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Lattice Socles and Radicals Described by a Galois Connection

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This paper deals with an extension of Stenström's concept of lattice socles and radicals. It is shown that in any modular lattice of finite length, the lower K-socle relative to an ideal $\mathcal{J}$ is equal to the upper one and that in modular algebraic lattices the socles are additive. The paper contains also some instructive counter-examples, as well as some results relating to the K-radicals which are immediate consequences of the theorems concerning the K-socles.

Throughout this paper L denotes a complete lattice.

If $\mathcal{J}$ is an ideal of a lattice T and $K \subset T \setminus \mathcal{J}$, the elements of the set $\mathrm{Ess}_{\mathcal{J}}(K) = \{t \in T \setminus \mathcal{J} \mid \forall\, k \in K\; t \cap k \notin \mathcal{J}\}$ are said to be *K-essential* relative to $\mathcal{J}$.

We omit the proofs of the statements in the following lemmas since they are straightforward.

Lemma 1. *If $K \subset M \subset T \setminus \mathcal{J}$, then*

(i) $\mathrm{Ess}^2_{\mathcal{J}}(K) \supset K$ [1])

(ii) $\mathrm{Ess}_{\mathcal{J}}(K) \supset \mathrm{Ess}_{\mathcal{J}}(M)$

(iii) $\mathrm{Ess}^3_{\mathcal{J}}(K) = \mathrm{Ess}_{\mathcal{J}}(K)$.

Corollary. *The correspondence $\xi : K \mapsto \mathrm{Ess}_{\mathcal{J}}(K)$ defines a Galois connection in T; the correspondence $\xi^2 : K \mapsto \mathrm{Ess}^2_{\mathcal{J}}(K)$ is a closure operation on $T \setminus \mathcal{J}$ and the closed subsets of $T \setminus \mathcal{J}$ form a complete lattice.*

If S is a subset of the lattice T, $[S)$ will denote the set $\{t \in T \mid \exists\, s \in S\; s \leqq t\}$; similarly $(S] = \{t \in T \mid \exists\, s \in S\; t \leqq s\}$.

Lemma 2. *In any lattice T,*

(i) $[\mathrm{Ess}_{\mathcal{J}} K)) = \mathrm{Ess}_{\mathcal{J}}(K)$

(ii) $[\mathrm{Ess}^2_{\mathcal{J}}(K)) = \mathrm{Ess}^2_{\mathcal{J}}(K)$.

Lemma 3. *If $K_\lambda \subset T \setminus \mathcal{J}$, $\lambda \in \Lambda$, then*

(i) $\mathrm{Ess}_{\mathcal{J}}(\mathbf{M}_{\lambda \in \Lambda} K_\lambda) = \mathbf{J}_{\lambda \in \Lambda} \mathrm{Ess}_{\mathcal{J}}(K_\lambda)$

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[1]) Here $\mathrm{Ess}^2_{\mathcal{J}}(K)$ means $\mathrm{Ess}_{\mathcal{J}}(\mathrm{Ess}_{\mathcal{J}}(K))$.

(ii) $\mathrm{Ess}^2{}^{\mathcal{J}}_T (\mathbf{M}_{\lambda\epsilon\Lambda} K_\lambda) \supset \mathbf{J}_{\lambda\epsilon\Lambda} \mathrm{Ess}^2{}^{\mathcal{J}}_T (K_\lambda)$. [2])

Corollary. *In any distributive lattice, the correspondence* ξ^2 *of Lemma 1 defines a topology.*

Proof of Corollary. By (ii), it remains to prove that $t \epsilon \mathrm{Ess}^2{}^{\mathcal{J}}_T (K_1 \mathbf{M} K_2)$ and $t \notin \mathrm{Ess}^2{}^{\mathcal{J}}_T (K_1) \mathbf{M} \mathrm{Ess}^2{}^{\mathcal{J}}_T (K_2)$ implies a contradiction. Indeed, in this case there are elements $u_i \epsilon \mathrm{Ess}^{\mathcal{J}}_T (K_i)$, $i = 1,2$, such that $t \cap u_i \epsilon \mathcal{J}$ and hence $t \cap (u_1 \cup u_2) \epsilon \mathcal{J}$. But by Lemma 2 (i) $u_1 \cup u_2 \epsilon \mathrm{Ess}^{\mathcal{J}}_T (K_i)$ for $i = 1,2$ and so we have, by (i), $u_1 \cup u_2 \epsilon \mathrm{Ess}^{\mathcal{J}}_T (K_1 \mathbf{M} K_2)$, $t \cap (u_1 \cup u_2) \notin \mathcal{J}$ which is a contradiction.

An element t of a lattice T *covers* an ideal $\mathcal{J}$ of T iff there is an $i \epsilon \mathcal{J}$ which is covered by t and $t \notin \mathcal{J}$. Thus, t covers $\mathcal{J}$ iff $\mathcal{J} \not\ni t \succ i \epsilon \mathcal{J}$. An element $k \epsilon T$ covering $\mathcal{J}$ is called a *$\mathcal{J}$-atom* iff it satisfies the condition $(k' \notin \mathcal{J}, k' \leqq k) \Rightarrow k' = k$.

The ideal $\mathcal{J}$ will be usually fixed in our considerations. This motivates the following definitions: The elements of $\mathcal{J}$ are called *elementary particles*. If an element $q \epsilon T$ is such that $q \leqq b$ for every $b \epsilon \mathrm{Ess}^{\mathcal{J}}_T (K)$, it is said to be an $\mathrm{Ess}^2{}^{\mathcal{J}}_T (K)$*-element*. Clearly, a $\mathcal{J}$-atom is an $\mathrm{Ess}^2{}^{\mathcal{J}}_T (K)$-element iff it belongs to $\mathrm{Ess}^2{}^{\mathcal{J}}_T (K)$.

For a subset K of $L \setminus \mathcal{J}$, the *upper K-socle* of the lattice L relative to the ideal $\mathcal{J}$ is the g.l.b. of the set $\mathrm{Ess}^{\mathcal{J}}_L (K)$ and is denoted by $\overline{\mathrm{Soc}}^{\mathcal{J}}_L (K)$; the *lower K-socle* of L relative to $\mathcal{J}$, denoted by $\underline{\mathrm{Soc}}^{\mathcal{J}}_L(K)$, is the l.u.b. of the set the elements of which are the $\mathcal{J}$-atoms and the elementary particles which are $\mathrm{Ess}^2{}^{\mathcal{J}}_L (K)$-elements. Thus, if $\mathcal{J} = \{0\}$, $\underline{\mathrm{Soc}}^{\{0\}}_L(K) = \cup\, a_\lambda$ where a_λ range over all atoms belonging to $\mathrm{Ess}^2{}^{\{0\}}_L (K)$. In this case we omit the phrase "relative to $\{0\}$" and, when no confusion can arise, we write $\underline{\mathrm{Soc}}_L (K)$ instead of $\underline{\mathrm{Soc}}^{\{0\}}_L (K)$ calling this element simply lower K-socle. Similar abbreviations are used for $\overline{\mathrm{Soc}}^{\{0\}}_L (K)$, $\mathrm{Ess}^{\{0\}}_L (K)$ etc. We say that a lattice L has a K-socle relative to $\mathcal{J}$ iff $\overline{\mathrm{Soc}}^{\mathcal{J}}_L (K) = \underline{\mathrm{Soc}}^{\mathcal{J}}_L (K)$.

Lemma 4. *In any lattice L,*

(i) $\underline{\mathrm{Soc}}^{\mathcal{J}}_L (K) \leqq \overline{\mathrm{Soc}}^{\mathcal{J}}_L (K)$

(ii) $K \supset M$ *implies* $\overline{\mathrm{Soc}}^{\mathcal{J}}_L (K) \geqq \overline{\mathrm{Soc}}^{\mathcal{J}}_L (M)$

(iii) $K \supset M$ *implies* $\underline{\mathrm{Soc}}^{\mathcal{J}}_L (K) \geqq \underline{\mathrm{Soc}}^{\mathcal{J}}_L (M)$

(iv) $\overline{\mathrm{Soc}}^{\{0\}}_L (\{k\}) = \underline{\mathrm{Soc}}^{\{0\}}_L (\{k\})$.

Proof. The first three assertions are obvious so we shall deal only with (iv). First consider the case $\overline{\mathrm{Soc}}_L (\{k\}) \neq 0$. Then $0 \prec \overline{\mathrm{Soc}}_L (\{k\}) \epsilon \mathrm{Ess}^2{}_L (\{k\})$ and it therefore follows, by (i), that $\overline{\mathrm{Soc}}_L (\{k\}) = \underline{\mathrm{Soc}}_L (\{k\})$. Next consider the case $\overline{\mathrm{Soc}}_L (\{k\}) = 0$. This time there are no atoms in $\mathrm{Ess}^2{}_L (\{k\})$, consequently $\underline{\mathrm{Soc}}_L (\{k\}) = 0$.

Now we shall formulate the key theorem which is a natural generalization of a very well known property of modular geometric lattices (cf. [2] and [3]).

Theorem 5. *Let L be a modular lattice of finite length and let $b \epsilon L$ be a join*

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[2]) $\mathbf{M}$ is the symbol for the set union and $\mathbf{J}$ for the set intersection.

of some $\mathcal{J}$-atoms. Let $r \notin \mathcal{J}$ be such that $r < b$. Then there exists an element $t \notin \mathcal{J}$ such that

$$\text{(i)} \quad r \cup t = b, \qquad \text{(ii)} \quad \mathcal{J} \ni s = r \cap t,$$

Corollary 1. *With the assumptions and notation of Theorem, for every i such that $\mathcal{J} \ni i < r$ there exists a $t_0 \notin \mathcal{J}$ such that*

$$\text{(i)} \quad r \cup t_0 = b, \qquad \text{(ii)} \quad \mathcal{J} \ni r \cap t_0 \geqq i.$$

Corollary 2. *If i_r is the greatest element of $\mathcal{J}$ with the property $i_r \leqq r$, then there exists an element $t_1 \notin \mathcal{J}$ such that*

$$\text{(i)} \quad r \cup t_1 = b, \qquad \text{(ii)} \quad r \cap t_1 = i_r.$$

Proof. Let $b = \xi_1 \cup \xi_2 \cup \ldots \cup \xi_k \cup r$, where ξ_j are $\mathcal{J}$-atoms such that k is the smallest possible. In particular, $\xi_m \not\leqq \bigcup\limits_{\substack{h=1\\h\neq m}}^{k} \xi_h$ for every $m = 1,2 \ldots, k$. Now we put $t = \bigcup\limits_{h=1}^{k} \xi_h$, $s = r \cap t$.

Then

$$s \leqq (r \cup \bigcup_{h=1}^{k-1} \xi_h) \cap \bigcup_{h=1}^{k} \xi_h = \bigcup_{h=1}^{k-1} \xi_h \cup (\xi_k \cap (r \cup \bigcup_{h=1}^{k-1} \xi_j))$$

by the modular law. Since $\xi_k \succ i_k \in \mathcal{J}$ is a $\mathcal{J}$-atom, it follows either

$$\xi_k = \xi_k \cap (r \cup \bigcup_{h=1}^{k-1} \xi_h) \quad \text{or} \quad i_k \geqq \xi_k \cap (r \cup \bigcup_{h=1}^{k-1} \xi_h).$$

The first alternative implies $b = r \cap \bigcup\limits_{h=1}^{k-1} \xi_h$ which contradicts the definition of k. Thus we must have $s \leqq \bigcup\limits_{h=1}^{k-1} \xi_h \cup i_k$. In the case $k = 1$ this implies $s \leqq i_k$, so we are through. From now on we assume $k > 1$. Then letting ${}^1\eta(j)$ denote the join $i_j \cup \bigcup\limits_{\substack{h=1\\h\neq j}}^{k} \xi_h$, we have by symmetry $s \leqq {}^1\eta(j)$, $j = 1, 2, \ldots, k$, and

$$1 \leqq j_1 \neq j_2 \leqq k \Rightarrow {}^1\eta(j_1) \neq {}^1\eta(j_2).$$

Indeed, if the implication were not true, there would exist two subscripts, say 1 and 2, such that ${}^1\eta(1) = {}^1\eta(2)$. This would imply

$$i_1 \cup \bigcup_{h=2}^{k} \xi_h = \xi_1 \cup \bigcup_{h=2}^{k} \xi_h = {}^1\eta(1)$$

and

$$i_1 \cap \bigcup_{h=2}^{k} \xi_h < \xi_1 \cap \bigcup_{h=2}^{k} \xi_h.$$

Since ξ_1 is a $\mathcal{J}$-atom we should have as above $\xi_1 \cap \bigcup\limits_{h=2}^{k} \xi_h \leqq i_1$, and consequently, $\xi_1 \cap \bigcup\limits_{h=2}^{k} \xi_h \leqq i_1 \cap \bigcup\limits_{h=2}^{k} \xi_h$, a contradiction.

New let ${}^c\eta(j_1 j_2 \ldots j_c)$ be the join $i_{j_1} \cup i_{j_2} \cup \ldots \cup i_{j_c} \cup \bigcup\limits_{\substack{h=1\\h\neq i_{j_1}, i_{j_2}, \ldots, i_{j_c}}}^{k} \xi_h$, the subscripts $i_{j_1}, j_{j_2}, \ldots, i_{j_c}$ being distinct. Assume

$$ {}^c\eta(j_1 j_2 \ldots j_c) = {}^c\eta(j_1' j_2' \ldots j_c'), \; c > 1, \tag{1}$$

where $j_1, j_2, \ldots, j_c$ and $j'_1, j'_2, \ldots, j'_c$ represent two different groups of subscripts. Hence there exists a j which does not appear in the right member of (1), and similarly, there exists a j' which is not on the left side of (1). Without loss of generality, we assume $j = j_c$, $j' = j'_c$. We shall distinguish two cases:
Case I. There exist two subscripts, say j_1 and j'_1, such that $j'_1 = j_1$. Then, by (1),

$$^{c-1}\eta\,(j_2 \ldots j_c) = {}^{c-1}\eta\,(j'_2 \ldots j'_c) \tag{2}$$

and the groups of subscripts in (2) are different.
Case II. $j_q \neq j_p$ for all p, $q = 1, 2, \ldots$, c. Then

$$^{c-1}\eta\,(j_1 j_2 \ldots j_{c-1}) = {}^{c-1}\eta\,(j'_1 j'_2 \ldots j'_{c-1}) \tag{3}$$

and the groups of subscripts in (3) are different.

Hence by induction on c, $^1\eta\,(j) = {}^1\eta\,(j')$ for some $j \neq j'$, which is a contradiction. Therefore

$$(j_1, j_2, \ldots, j_c) \neq (j'_1, j'_2, \ldots, j'_c) \Rightarrow {}^c\eta\,(j_1 j_2 \ldots j_c) \neq {}^c\eta\,(j'_1 j'_2 \ldots j'_c).$$

Let $w_1 = {}^c\eta\,(j_1 j_2 \ldots j_c)$, $w_2 = {}^{c-1}\eta\,(j_2 j_3 \ldots j_c)$, $w_3 = {}^{c-1}\eta\,(j_1 j_3 \ldots j_c)$ where $k \geqq c \geqq 2$. We have $w_2 \neq w_3$, $w_2 \succeq w_1$ and $w_3 \succeq w_1$. Now let us suppose $w_2 = w_1$. Then $\xi_{j_1} \cup \lambda = i_{j_1} \cup \lambda$, λ denoting the element $i_{j_2} \cup i_{j_3} \cup \ldots \cup i_{j_c} \cup \bigcup_{1 \leqq h \neq j_1, j_2, \ldots, j_c \leqq h} \xi_h$.
But $\xi_1 = i_{j_1} \cup (\xi_{j_1} \cap \lambda)$ is a $\mathcal{J}$-atom; hence $\xi_{j_1} \cap \lambda = \xi_{j_1}$ which is impossible. Thus we see that $w_2 \succ w_1$, $w_3 \succ w_1$, $w_2 \neq w_3$ and therefore $w_2 \cap w_3 = w_1$. Specializing to the case $w_2 = {}^1\eta\,(j_2)$, $w_3 = {}^1\eta\,(j_1)$, we obtain by the preceding results $s \leqq {}^2\eta\,(j_1 j_2)$. Again, by induction on c, we find that $s \leqq {}^c\eta\,(j_1 j_2 \ldots j_c)$; consequently

$$s \leqq {}^k\eta\,(1\,2 \ldots k) = i_1 \cup i_2 \cup \ldots \cup i_k \in \mathcal{J}.$$

This completes the proof.
Proof of Corollaries. Put $t_0 = t \cup i$ and $t_1 = t \cup i_r$. Then $L \ni r \cap t_1 \geqq i_r$ implies $i_r = r \cap t_1$.

Our main result can be stated as:

Theorem 6. *Any modular lattice L of finite length has a K-socle relative to $\mathcal{J}$ for every $K \subset L \setminus \mathcal{J}$.*

Proof. We claim that for each $K \subset L \setminus \mathcal{J}$ $\underline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K) = \overline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K)$. For suppose this is not true, so that $\underline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K) < \overline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K)$ for some $K \subset L \setminus \mathcal{J}$ and let $\bar{i}$ and $\underline{i}$ be the greatest elementary particles satisfying $\bar{i} \leqq \overline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K)$ and $\underline{i} \leqq \underline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K)$ respectively. Then $\bar{i}$ is an $\mathrm{Ess}^{2\,\mathcal{J}}_{L}(K)$-element and $\bar{i} \leqq \underline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K)$. Moreover, $\bar{i} = \underline{i}$. Next we prove that $\underline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K) \in \mathcal{J}$ implies $\overline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K) \in \mathcal{J}$. Suppose not and let f be an element such that $\mathcal{J} \ni i \prec f \leqq \overline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K)$. Let f_0 denote a $\mathcal{J}$–atom which satisfies $f_0 \leqq f$. It is clear that f_0 is an $\mathrm{Ess}^{2\,\mathcal{J}}_{L}(K)$-element; hence $\mathcal{J} \not\ni f_0 \leqq \underline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K)$, which is in contradiction to $\underline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K) \in \mathcal{J}$. But $\overline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K) \in \mathcal{J}$ implies $\bar{i} = \overline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K) \geqq \underline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K) \geqq \underline{i}$, so we are in this case done.

Finally, let $\underline{\mathrm{Soc}}{}^{\mathcal{J}}_{L}(K) \notin \mathcal{J}$ and choose a $b \in \mathrm{Ess}^{\mathcal{J}}_{L}(K)$. Let b_k denote a $\mathcal{J}$-atom such that $b_k \leqq b \cap k$, $k \in K$. $b_0 = \bigcup_{k \in K} b_b$ is easily seen to be an element of $\mathrm{Ess}_L^{\mathcal{J}}(K)$, i.e. $b_0 \geqq \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K)$. By Corollary 2 of Theorem 5 there exists a t_1 such that

$$t_1 \cup \underline{\mathrm{Soc}}_L^{\mathcal{J}}(K) = b_0, \qquad \mathcal{J} \ni t_1 \cap \underline{\mathrm{Soc}}_L^{\mathcal{J}}(K) = \bar{i}.$$

By $t_1 \cup \underline{\mathrm{Soc}}_L^{\mathcal{J}}(K) = t_1 \cup \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K) = b_0$, we have $\bar{i} = t_1 \cap \underline{\mathrm{Soc}}_L^{\mathcal{J}}(K) < t_1 \cap \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K)$ by modularity. Since $\bar{i} < t_1 \cap \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K) \leqq \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K)$, it follows that $t_1 \cap \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K) \notin \mathcal{J}$. Hence there exists a ξ_0 such that $\bar{i} \prec \xi_0 \leqq t_1 \cap \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K)$; then of course $\xi_0 \not\leqq \underline{\mathrm{Soc}}_L^{\mathcal{J}}(K)$. On the other hand, it is easily to be check that each $\mathrm{Ess}^{2\,\mathcal{J}}_L(K)$-element covering the ideal $\mathcal{J}$ is equal to the join of a $\mathcal{J}$-atom and an elementary particle both being $\mathrm{Ess}^{2\,\mathcal{J}}_L(K)$ — elements and it is evident that ξ_0 has this property. This leads to the contradiction $\xi_0 \leqq \underline{\mathrm{Soc}}_L^{\mathcal{J}}(K)$ and our proof of Theorem 6 is complete.

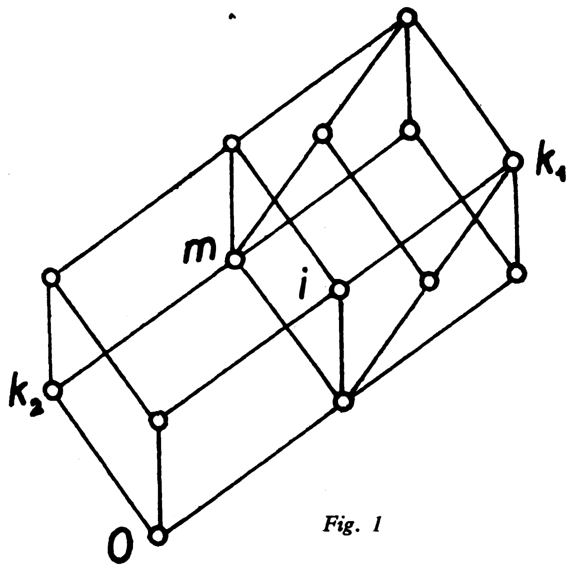

Fig. 1

The elementary particles mentioned in the definition of $\overline{\mathrm{Soc}}_L^{\mathcal{J}}(K)$ cannot be left out. This shows Figure 1 where we choose $K = \{k_1, k_2\}$, $\mathcal{J} = (i]$. Clearly, $\overline{\mathrm{Soc}}_L^{\mathcal{J}}(K) = m$ but the join of all J-atoms of $\mathrm{Ess}^{2\,\mathcal{J}}_L(K)$ is equal to k_2.

We shall now investigate the additivity of the upper K-socles relative to $\mathcal{J}$.

Theorem 7. *Let L be a distributive lattice of finite length. Then*

$$\overline{\mathrm{Soc}}_L^{\mathcal{J}}(\mathbf{M}_{\lambda \in \Lambda} K_\lambda) = \bigcup_{\lambda \in \Lambda} \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K_\lambda)$$

where $K_\lambda \subset L \setminus \mathcal{J}$.

Proof. Assume that the assertion does not hold. By Lemma 4 (ii) this implies that $\overline{\mathrm{Soc}}_L^{\mathcal{J}}(\mathbf{M}\, K_\lambda) > \bigcup \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K_\lambda)$. Then there exists an element c which is maximal among the elements having the property $c \geqq \bigcup \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K_\lambda)$, $c \ngeqq \overline{\mathrm{Soc}}_L^{\mathcal{J}}(\mathbf{M}\ K_\lambda)$. Suppose $c \in \mathbf{J}_{\lambda \in \Lambda} \mathrm{Ess}_L^{\mathcal{J}}(K_\lambda)$. Then by Lemma 3 $c \in \mathrm{Ess}_L^{\mathcal{J}}(\mathbf{M}\, K_\lambda)$, i.e. $c \geqq \overline{\mathrm{Soc}}_L^{\mathcal{J}}(\mathbf{M}\, K_\lambda)$. By this contradiction there exists a λ_0 such that $c \notin \mathrm{Ess}_L^{\mathcal{J}}(K_{\lambda_0})$. Hence $x \not\leqq c$ for all $x \in \mathrm{Ess}_L^{\mathcal{J}}(K_{\lambda_0})$ and therefore $x \cup c \geqq \overline{\mathrm{Soc}}_L^{\mathcal{J}}(\mathbf{M}\, K_\lambda)$. By distributivity $\overline{\mathrm{Soc}}_L^{\mathcal{J}}(\mathbf{M}\, K_\lambda) \leqq \leqq \bigcap (x_\lambda \cup c) = c \cup \bigcap x_\lambda$, the meet being taken over all elements x_λ of $\mathrm{Ess}_L^{\mathcal{J}}(K_{\lambda_0})$. But $c \cup \bigcap x_\lambda = c \cup \overline{\mathrm{Soc}}_L^{\mathcal{J}}(K_{\lambda_0}) = c$ which gives a contradiction to the choice of c.

By inspecting Figure 2 we see that the conclusion of Theorem 7 is not true for modular lattices: It is obvious that $\overline{\mathrm{Soc}}_L^{(i]}(\{k_1\}) = 0$, $\overline{\mathrm{Soc}}_L^{(i]}(\{k_2\}) = i_0$ but $\overline{\mathrm{Soc}}_L^{(i]}(\{k_1, k_2\}) = k_1 \neq i_0 \cup 0$. (Since the lattice L which is sketched in Figure 2 is an amalgam of a special type of two direct produts of modular lattices, it is clearly modular (cf. [1]).)

However, in the special case $\mathcal{J} = \{0\}$, the K-socles are additive in all algebraic (cf. [2], p. 187) modular lattices, as the following result shows.

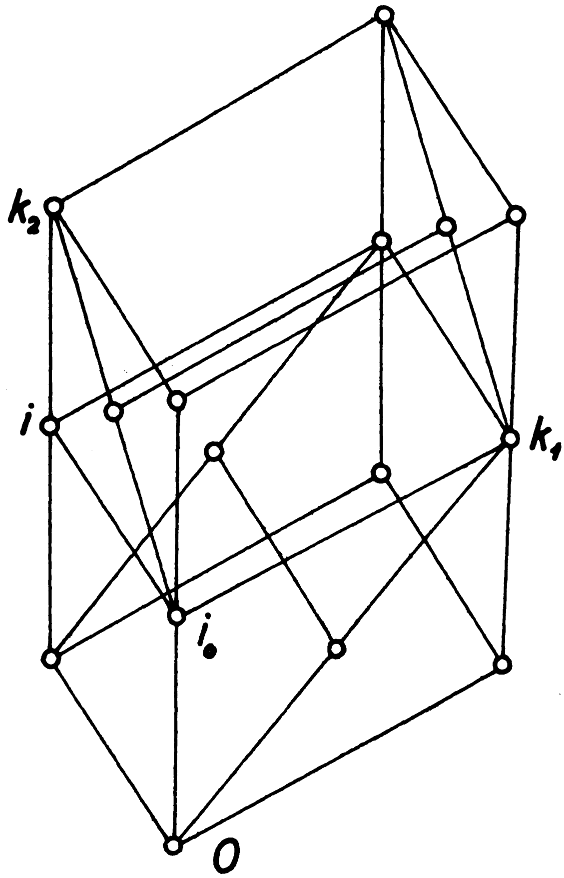

Fig. 2

Theorem 8. *Let L be an algebraic modular lattice. Then*

$$\overline{\mathrm{Soc}}_L^{\{0\}}\big(\mathop{\mathbf{M}}_{\lambda\in\Lambda} K_\lambda\big) = \bigcup_{\lambda\in\Lambda} \overline{\mathrm{Soc}}_L^{\{0\}}(K_\lambda)$$

where $K_\lambda \subset L \setminus \{0\}$.

Corollary. *In any algebraic modular lattice L,*

$$\overline{\mathrm{Soc}}_L^{\{0\}}(K) = \underline{\mathrm{Soc}}_L^{\{0\}}(K) \qquad (4)$$

for all $K \subset L \setminus \{0\}$.

Proof. Suppose by way of contradiction that $\overline{\mathrm{Soc}}_L(\mathbf{M}\, K_\lambda) > \bigcup \overline{\mathrm{Soc}}_L(K_\lambda)$.[3]) Since L is algebraic there exists a compact element k such that $k \leqq \overline{\mathrm{Soc}}_L(\mathbf{M}\, K_\lambda)$ and

$k \not\leqq \bigcup \overline{\mathrm{Soc}}_L (K_\lambda)$. Let m denote an element which is maximal with respect to $m \geqq \bigcup \overline{\mathrm{Soc}}_L (K_\lambda)$ and $m \not\geqq k$. Then $n \notin \mathrm{Ess}_L (\mathbf{M}\, K_\lambda)$ and there exists a λ_0 such that $m \cap h_0 = 0$ for some $h_0 \in K_{\lambda_0}$. Let a $= \bigcap_{\mu \in M} h_\mu$ be the meet of all compact elements which are such that $0 < h_\mu \leqq h_0, \mu \forall \in M$. If $a \neq 0$, then $0 \prec a \leqq b' \cap h_0$ for all $b' \in \mathrm{Ess}_L (K_{\lambda_0})$. Thus $a \leqq \overline{\mathrm{Soc}}_L (K_{\lambda_0})$ and

$$0 = m \cap h_0 \geqq \overline{\mathrm{Soc}}_L (K_{\lambda_0}) \cap h_0 \geqq a \cap h_0 = a,$$

a contradiction. Now let $a = 0$. Since $[m, m \cup h_0]$ and $[0, h_0]$ are transposes, it follows that

$$k \leqq \bigcap_{\mu \in M} (m \cup h_\mu) = m \cup a = m,$$

a contradiction. This completes the proof of this result.[3]

Proof of Corollary. $\overline{\mathrm{Soc}}_L (K) = \overline{\mathrm{Soc}}_L (\mathbf{M}_{k \in K} \{k\}) =$

$= \bigcup_{k \in K} \overline{\mathrm{Soc}}_L (\{k\})$	by Theorem 8
$= \bigcup_{k \in K} \underline{\mathrm{Soc}}_L (\{k\})$	by Lemma 4 (iv)
$\leqq \underline{\mathrm{Soc}}_L (\mathbf{M} \{k\}) = \underline{\mathrm{Soc}}_L (K)$	by Lemma 4 (iii)
$\leqq \overline{\mathrm{Soc}}_L (K)$	by Lemma 4 (i).

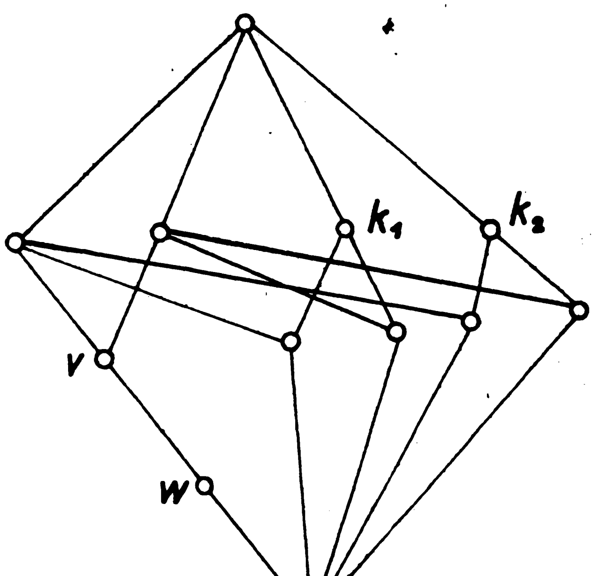

Fig. 3

[3]) The technique of separation of these elements illustrated in the proof is essentially that of [4].

The Corollary does not hold in all lattices: The lattice L shown in Figure 3 is such that $\overline{\mathrm{Soc}}_L\,\{k_1, k_2\} = v$ but $\underline{\mathrm{Soc}}_L\,(\{k_1, k_2\}) = w$.

Theorem 9. *Let L be a lattice satisfying the descending chain condition. Assume that either*

(a) $(\overline{\mathrm{Soc}}_L^{\{0\}}\,(K)]$ *is a complemented lattice*

or

(b) $\overline{\mathrm{Soc}}_L^{\{0\}}\,(K) \in \mathrm{Ess}_L^{\{0\}}\,(K)$.

Then L has a K-socle for every $K \subset L \setminus \{0\}$.

Corollary. *Let L be a relatively complemented lattice which satisfies the descending chain condition. Then* (4) *holds.*

Proof. (a): We show that $\overline{\mathrm{Soc}}_L\,(K) = \underline{\mathrm{Soc}}_L\,(K)$. If not, then we can find a complement c of $\underline{\mathrm{Soc}}_L\,(K)$ in $[0, \overline{\mathrm{Soc}}_L\,(K)]$ and an atom c_0 such that $0 \prec c_0 \leqq c$. Then $c_0 \in \mathrm{Ess}_L^2\,(K)$ and this contradicts the fact that $c_0 \not\leqq \underline{\mathrm{Soc}}_L\,(K)$.
(b): By assumption $0 \neq k \cap \overline{\mathrm{Soc}}_L\,(K)$ for $k \in K$. If k_0 is such that $0 \prec k_0 \leqq k \cap \cap\, \overline{\mathrm{Soc}}_L\,(K)$, than $k_0 \in \mathrm{Ess}_L^2\,(K)$ and this yields $k \cap \underline{\mathrm{Soc}}_L\,(K) \neq 0$ for every $k \in K$. Therefore $\underline{\mathrm{Soc}}_L\,(K) \in \mathrm{Ess}_L\,(K)$, $\underline{\mathrm{Soc}}_L\,(K) \geqq \overline{\mathrm{Soc}}_L\,(K)$ and hence by Lemma 4 (i) $\underline{\mathrm{Soc}}_L\,(K) = \overline{\mathrm{Soc}}_L\,(K)$.

We remark in passing that the condition (b) which means that $\mathrm{Ess}_L\,(K)$ is closed under meet need not hold even if we assume that L is distributive. This may be shown by the direct product $2 \otimes 2 \otimes 3$ consisting of all triplets (m, n, p) where $m = 0,1$, $n = 0,1$, $p = 0,1,2$: If one chooses $K = \{(1,1,0), (0,0,2)\}$, then $\mathrm{Soc}_L(K) = (0, 0, 1) \notin \mathrm{Ess}_L\,(K)$.

A subset $H \subset L \setminus \{0\}$ is said to be a *seave* of L iff $h \in H, 0 \prec h_0 \leqq h$ implies $h_0 \in H$.

Theorem 10. *In any dually algebraic lattice L,*

$$\overline{\mathrm{Soc}}_L^{\{0\}}\,(\mathbf{M}_{\lambda\in\Lambda}\, H_\lambda) = \bigcup_{\lambda\in\Lambda} \overline{\mathrm{Soc}}_L^{\{0\}}\,(H_\lambda)$$

for any system $\{H_\lambda\}_{\lambda\in\Lambda}$ of seaves such that $H_\lambda \subset L \setminus \{0\}$.

Corollary. *If L is dually algebraic lattice, then*

$$\overline{\mathrm{Soc}}_L^{\{0\}}\,(H) = \underline{\mathrm{Soc}}_L^{\{0\}}\,(H)$$

for any seave H of L.

Proof. Suppose we do not have $\overline{\mathrm{Soc}}_L\,(\mathbf{M}\, H_\lambda) = \bigcup \overline{\mathrm{Soc}}_L\,(H_\lambda)$. Then $\overline{\mathrm{Soc}}_L\,(\mathbf{M}\, H_\lambda) > \bigcup \overline{\mathrm{Soc}}_L\,(H_\lambda)$ and there exists a dually compact element k such that $k \geqq \bigcup \overline{\mathrm{Soc}}_L\,(H_\lambda)$ and $\overline{\mathrm{Soc}}_L\,(\mathbf{M}\, H_\lambda) \not\geqq k \notin \mathrm{Ess}_L\,(\mathbf{M}\, H_\lambda)$. Thus there exists an element $h \in H_{\lambda_0}$ which is minimal such that $h \cap k = 0$, $0 \neq$ h. Clearly $0 \prec h \in H_{\lambda_0} \subset \mathrm{Ess}_L^2\,(H_{\lambda_0})$ so $h \leqq \bigcup.\ \overline{\mathrm{Soc}}_L\,(H_\lambda) \leqq k$, a contradiction.

If $D(L)$ denotes the dual of L and D is an ideal of $D(L)$, $K \subset D(L) \setminus D$, the *upper K-radical* of L relative to the dual ideal D of L, denoted by $\overline{\mathrm{Rad}}_L^D\,(K)$ is defined to be the lower K-socle $\underline{\mathrm{Soc}}_{D(L)}^D\,(K)$. Similarly, $\overline{\mathrm{Soc}}_{D(L)}^D\,(K)$ is called a *lower*

K-radical of L relative to D and is denoted by $\underline{\mathrm{Rad}}_L^D(K)$. If $\overline{\mathrm{Rad}}_L^D(K) = \underline{\mathrm{Rad}}_L^D(K)$, we say that L has a *K-radical* relative to D.

Using the previous results concerning the K-socles we may obtain by duality corresponding theorems for K-radicals. Here we mention the following typical result:

Theorem 11. *If L is a modular lattice of finite length, then L has a K-radical relative to D for any $K \subset L \setminus D$.*

If L is an algebraic lattice, then

$$\underline{\mathrm{Rad}}_L^{\{1\}}\left(\mathbf{M}_{\lambda\in\Lambda} H_\lambda\right) = \overline{\mathrm{Rad}}_L^{\{1\}}\left(\mathbf{M}_{\lambda\in\Lambda} H_\lambda\right) = \bigcap_{\lambda\in\Lambda} \overline{\mathrm{Rad}}_L^{\{1\}}(H_\lambda) = \bigcap_{\lambda\in\Lambda} \underline{\mathrm{Rad}}_L^{\{1\}}(H_\lambda)$$

for any system $\{H_\lambda\}_{\lambda\in\Lambda}$ of dual seaves of L.

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