 18:** Let $G$ be a $n$-groupoid. Then the following conditions are equivalent:

1) There exists $i, 1 < i < n$, such that $G$ is $A^t$-transitive and $G$ has an $i$-unit $e$ which satisfies the $\nu$-condition.

2) $G$ is $A^1$ and $A^n$-transitive and $G$ has a unit $g$ which satisfies the $\nu$-condition.

3) There is a group $G(o)$ such that for every $x_1, \ldots, x_n \in G$, $(x_1, \ldots, x_n) = \lambda_1(e) \ldots \lambda_n(e)$.

**Proof:** 1) implies 3). By Lemma 24, $G$ is $\sigma$-groupoid for $\sigma = (1, \ldots, n, i-1, \ldots, 1)$. Hence, by Lemma 23, $G$ is $A^{\sigma(n)}$-transitive. But $\sigma(n) = i$. Hence $G$ is $A^1$-transitive. The element $e$ is, by Lemma 22, a unit of $G$. Hence, by Lemma 24, $G$ is $e$-groupoid for $e = (1, 2, \ldots, n)$.

Since $e$ is the identity permutation, there is a group $G(o)$ such that for every $x_1, \ldots, x_n \in G$,

$$(x_1, \ldots, x_n) = \lambda_1(e) \ldots \lambda_n(e) = (x_1 \circ x_2 \circ \ldots \circ x_n).
$$

3) implies 2) and 2) implies 1) evidently.

**Theorem 19:** Let $G$ be a $n$-groupoid. Then the following conditions are equivalent:

1) There exists $i, 1 < i < n$, such that $G$ is $A^t$-transitive and $G$ has an $i$-unit $e$, which satisfies the $\nu$-condition.

2) $G$ is $A^j$-transitive for all $j, 1 \leq j \leq n$. $G$ has a unit $g$ and an arbitrary element of $G$ satisfies the $\nu$-condition.

3) There is an Abelian group $G(\cdot)$ such that for every $x_1, \ldots, x_n \in G$,

$$(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n.
$$

**Proof:** 1) implies 3). By Theorem 18, there is a group $G(\cdot)$ such that for every $x_1, \ldots, x_n \in G$,

$$(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n.$$

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Let $\lambda \in A_G^n$. Then $\lambda(x) = \lambda(x, e, \ldots, e) = (x, e, \ldots, e, \lambda(e), e, \ldots, e) = x + e + \ldots + \lambda(e) + e + \ldots + e = x + \lambda(e)$. Hence for every $x_1, \ldots, x_n \in G$ we have $\lambda(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n + \lambda(e) = (x_1, \ldots, x_{i-1}, \lambda(x_i), x_{i+1}, \ldots, x_n) = x_1 + \ldots + x_{i-1} + x_i + \lambda(e) + x_{i+1} + \ldots + x_n.$

Since $1 < i < n, i + 1 \leq n$. Hence $x_{i+1} + \ldots + x_n + \lambda(e) = \lambda(e) + x_{i+1} + \ldots + x_n$, and hence, $\lambda(e) + x = x + \lambda(e)$ for all $x \in G$. Using the $A^t$-transitivity, we get that $G(+) \text{ is commutative.}$

3) implies 2) and 2) implies 1) evidently.