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Trajectories and Natural Numbers

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Introduction

In the theory of sets, in which the existence of natural numbers is assumed, the investigation of trajectories and their classification are the easy problems. In this article we approach an investigation of trajectories not being acquainted with natural numbers. It is shown here, that the classification of trajectories is possible to establish without the notion of the natural number, using only the considerations based on sets, which may be a methodical contribution for the development of sets theory. In the second part we have shown, how the notion of the natural number can be introduced when the classification of trajectories is already established and how being familiar with trajectories, the basic properties of natural numbers can be derived easily.

In the article the Morse's theory is used. All the considerations carried out in it, is possible to transform into the Gödel – Bernays's system, since the only class defined by innormal formula, can be defined by the different manner too, namely using the normal formula, how it is described in the Appendix. At the beginning of the article the use of the Axiom of Infinity is omitted. At the place, where the introduction of this Axiom is desirable, one of its possible formulations is given.

I. Trajectories

1. Transformation of the class into itself

Assume, in all this article, A is a class, F a correspondence of A into A, X a subclass of A.

Definition. *We shall say* the class X is closed (*with respect to the mapping* F) *if for every* $z \in X$ *follows* $F(z) \in X$.

Definition. *Suppose* $x \in A$. *The class of all* $z \in A$, *such that for every closed class* X, *containing* x, z *belongs to* X, *will be called* the trajectory of x (*denoted* $Tr^F(x)$ *or briefly* $Tr(x)$). Then

$$Tr(x) = \{z; (\forall X) [(X \text{ is closed}) \wedge (x \in X)] \Rightarrow (z \in X)\}$$

Lemma 1. *It holds*

a) $x \in Tr(x)$

b) *If X is closed, $x \in X$, then $Tr(x) \subseteq X$.*

c) *$Tr(x)$ is closed.*

Proof: a) follows immediately from the definition of $Tr(x)$.

b) Let us suppose that X is a closed class, such that $x \in X$, and let $a \in Tr(x)$, than a belongs to every closed subclass, containing x; thus $a \in X$.

c) Let $b \in Tr(x)$, then, according to the definition of $Tr(x)$, b belongs to any closed subclass of the class A, containing x. Then $F(b)$ belongs to every closed subclass, containing x. Hence $F(b) \in Tr(x)$ and $Tr(x)$ is therefore a closed class.

Lemma 2. *$y \in Tr(x)$ if and only if $Tr(y) \subseteq Tr(x)$.*

Proof: Let us suppose $y \in Tr(x)$ and assume $z \in Tr(y)$. Then, according to the definition of $Tr(x)$, there is $y \in U$, provided U is an arbitrary closed subclass of A, containing x. Then U is a closed subclass, containing y and since $z \in Tr(y)$, it implies $z \in U$. Hence z belongs to every closed subclass, containing x and therefore $Tr(y) \subseteq Tr(x)$. Let us consider now that $Tr(y) \subseteq Tr(x)$, then according to Lemma 1, there is $y \in Tr(y)$ and so $y \in Tr(x)$.

Consequence. $Tr\,(F(x)) \subseteq Tr(x)$.

Note. From Lemma 2 it follows $a \in Tr(b)$ if and only if $Tr(a) \cap Tr(b) = Tr(a)$.

Lemma 3. $Tr(x) = \{x\} \cup Tr(F(x))$.

Proof: $Tr(F(x))$ is closed, according to Lemma 1; also $\{x\} \cup Tr(F(x))$ is closed because, according to Lemma 1, there is $F(x) \in Tr(F(x))$ and thus $F(x) \in \{x\} \cup \cup Tr(F(x))$. Since $x \in \{x\} \cup Tr(F(x))$ is trivial then, according to Lemma 1 and the above consideration, $Tr(x) \subseteq \{x\} \cup Tr(F(x))$.

Let $y \in \{x\} \cup Tr(Fx))$. If $y \in \{x\}$, then $y = x$ and Lemma 1 implies $y \in Tr(x)$. If $y \in Tr(F(x))$, then the consequence of Lemma 2 implies $y \in Tr(x)$. Thus $Tr(x) \supseteq \supseteq \{x\} \cup Tr(F(x))$ which, together with the above mentioned result, completes the proof.

Consequence 1. *$y \in Tr(x)$, $y \neq x$ implies $y \in Tr(F(x))$.*

Consequence 2. *If $Tr(u) \subseteq Tr(v)$* and *$u \neq v$, then $Tr(u) \subseteq Tr(F(v))$.*

Proof: If $Tr(u) \subseteq Tr(v)$ then, according to Lemma 2, there is $u \in Tr(v)$, where $Tr(v) = \{v\} \cup Tr(F(v))$. Since $u \neq v$, then $u \in Tr\,(F(v))$.

Lemma 4. *If $y \in Tr(x)$, then just one holds: either $Tr(x) - Tr(y) = \emptyset$, or $Tr(x) - - Tr(y)$ is not closed.*

Proof: The assertion is obvious provided that $B = Tr(x) - Tr(y)$ is empty. Let $B \neq \emptyset$. Suppose that B is closed. It holds, $x \in B$ (obviously $x \in Tr(x)$ and if in addition $x \in Tr(y)$ then, according to Lemma 2, there is $Tr(x) \subseteq Tr(y)$ and hence $B = \emptyset$, which contradicts the assumption $B \neq \emptyset$). Therefore B is a closed class, containing x; according to Lemma 1, there is $Tr(x) \subseteq B$. From this and from the assumption $y \in Tr(x)$, $y \in B$ follows. It is in the contradiction with Lemma 1, according to which $y \in Tr(y)$.

Lemma 5. *Let us assume* $Y = \emptyset$, $Y \subseteq Tr(x)$, Y *is a closed class. Then there exists* $y \in Tr(x)$ *such that* $Y = Tr(y)$.

Proof: *Let* $Y \neq \emptyset$, $Y \subseteq Tr(x)$, Y closed. Let us denote

$$Z = \{y; (y \in Tr(x)) \wedge (Y \subseteq Tr(y))\}.$$

The class Z is not empty, since $x \in Z$. We shall prove that Z is closed. Suppose $y \in Z$. From the definition of Z there is $Y \subseteq Tr(y)$ and so Lemma 3 implies $Y \subseteq \{y\} \cup \cup Tr(F(y))$. If $y \in Y$, then (according to Lemma 1) $Tr(y) \subseteq Y$. From the definition of Z we know, that $Y \subseteq Tr(y)$ and therefore $Y = Tr(y)$; thus Lemma 5 is proved in this case. Let us suppose now $y \notin Y$. Then $Y \subseteq Tr(F(y))$. Since $y \in Tr(x)$, it also holds $F(y) \in Tr(x)$ and with the use of $Y \subseteq Tr(F(y))$, there is $F(y) \in Z$. Thus Z is closed and $x \in Z$. According to Lemma 1, there is $Tr(x) \subseteq Z$. At the same time the definition of Z implies $Z \subseteq Tr(x)$ and therefore $Z = Tr(x)$ holds. So we have shown that for any $v \in Tr(x)$, $Y \subseteq Tr(v)$ follows. Let us choose an arbitrary $a \in Y$. Then there is $Tr(a) \subseteq Y$ (according to Lemma 1). Hence $Y = Tr(a)$.

Lemma 6. *If* $Tr(x) \cap Tr(y) \neq \emptyset$, *then there exists* z, *such that* $Tr(x) \cap Tr(y) = = Tr(z)$.

Proof: Let us denote $Tr(x) \cap Tr(y) = Y$. If $z \in Y$ then $z \in Tr(x)$ and $z \in Tr(y)$ at the same time, and there are $F(z) \in Tr(x)$ and $F(z) \in Tr(y)$ (according to Lemma 1), from which $F(z) \in Tr(x) \cap Tr(y)$ follows. Thus Y is closed and non-empty according to the assumption. Obviously $Y \subseteq Tr(x)$. Thus the assumptions of Lemma 5 hold. It means that $a \in Tr(x)$ exists, such that $Y = Tr(a)$.

Lemma 7. *Let* $y, z \in Tr(x), y \notin Tr(x)$. *Then* $z \in Tr(y)$.

Proof: Let us suppose the assumptions of the Lemma under consideration are satisfied. Denote

$$Y = \{u; (u \in Tr(x)) \wedge (u \notin Tr(z)) \wedge (Tr(u) \cap Tr(z) = Tr(y) \cap Tr(z))\}.$$

The class Y is non-empty, since $y \in Y$. Let us prove, that $x \in Y$. Suppose on the contrary $x \notin Y$. Then $x \in Tr(x) - Y$. Let us show $Tr(x) - Y$ is closed. Let $u \in Tr(x) - Y$ and suppose further $F(u) \notin Tr(x) - Y$, i.e. $F(u) \in Y$. Then the definition of Y implies that $Tr(F(u)) \cap Tr(z) = Tr(y) \cap Tr(z)$. Moreover $Tr(u) \cap Tr(z) = (Tr(F(u)) \cup \{u\}) \cap Tr(z) \supseteq Tr(F(u)) \cap Tr(z) = Tr(y) \cap Tr(z)$. It is true even that $Tr(u) \cap Tr(z) \supset Tr(y) \cap Tr(z)$ (in the opposite case there is $u \in Y$ which leads to the contradiction to the assumption, that $u \in Tr(x) - Y$). Thus $Tr(F(u)) \cap Tr(z) \subset Tr(u) \cap Tr(z) = (Tr(F(u)) \cup \{u\}) \cap Tr(z)$, which implies $u \in Tr(z)$ and therefore $F(u) \in Tr(z)$. But from the definition of $Y, F(u) \notin Y$ follows, which contradicts the above considerations. Therefore $Tr(x) - Y$ is closed, and $x \in Tr(x) - Y$. According to Lemma 1, $Tr(x) \subseteq Tr(x) - Y$ holds and consequently $Y = \emptyset$. We have come to the contradiction to the fact that $y \in Y$, shown above. Hence $x \in Y$ and consequently $Tr(x) \cap Tr(z) = Tr(y) \cap Tr(z)$. Since $z \in Tr(x)$ holds, there is $Tr(x) \cap Tr(z) = Tr(z)$. It follows finally, that $Tr(y) \cap Tr(z) = = Tr(z)$ and $z \in Tr(y)$.

2. Classification of trajectories

Definition. *A set x is called* an invariant set, *if* $F(x) = x$.

Lemma 8. *If x is an invariant set, then* $Tr(x) = \{x\}$.

Definition. *A trajectory* $Tr(x)$ *is called* a cycle *if there exists* $y \neq x$, *such that* $Tr(x) = Tr(y)$.

Note. $Tr(x)$ is not cycle implies $Tr(x) = Tr(y)$ if and only if $x = y$.

Lemma 9. *Let* $Tr(x)$ *is a cycle,* $z \in Tr(x)$. *Then* $Tr(z) = Tr(x)$.

Proof: Denote

$$Y = \{u; (u \in Tr(x)) \wedge (Tr(u) = Tr(x))\}.$$

Since $x \in Y$, Y is non-empty. Let us prove that Y is closed. Suppose $u \in Y$. According to Lemma 3, there is $Tr(u) = Tr(F(u)) \cup \{u\} = Tr(x)$. If $u \in Tr(F(u))$, then $\{u\} \cup Tr(F(u)) = Tr(F(u)) = Tr(x)$, which means that $F(u) \in Y$. Let now $u \notin Tr(F(u))$. Since $Tr(x)$ is a cycle, then there exists $y \in Tr(x)$ such that $x \neq y$. Since $x \in Tr(x)$, u must be distinct from at least one of the sets x, y. Without any restriction of generality we can suppose, that for example $x \neq u$. Because $x \in Tr(u) = \{u\} \cup Tr(F(u))$, there is $x \in Tr(F(u))$. Utilizing Lemma 2 upon the previous result we receive $Tr(x) \subseteq Tr(F(u))$. Since $u \in Tr(x)$ and therefore $F(u) \in Tr(x)$, then there is also $Tr(F(u)) \subseteq Tr(x)$; so we have $Tr(x) = Tr(F(u))$, which means that $F(u) \in Y$ again. Hence the class Y is closed. Following the fact, that $x \in Y$ and taking in the account Lemma 1 we obtain $Tr(x) \subseteq Y$ and from the definition of Y there is $Y \subseteq Tr(x)$ and consequently $Y = Tr(x)$. For every $z \in Tr(x)$ there is then $Tr(z) = Tr(x)$.

Lemma 10. *Let us assume* $Tr(x)$ *is a cycle, and* X *is an arbitrary non-empty, closed subclass* $Tr(x)$. *Then* $X = Tr(x)$.

Proof: The class X is non-empty; so there is some z in X. But $z \in Tr(x)$ and therefore the preceeding Lemma implies $Tr(z) = Tr(x)$. Thus $X \subseteq Tr(z)$. From the Lemma 1 follows that $Tr(z) \subseteq X$ and finally $X = Tr(z) = Tr(x)$.

Consequence. *If* $Tr(x)$, $Tr(y)$ *are such cycles, that* $Tr(x) \subseteq Tr(y)$. *Then* $Tr(x) = Tr(y)$.

Lemma 11. *Any trajectory contains at most one cycle or at most one invariant set and these two possibilities are mutually exclusive.*

Proof: Let us have $Tr(z)$, $Tr(y)$ two distinct cycles upon $Tr(x)$. Since, according to Lemma 2, there is $y \in Tr(x)$, then Lemma 7 implies that just one from both cases holds: either $y \in Tr(z)$ or $z \in Tr(y)$. We can suppose without any restriction of generality that $y \in Tr(z)$ is valid. Then Lemma 2 leads to the conclusion that $Tr(y) \subseteq Tr(z)$. Because $Tr(y)$ is non-empty and closed set and $Tr(z)$ is a cycle, Lemma 10 gives $Tr(y) = Tr(z)$, which contradicts the assumption about the distinction of both trajectories.

Suppose now, that $Tr(x)$ contains a cycle $Tr(y)$ and an invariant set a. Then, according to Lemma 9, the set a cannot belong to $Tr(y)$ (otherwise $Tr(y) = Tr(a) =$

$= \{a\}$) and therefore y belongs to $Tr(a) = \{a\}$, according to Lemma 7. Thus $y = a$ which contradicts the assumption.

Assume finally that $Tr(x)$ contains two invariant sets a and b, where $a \neq b$. If for instance $a \notin Tr(b)$ then, according to Lemma 7, there is $b \in Tr(a) = \{a\}$ and therefore $a = b$.

Definition. *Any* set $a \in Tr(x)$ has an antecedent *in* $Tr(x)$ *if and only if there exists* $b \in Tr(x)$ *such that* $F(b) = a$.

Lemma 12. *If* $Tr(x)$ *is not a cycle and* x *is not an invariant set, then* x *has no antecedent in* $Tr(x)$.

Proof: Let us suppose that the assumptions of Lemma 12 are satisfied and let there exists $y \in Tr(x)$ such that $F(y) = x$. Obviously $F(y) \in Tr(x)$, so that $x \in Tr(y)$. Therefore Lemma 2 implies $Tr(x) \subseteq Tr(y)$ and $Tr(y) \subseteq Tr(x)$ at the same time. Since $y \neq x$ (otherwise x is an invariant set), then $Tr(x) = Tr(y)$ is a cycle which contradicts the assumptions.

Note. If x is an invariant set, then x has just one antecedent in $Tr(x)$.

Lemma 13. *Let* $y \in Tr(x)$. $y \neq x$. *Then* y *has at least one antecedent in* $Tr(x)$.

Proof: Let $y \in Tr(x), y \neq x$ and y has no antecedent in $Tr(x)$. Then $Tr(x) - \{y\}$ is non-empty (it contains x) and closed class, because for every $u \in Tr(x) - \{y\}$ there is $F(u) \in Tr(x)$ and also according to the assumption, $F(u) \neq y$ for every u, i.e. $F(u) \notin \{y\}$. Hence, according to Lemma 1, $Tr(x) \subseteq Tr(x) - \{y\}$, from where $y \notin Tr(x)$, which implies the contradiction.

Lemma 14. *If* $Tr(x)$ *is a cycle, then any* $a \in Tr(x)$ *has at least one antecedent in* $Tr(x)$.

Proof: For any $y \neq x$ the assertion follows immediately from Lemma 13. Since $Tr(x)$ is a cycle, there exists $u \neq x, u \in Tr(x)$ such that $Tr(u) = Tr(x)$. According to Lemma 13, any $b \in Tr(u)$, $b \neq u$ has an antecedent in $Tr(x)$. Thus x has also an antecedent in $Tr(x)$.

Lemma 15. *Let* $Tr(x)$ *contains an invariant set. Then there are no* $u, v, y \in Tr(x)$ *mutually distinct sets, to be* $F(u) = F(v) = y$.

Proof: Let us suppose on the contrary, that there are mutually distinct $u, v, y \in Tr(x)$ for which $F(u) = F(v) = y$. It follows, that $y \in Tr(u)$, $y \in Tr(v)$. Since $u, v \in Tr(x)$, there can be for example $u \in Tr(v)$ (see Lemma 7). Then $Tr(v) = \{v\} \cup Tr(F(v)) = \{v\} \cup Tr(y)$. Since $u \neq v$, there is $u \in Tr(y)$ and since also $y \in Tr(u)$ holds, u and y belong to the cycle. It includes the contradiction, since $Tr(x)$ has an invariant set and, according to Lemma 11, it cannot contain a cycle at the same time.

Lemma 16. *Let* p *is an invariants set of trajectory* $Tr(x)$. *Then for any* $a \in Tr(x)$, $a \neq p$, $a \neq x$ *just one antecedent exists in* $Tr(x)$.

Proof: According to Lemma 13, for every $a \neq x$, $a \in Tr(x)$ there is at least one antecedent in $Tr(x)$. If $a \neq p$, $a \neq x$, then, according to Lemma 15, a has just one antecedent in $Tr(x)$.

Lemma 17. *If* $p \neq x$, *where* p *is an invariant set in* $Tr(x)$, *then* p *has just two distinct antecedents in* $Tr(x)$.

Proof: According to Lemma 13, p has at least one antecedent in $Tr(x)$, which is equal to p. Let us suppose p has no other antecedent in $Tr(x)$. Then for any $z \in Tr(x)$, $z \neq p$ there is $F(z) \neq p$. It implies, the class $Y = Tr(x) - \{p\}$ is closed and obviously Y contains x. Hence, according to Lemma 1, there is $Tr(x) \subseteq Y$. But $p \in Tr(x)$ and $p \notin Y$ at the same time; it contradicts the condition $Tr(x) \subseteq Y$. The fact p has just two distinct antecedents in $Tr(x)$ follows from Lemma 15.

Lemma 18. *If $Tr(x)$ contains an invariant set, then $Tr(x)$ is a set.*

Proof: Let us suppose p is an invariant set in $Tr(x)$. If $p = x$, then $Tr(x) = \{x\} = \{x, x\}$; but any unordered couple of sets is a set, thus $Tr(x)$ is a set. Let now $p \neq x$. According to Lemma 11, $Tr(x)$ has only one invariant set. Let us denote

$$Y = \{y;\ (\exists u)\ ((x \in u) \wedge (y \in u) \wedge (u \subseteq Tr(x))) \wedge (\forall z)\ [(z \in u) \wedge (z \neq y)] \Rightarrow \Rightarrow (F(z) \in u)\}.$$

The class Y is non-empty, since $x \in Y$ (it is enough to take $u = \{x\}$). Show that the class Y is also closed, i.e. if $y \in Y$, then $F(y) \in Y$. In this case it is enough to put $u' = u \cup \{F(u)\}$; u' has the properties requested by the definition of the class Y (since u' is a union of two sets, u' is obviously a set). According to Lemma 1, there is $Tr(x) \subseteq Y$ and the definition of Y implies $Y \subseteq Tr(x)$. Thus $Tr(x) = Y$. It holds, that $p \in Tr(x) = Y$. Hence u_p exists, having the properties required by the definition of Y. We shall show now that u_p is closed. There are $x, p \in u_p$. If $z \neq p$, $z \in u_p$, then $F(z) \in u_p$ (from the definition of Y). If $z = p$, there is $z \in u_p$ and also $F(z) \in u_p$, because $F(z) = F(p) = p$. Thus u_p is a closed set, containing x. According to Lemma 1, then there is $Tr(x) \subseteq u_p$. Inclusion $u_p \subseteq Tr(x)$ follows from the definition of Y and from the choice of u_p; therefore $u_p = Tr(x)$ which means, that $Tr(x)$ is a set.

Note. In the further concept let us use the following denotation. If H is the given correspondence, $Tr^H(x)$ means the trajectory of x, $x \in A$, under the correspondence H. For the purpose of the abbreviation we shall use sometimes $Tr(x)$ instead of $Tr^F(x)$, where F is the mapping mentioned at the beginning of this article.

Lemma 19. *Let $Tr^F(x)$ is a cycle, $y \in Tr^F(x)$ and suppose H is the mapping of the class A into A, defined as follows: for any $a \in A$, $a \neq y$ there is $H(a) = F(a)$ and $H(y) = y$. Then $Tr^H(F(y)) = Tr^F(F(y)) = Tr^F(x)$ holds, where $Tr^H(F(y))$ is a trajectory with an invariant set y.*

Proof: Since $Tr^F(x)$ is, by the assumption, a cycle then, according to Lemma 9, there is $Tr^F(y) = Tr^F(F y)) = Tr^F(x)$, which implies $y \in Tr^F(F(y))$. Thus there exists $z \in Tr^F(F(y))$ such that $F(z) = y$ where $z \neq y$ (see Lemma 11). Since both the mappings F and H on $A - \{y\}$ are equal to each other, there is $Tr^F(F(y)) - \{y\} = Tr^H(F(y)) - \{y\}$. Since $y \in Tr^F(F(y))$ and $H(z) = F(z) = y$, there is also $y \in Tr^H(F(y))$ and then $Tr^F(F(y)) = Tr^H(F(y))$ holds. In addition, $Tr^H(F(y))$ is obviously a trajectory with an invariant set y.

Lemma 20. *Let $Tr(x)$ is a cycle, then $Tr(x)$ is a set.*

Proof: Let $Tr^F(x)$ is cycle, H is the mapping from the previous Lemma. Then

$Tr^F(x) = Tr^H(F(y))$, but $Tr^H(F(y))$ is a set, according to Lemma 18, hence $Tr^F(x)$ is a set.

Lemma 21. *If $Tr(x)$ is a cycle, any $a \in Tr(x)$ possesses in $Tr(x)$ just one antecedent.*

Proof: If the cycle $Tr(x)$ is a couple, the assertion of the Lemma is obvious. Suppose now that $Tr(x)$ is not a couple and let a is an arbitrary set, $a \in Tr(x)$. Denote $F(a) = y$ (according to Lemma 21, there is $a \neq y$). Since $Tr(x)$ is not a couple, there exists $v \in Tr(x)$, such that $v \neq y$ and $F(v) = a$ (and therefore by Lemma 11 there is $v \neq a$). Let us construct the mapping from Lemma 19 for the set a. Then $Tr^H(F(a)) = Tr^H(y) = Tr(x)$ and the set a itself and v are both the antecedents of a in the correspondence H, while there is no other antecedent of the set a (see Lemma 15). If we proceed to the mapping F, there is a unique antecedent of the set a and it is the set v. Since a was an arbitrary set from $Tr(x)$, Lemma 21 is proved.

Lemma 22. *For an arbitrary trajectory $Tr(x)$ there is not possible to find any u, v, w, $y \in Tr(x)$ mutually distinct such that $F(u) = F(v) = F(w) = y$.*

Proof: Suppose the contrary holds, i.e. u, v, w, y mentioned in Lemma 22 exist. Then $y \in Tr(u)$, $y \in Tr(v)$, $y \in Tr(w)$. Since $u, v, w \in Tr(x)$ then there is, according to Lemma 7, for instance $u \in Tr(v)$. Then $u, y \in Tr(v)$ and there is $u \in Tr(F(v)) = Tr(y)$ (according to the Consequence of Lemma 3). Thus $u \in Tr(y)$ and $y \in Tr(u)$ at the same time. According to Lemma 2, then there is $Tr(u) = Tr(y)$ and since $u \neq y$, this trajectory is a cycle. Similarly $v, w \in Tr(x)$. Let us suppose for instance $w \in Tr(v)$ (thus $w, y \in Tr(v)$). By the same way we can obtain that w, y belong to a cycle. According to Lemma 11, there is no trajectory having two distinct cycles and therefore u, w, y belong to the same cycle and there is $F(u) = F(w) = y$. It contradicts 21.

Lemma 23. *Any trajectory possesses no more than one set, having two distinct antecedents.*

Proof: Let $i = 1,2$ $y_i \in Tr(x)$, $y_1 \neq y_2$, $u_i \neq v_i$ such that $F(u_i) = F(v_i) = y_i$. If $y_i = u_i$ or $y_i = v_i$, then y_i are invariant sets. But, according to Lemma 11, there is at most one invariant set in $Tr(x)$; it implies the contradiction. If $v_1 \neq y_1 \neq u_1$ then $y_1 \in Tr(u_1)$, $y_1 \in Tr(v_1)$. Since $u_1, v_1 \in Tr(x)$ Lemma 7 implies either $u_1 \in Tr(v_1)$ or $v_1 \in Tr(u_1)$. Let for instance $u_1 \in Tr(v_1)$, then $Tr(v_1) = \{v_1\} \bigcup Tr(F(v_1)) = \{v_1\} \bigcup Tr(y_1)$ and since $v_1 \neq u_1$ there is $u_1 \in Tr(y_1)$, which together with the preceding result $y_1 \in Tr(u_1)$ means that $Tr(y_1) = Tr(u_1)$. Since $u_1 \neq y_1$, u_1, y_1 belong to a cycle. We obtain the similar result for $u_2 \neq y_2 \neq v_2$. Thus $Tr(x)$ contains two mutually distinct cycles (because $y_1 \neq y_2$), which contradicts Lemma 11. Suppose now the further possible case, in which for instance $y_1 = u_1$, $u_1 \neq v_1$, and y_2, u_2, v_2 are mutually distinct. Then, which follows from above, $Tr(x)$ contains an invariant set y_1 and a cycle to which for example y_2, u_2 belong. We received the contradiction with Lemma 11 again. Similarly in the remaining cases.

Lemma 24. *Let $Tr(x)$, which itself is not a cycle, contains a cycle $Tr(a)$. Then there exists $y \in Tr(x)$ such that y has two distinct antecedents in $Tr(x)$.*

Proof: From the assumptions follows, that $Tr(x) — Tr(a) \neq \emptyset$ and, according to Lemma 4, $Tr(x) — Tr(a)$ is not closed. There exists $y \in Tr(x) — Tr(a)$ such that $F(y) \notin Tr(x) — Tr(a)$; it means $F(y) \in Tr(a)$ (because $F(y) \in Tr(x)$). According to Lemma 9, there is $Tr(a) = Tr(F(y))$ and by Lemma 21 $F(y)$ has just one antecedent in $Tr(a)$ – let us denote it y'. Thus $F(y') = F(y)$ and $y \neq y'$ at the same time, because $y \notin Tr(\mathrm{a})$ and $y' \in Tr(a)$. Hence $F(y)$ has two distinct antecedents in $Tr(x)$.

Lemma 25. *Any trajectory, containing a cycle and not being a cycle itself, has just one set with two distinct antecedents.*

Proof: It follows immediately from Lemmas 23, 24.

Lemma 26. *If $Tr(x)$ contains a set y with two antecedents u, v such that u, v, y are mutually distinct, then $Tr(x)$ contains a cycle and it is not a cycle itself.*

Proof: According to Lemma 23, at most one such a y exists. Let $y \in Tr(x)$ is a set, for which $F(u) = F(v) = y$ holds, where u, v, y are mutually distinct. Then $F(u) = y$, $y \in Tr(u)$ and $F(v) = y$, $y \in Tr(v)$ at the same time, from where $Tr(y) \subseteq$ $\subseteq Tr(v)$ and $Tr(y) \subseteq Tr(u)$ follows. Then there is also $Tr(u) = \{u\} \cup Tr(F(u)) =$ $= \{u\} \cup Tr(y)$. Since $u, v \in Tr(x)$ then, according to Lemma 7, either $u \in Tr(v)$ or $v \in Tr(u)$ holds. Suppose, both the conditions $u \in Tr(v)$ and $v \in Tr(u)$ hold at the same time. Then $Tr(u) = Tr(v)$ is a cycle, but $F(u) = F(v) = y$, $F(u) \in Tr(u)$ so that y belongs to the cycle $Tr(u)$ and there are two distinct antecedents of y in $Tr(x)$. We are coming to the contradiction with Lemma 21. Let us have now $v \in Tr(u)$ then $v \in \{u\} \cup Tr(F(u)) = \{u\} \cup Tr(y)$. Because $v \neq u$, there is $v \in Tr(y)$ and so $Tr(v) \subseteq Tr(y)$. But there is also $Tr(y) \subseteq Tr(v)$. Thus $Tr(y) = Tr(v)$ for $v \neq y$. Trajectory $Tr(y)$ is therefore a cycle. From Lemma 21 it follows, that $Tr(x)$ cannot be a cycle, since otherwise for any $y \in Tr(x)$ there is just one antecedent in $Tr(x)$.

Lemma 27. *If $Tr(x)$ contains a set with two distinct antecedents, then $Tr(x)$ is a set.*

Proof: Let $Tr(x)$ satisfies the assumptions of the above Lemma, then by Lemma 26 there is a cycle in $Tr(x)$; let us denote it $Tr(a)$. $Tr(a)$ is a set, according to Lemma 18. Examine $Tr(x) — Tr(a)$. Let us show that it is a set, too. Let $y \in Tr(x)$ and suppose that y has two distinct antecedents in $Tr(x)$. Let us define the mapping $G : G(x) = F(x)$ for every $x \in Tr(x) — Tr(a)$, $G(y) = y$. $G((Tr(x) — Tr(a)) \cup \{y\})$ is a trajectory with the invariant set y and then (according to Lemma 18) this trajectory is also a set. Thus $Tr(x) = G([(Tr(x) — Tr(a)] \cup \{y\}) \cup Tr(a)$ is the union of the two sets and therefore $Tr(x)$ is a set.

Definition. $Tr(x)$ is called of

the type 1) *if and only if $Tr(x)$ contains an invariant set and does not contain any set with two distinct antecedents*;

the type 2) *if and only if $Tr(x)$ contains an invariant set and just one set with two distinct antecedents*;

the type 3) *if and only if $Tr(x)$ contains a cycle and $Tr(x)$ does not contain any set with two distinct antecedents*;

the type 4) *if and only if $Tr(x)$ contains a cycle and just one set with two distinct antecedents*;

the type 5) *if and only if $Tr(x)$ contains neither a cycle nor an invariant set.*

Theorem. *Any trajectory is a trajectory of just one type* 1) – 5).

Proof: Let $Tr(x)$ contains an invariant set; denote it s. If $s = x$, then $Tr(x) = \{x\}$ by Lemma 8 and thus $Tr(x)$ contains no set with two distinct antecedents; $Tr(x)$ is of the type 1). If $s \neq x$, then the set s has just two distinct antecedents in $Tr(x)$ (according to Lemma 17). By Lemma 23 there is no other set in $Tr(x)$ with two distinct antecedents, which implies that $Tr(x)$ is of the type 2). Both above mentioned cases are mutually exclusive, and therefore $Tr(x)$ cannot be of the type 1) and 2) at the same time. Let $Tr(x)$ contains a cycle. If $Tr(x)$ is a cycle itself, then any $y \in Tr(x)$ has just one antecedent in $\mathrm{T}r(x)$ (see Lemma 21). Hence there is no set with two distinct antecedents in $Tr(x)$. $Tr(x)$ is therefore a trajectory of the type 3). If $Tr(x)$ is not a cycle itself and it contains a cycle, then there exists, according to Lemma 25, just one set with two distinct antecedents in it; $Tr(x)$ is therefore of the type 4). Both described cases are mutually exclusive again, i.e. no trajectory can be a trajectory of the type 3) and 4) at the same time. According to Lemma 11, any trajectory contains at most one cycle or at most one invariant set, where both the cases are incompatible; hence any trajectory cannot be the trajectory of more than one of these described types 1) – 4), at the same time. Let $Tr(x)$ contains neither a cycle nor an invariant set, than $Tr(x)$ is of the type 5) and obviously it cannot be a trajectory of any of the types 1) – 4). Hence any trajectory is a trajectory of just one of the types 1) – 5).

Lemma 28. *Let $Tr(x)$ is of the type 5). Then any $y \in Tr(x)$, $y \neq x$ has just one antecedent in $Tr(x)$. The set x has no antecedent in $Tr(x)$.*

Proof: According to Lemma 13, any $y \in Tr(x)$, $y \neq x$ has at least one antecedent in $Tr(x)$. Suppose there exists a set z in $Tr(x)$ with two distinct antecedents u, v. If $u = z$ or $v = z$, $Tr(x)$ contains an invariant set, which violates the properties of the trajectory of the type 5). If u, v, y are mutually distinct, then by Lemma 26 $Tr(x)$ contains a cycle, which violates the same properties as above. Thus there exists no set with two distinct antecedents in $Tr(x)$ and any $y \in Tr(x)$, $y \neq x$ has at least one antecedent in $Tr(x)$. From that it follows, that every $y \in Tr(x)$, $y \neq x$ has just one antecedent in $Tr(x)$. By Lemma 12 ($Tr(x)$ is itself neither a cycle nor an invariant set) x has no antecedent in $Tr(x)$.

Definition. $Tr(x)$ is of

the type 1) *if and only if x is an invariant set;*

the type 2) *if and only if it contains an invariant set $p \neq x$;*

the type 3) *if and only if it is a cycle itself;*

the type 4) *if and only if it contains a cycle and it is not a cycle itself, which happens provided there exists $y \in Tr(x)$ with two antecedents u, v, where u, y, v are mutually distinct;*

the type 5) *if and only if any $y \in Tr(x)$, $y \neq x$ has just one antecedent in $Tr(x)$ and x has no antecedent there.*

Lemma 29. *Both the mentioned definitions of particular types of the trajectories are equivalent to each other.*

Proof: 1) Let $Tr(x)$ is a trajectory, which contains an invariant set p and it does not contain any set with two distinct antecedents. Then $F(p) = p$. If $p = x$, then x is an invariant set and the proof is complete. If $p \neq x$, then there exists $y \in Tr(x)$, $y \neq p$ such that $F(y) = p$. It means, p has two distinct antecedents p, y in $Tr(x)$, which contradicts the assumptions. If x is an invariant set, then $F(x) = x$ and, according to Lemma 8, there is $Tr(x) = \{x\}$. Thus there is no $y \in Tr(x)$ with two distinct antecedents.

2) If $Tr(x)$ contains an invariant set p and just one set with two distinct antecedents, then either $p = x$, which implies $Tr(x) = \{x\}$ and thus there is no set in $Tr(x)$ with two distinct antecedents, which violates the assumptions, or $p \neq x$, which is just the desired result. On the contrary, according to Lemma 17, p has in $Tr(x)$ two distinct antecedents and by Lemma 23 just one such a set exists in $Tr(x)$.

3) Let $Tr(x)$ contains a cycle $Tr(y)$ and it does not contain any set with two distinct antecedents in $Tr(x)$ and let $Tr(x) \neq Tr(y)$. Thus $Tr(y) \subseteq Tr(x)$ and $Tr(x) \nsubseteq Tr(y)$ at the same time. Therefore there exists $z \in Tr(x)$ such that $z \notin Tr(y)$, i.e. $z \in Tr(x) — Tr(y)$. From Lemma 4 it follows, that $Tr(x) — Tr(y)$ is not closed. Thus there exists $a \in Tr(x) — Tr(y)$ such that $F(a) \in Tr(y)$. But $Tr(y)$ is a cycle, $F(a)$ has therefore an antecedent in $Tr(y)$. Eventually $F(a)$ has an antecedent in $Tr(x) — Tr(y)$; thus there exists the set $F(a) \in Tr(x)$ with two distinct antecedents. It leads to the contradiction. If $Tr(x)$ is a cycle itself, any $y \in Tr(x)$ has just one antecedent in $Tr(x)$ (see Lemma 21), i.e. there is no $y \in Tr(x)$ with two distinct antecedents in $Tr(x)$.

4) If $Tr(x)$ contains a cycle and just one set with two distinct antecedents, then $Tr(x)$ is not a cycle itself, since otherwise, according to Lemma 21, any $y \in Tr(x)$ has just one antecedent in $Tr(x)$. Hence there exists $y \in Tr(x)$ with two distinct antecedents u, v. If $u = y$ or $v = y$, $Tr(x)$ contains an invariant set y, which is not possible (see Lemma 11), because $Tr(x)$ contains a cycle. Thus u, v, y are mutually distinct. Suppose $Tr(x)$ contains a cycle and it is not a cycle itself, then by Lemmas 24, 25 $Tr(x)$ contains just one set y with two distinct antecedentts u, v. Similarly to the previous u, v, y must be mutually distinct.

5) If $Tr(x)$ contains neither a cycle nor an invariant set then, according to Lemma 28, every $y \in Tr(x)$, $y \neq x$ has just one antecedent in $Tr(x)$ and x has no antecedent there. On the other hand, if every $y \in Tr(x)$, $y \neq x$ has just one antecedent in $Tr(x)$ and x has no antecedent, then x is not invariant itself and there is no other set $p \neq x$, p being an invariant set in $Tr(x)$. Therefore $Tr(x)$ does not contain any invariant set. If $Tr(x)$ contains a cycle, then by Theorem either it is itself a cycle and thus x has an antecedent, which contradicts the above assumptions, or $Tr(x)$ contains a cycle while it is not a cycle itself. But in this case there exists (see Lemma 24) a set with two distinct antecedents in $Tr(x)$, which violates the assumptions again.

Note. About the trajectories of the type 5) such assertion can be proved, provided so called Axiom of Infinity is given. One of all its possible mutually equivalent formulations follows.

Axiom of Infinity. *Every trajectory of the type* 5) *is a set.*

Lemma 30. *Be* $Tr(x)$ *of the type* 5). *Then the relation* R *defined in* $Tr(x)$ *by*

$$(\forall\, y, z)\ [(y \in Tr(x)) \wedge (z \in Tr(x))] \Rightarrow [y\, R\, z \Leftrightarrow (z \in Tr(y))]$$

is a relation of the linear ordering in $Tr(x)$.

Proof: The relation R is reflexive, since for any $a \in Tr(x)$, $a\, R\, a$ holds if and only if $a \in Tr(a)$ which is obvious. If $y\, R\, z$ and $z\, R\, y$ at the same time, it means that $z \in Tr(y)$ and $y \in Tr(z)$, which implies $Tr(z) \subseteq Tr(y)$ and $Tr(y) \subseteq Tr(z)$. Thus $Tr(y) = Tr(z)$ holds. Since $Tr(x)$ is of the type 5), it does not contain a cycle and the equality $Tr(y) = Tr(z)$ holds just in the case, in which $y = z$. Hence the relation R is antisymmetric. We shall prove now that R is transitive. Suppose $y\, R\, z$ and $z\, R\, u$ hold. Then $z \in Tr(y)$, $u \in Tr(z)$ and therefore $Tr(z) \subseteq Tr(y)$, $Tr(u) \subseteq Tr(z)$, which implies $Tr(u) \subseteq Tr(y)$. It means $u \in Tr(y)$ and hence $y\, R\, u$. For R to be a linear ordering in $Tr(x)$, it must satisfy in addition the following condition: $(\forall\, a, b)$ $(a, b \in Tr(x) \Rightarrow (a\, R\, b \vee b\, R\, a)$, i.e. $(\forall\, a, b)$ $(a, b \in Tr(x) \Rightarrow (b \in Tr(a) \vee$ $\vee\ a \in Tr(b))$.

Suppose the above condition is not satisfied. Let there exist $a, b \in Tr(x)$ such that, $b \notin Tr(a)$ and $a \notin Tr(b)$ at the same time. If for example $b \notin Tr(a)$ there is $a \in Tr(b)$ (see Lemma 7) and we are getting the contradiction. Similarly in the case $a \notin Tr(b)$. Hence R is a linear ordering in $Tr(x)$.

Lemma 31. *Every trajectory of the type* 5) *is by the relation from Lemma 30 well – ordered.*

Proof: Let us suppose R is a relation described in Lemma 30 defined in $Tr(x)$ of the type 5). Let M is an arbitrary non-empty subclass of $Tr(x)$. We shall show, that there is the least set in M. Denote

$$Z = \{z; (\exists\, y)\, (y \in M \wedge y\, R\, z)\}.$$

The class Z is non-empty, because $M \subseteq Z$. For this purpose it is enough to show that for any a, $a \in M$ implies $a \in Z$. To be $a \in Z$, there must exists $y \in M$ such that $y\, R\, a$; clearly it is enough to put $y = a$. Let us prove that the class Z is closed. If $z \in Z$, then there exists $y \in M$ such that $y\, R\, z$. Since $F(z) \in Tr(z)$, there is $z\, R\, F(z)$. From the transitive property of relation R, there is $y\, R\, F(z)$, which means that $y' = y$ exists such that $y' \in M$ and $y'\, R\, F(z)$. Hence $F(z) \in Z$ and the class Z is therefore closed. Thus $Z \neq \emptyset$, Z is closed and $Z \subseteq Tr(z)$. According to Lemma 5, there exists $u \in Tr(x)$ such that $Z = Tr(u)$. Let us prove, u is the least set in M. For this purpose we have to show first that $u \in M$. Since $Tr(u) = Z$, there is $u \in Z$ and therefore y_1 exists in M, such that $y_1\, R\, u$, which means that $u \in Tr(y_1)$ and thus $Tr(u) \subseteq Tr(y_1)$. Simultaneously with that there is also $u, y_1 \in Z = Tr(u)$. If $u = y_1$ the above condition is proved while the assumption $u \neq y_1$ leads to the contradiction. If $u \neq y_1$, there is $y_1 \in Tr(u)$ which implies $Tr(y_1) \subseteq Tr(u)$. From the above consideration it follows, that $Tr(y_1) = Tr(u)$ and because $Tr(x)$ does not contain a cycle, there is necessarily $u = y_1$, which violates the assumption $u \neq y_1$. Let us consider $t \in M$, $t R u$. We shall show that $t = u$. If $t \in M$, there is $t \in Z = Tr(u)$ and therefore $Tr(t) \subseteq Tr(u)$. Since $t R u$, we are getting $u \in Tr(t)$ and thus $Tr(u) \subseteq \subseteq Tr(t)$. From both the above results there is $Tr(u) = Tr(t)$ and because $Tr(x)$ is of the type 5), there is $u = t$ ($Tr(x)$ contains no cycle). Thus we have proved, that u is the least set in M.

II. Natural Numbers

Definition. *Let F is a mapping defined as follows: $F(x) = x \cup \{x\}$ for any $x \in V$*). Let us form a class $Tr^F(\emptyset)$ (sometimes we shall write briefly $Tr(\emptyset)$). If $n \in Tr^F(\emptyset)$, n will be called* natural number. *The class $Tr^F(\emptyset)$ will be called the class of natural numbers and let us denote it sometimes the symbol N.*

Note. In the further text $Tr(x)$ will denote $Tr^F(x)$, where $F(x) = x \cup \{x\}$.

Lemma 1. *Let $m, n \in N$, then $n \in m$ implies $m \in Tr(n)$.*

Proof: Denote

$$X = \{x; (x \in N) \wedge (\forall n)(n \in N) \Rightarrow [(n \in x) \Rightarrow (x \in Tr(n))]\}.$$

Let us show, that $X = N$. There is $\emptyset \in N$ and since $n \notin \emptyset$ for any $n \in N$, there holds $\emptyset \in X$. Prove that X is closed. Let $x \in X$. Since N is closed, $x \in N$, there is $F(x) \in N$. If $n \in N$, $n \in F(x)$ and $n \in x$ at the same time, then (by the definition of X) there is $x \in Tr(n)$ and from the closure of $Tr(n)$ also $F(x) \in Tr(n)$ holds and therefore $F(x) \in X$. If $n \in F(x) = x \cup \{x\}$ and $n \notin x$, $n \in \{x\}$ holds, from which $n = x$ follows. Consequently $x \in Tr(n)$ and $F(x) \in Tr(n)$ at the same time, which means $F(x) \in X$. In the case $n \notin F(x)$, there is $F(x) \in X$ clearly. Thus $\emptyset \in X$, $X \subseteq N$ and X is closed. According to Lemma 1, Part I, there is $X \supseteq N$ and thus $X = N$.

Lemma 2. *The class $Tr^F(\emptyset)$ is a trajectory of the type 5).*

Proof: We need to show, that $Tr^F(\emptyset)$ contains neither a cycle nor an invariant set. Let us prove first, that $Tr^F(\emptyset)$ itself is neither a cycle nor an invariant set. Otherwise $x \in Tr(\emptyset)$ exists such that $F(x) = \emptyset$, i.e. $x \cup \{x\} = \emptyset$, which is impossible ($x \cup \{x\}$ contains at least one set). Let $Tr(\emptyset)$ contains a cycle. Then $Tr(\emptyset)$ is of the type 4) (since $Tr(\emptyset)$ is not a cycle itself). According Lemma 29, Part I, there exist $x, y, z, \in Tr(\emptyset)$ mutually distinct such that $F(x) = F(y) = z$. Let us suppose, for instance, that $Tr(y) = Tr(z)$ is a cycle. Then $x \notin Tr(y)$. If $x \in Tr(y)$, then $Tr(x) = = Tr(y) = Tr(z)$ is a cycle. But $F(x) = F(y) = z$, i.e. there exists a set in a cycle, having two distinct antecedents, which violates Lemma 2, Part I. Since $F(x) = F(y)$, $x \cup \{x\} = y \cup \{y\}$ holds. It implies $y \in x \cup \{x\}$, but $y \neq x$ (otherwise $x \in Tr(y)$), so then $y \in x$. Lemma 1 implies $x \in Tr(y)$, which contradicts $x \notin Tr(y)$. It remains to show, that $Tr(\emptyset)$ contains no invariant set. Let $z \in Tr(\emptyset)$ is an invariant set. According to Theorem, Part I, $Tr(\emptyset)$ is of the type 2) (since $Tr(\emptyset)$ is not an invariant set itself). Thus $x \in Tr(\emptyset)$, $x \neq z$, exists such that $F(x) = F(y) = z$, which means $x \cup \{x\} =$

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*) V is defined by: $V = \{x; x = x\}$.

$= z \cup \{z\}$ and $x \notin Tr(z)$. Thus $z \in x \cup \{x\}$ holds and since $z \neq x$ then $z \in x$, which implies (by Lemma 1) $x \in Tr(z)$, which is leading to the contradiction again.

Consequence 1. *The class N is a set.*
(Since any trajectory of the type 5) is a set)

Consequence 2. $(\forall n)(n \in N) \Rightarrow (n \neq F(n))$.
(If n $= F(n)$, N contains an invariant set).

Lemma 3. *For any $m, n \in N$ $(m \in Tr(F(n)) \Leftrightarrow n \in m)$* holds.

Proof: Let us prove first $m \in Tr(F(n)) \Rightarrow n \in m$.

Denote

$$X = \{x; (x \in N) \wedge (\forall n)(n \in N) \Rightarrow [(x \in Tr(F(n)) \Rightarrow (n \in x)]\}.$$

Obviously $\emptyset \in X$, because $\emptyset \in N$ and $\emptyset \notin Tr\,(F(n))$ for any n; otherwise there exists $z \in N$ such that $z \cup \{z\} = \emptyset$, which is impossible. Let us show furthermore, that X is closed. Let $x \in X$, then $x \in N$ and thus $F(x) \in N$. Suppose further $n \in N$ and $F(x) \notin Tr(F(n))$, then $F(x) \in X$ and the proof is complete. If $F(x) \in Tr(F(n))$, $Tr(F(x)) \subseteq Tr(F(n))$ holds. Let in addition $x \in Tr(F(n))$ holds. By the definition of X it follows, that $n \in x$ and thus $n \in x \cup \{x\} = F(x)$, therefore $F(x) \in X$. It remains to inquire the case of $F(x) \in Tr(F(n))$ and $x \notin Tr(F(n))$ at the same time. Let us show that $Tr(F(x)) = Tr(F(n))$. Since $F(x) \in Tr(F(n))$, then $Tr(F(x)) \subseteq Tr(F(n))$, and it remains to prove that $Tr(F(n)) \subseteq Tr(F(x))$. Both the sets x, $F(n)$ belong to $Tr(\emptyset)$. Since $x \notin Tr(F(n))$ then, according to Lemma 7, Part I, $F(n) \in Tr(x) = \{x\} \cup \cup Tr(F(x))$. If $F(n) \in Tr(F(x))$, then $Tr(F(n)) \subseteq Tr(F(x))$ and the above mentioned inclusion is proved. If $F(n) \in \{x\}$, then $F(n) = x$ and $x \in Tr(F(n))$, which contradicts the assertion $x \notin Tr(F(n))$. Thus $Tr(F(x)) = Tr(F(n))$. If $F(n) \neq F(x)$, it means that $Tr(\emptyset)$ contains a cycle, which violates Lemma 2. Therefore $F(n) = F(x)$ holds, which implies $n \cup \{n\} = F(x)$ and thus $n \in F(x)$; therefore $F(x) \in X$. Thus it is proved that the class X contains $\emptyset$ and X is closed. Thus $X \supseteq N$ and $X \subseteq N$ at the same time. Hence $X = N$.

It remains to prove the opposite implication of Lemma 3, i.e. the assertion $n \in m \Rightarrow m \in Tr(F(n))$. Denote

$$Y = \{y; (y \in N) \wedge (\forall n)(n \in N) \Rightarrow [(n \in y) \Rightarrow (y \in Tr(F(n)))]\}.$$

Since $\emptyset \in N$ and $n \notin \emptyset$ for any $n \in N$, then $\emptyset \in Y$ holds. Let $y \in Y$; let us prove that $F(y) \in Y$, i.e. Y is a closed class. Certainly $y \in N$ and $F(y) \in N$. If $n \in N$ and $n \notin F(y)$, there is $F(y) \in Y$; if $n \in F(y)$, $n \in N$ and $n \in y$, then (from the definition of Y) $y \in Tr(F(n))$ and $F(y) \in Tr(F(n))$ hold. It means that $F(y) \in Y$. It remains to examine the case $n \in F(y)$ and $n \notin y$ at the same time. Then $n \in y \cup \{y\}$, which implies $n = y$ and thus $F(n) = F(y)$. Since $F(n) \in Tr(F(n))$, $F(y) \in Tr(F(n))$ holds and we are getting $F(y) \in Y$ again. Similarly as in the previous case there holds $Y = N$, which completes the proof of Lemma 3.

Lemma 4. *For any natural number n, $n \notin n$ holds.*

Proof: Let there exists a natural number n such that $n \in n$. By Lemma 3 there is

$n \in Tr(F(n))$ and therefore $Tr(n) \subseteq Tr(F(n))$. From the definition of trajectory it holds that $Tr(F(n)) \subseteq Tr(n)$, from which $Tr(n) = Tr(F(n))$ follows. If $n = F(n)$ then N contains an invariant set and if $n \neq F(n)$, N contains a cycle. In both the cases we are getting the contradiction with Lemma 2.

Consequence. $(\forall\, m, n)\ (m, n \in N) \Rightarrow [(n \in m) \Rightarrow (n \neq m)]$.
(If $n \in m$ and $n = m$, there is $n \in n$ which violates Lemma 4).

Lemma 5. *For any* $m, n \in N$ *it holds that* $(m \in Tr(n) \Leftrightarrow [(n \in m) \vee (n = m)]$, *where both the cases are mutually exclusive.*

Proof: Let $m \in Tr(n) = \{n\} \cup Tr(F(n))$. If $m \in \{n\}$, then $m = n$ and if $m \in Tr(F(n))$, then by Lemma 3 there is $n \in m$. Both the cases are clearly mutually exclusive. If $n \in m$ and $n = m$ at the same time, then $n \in n$ which contradicts Lemma 4. Let us assume that $n \in m$, then, according to Lemma 1, there is $m \in Tr(n)$. If $m = n$, then $m \in Tr(n)$ is obvious.

Note. We have shown already, that N is the trajectory of the type 5). According to Lemmas 30 and 31, Part I, there is possible to define the relation R of well-ordering of N by the following manner:

$$(\forall\, m, n \in N)\ \ m\, R\, n \Leftrightarrow n \in Tr(m).$$

Let us look for the interpretation of R. From Lemma 5 it follows:

$$n \in Tr(m) \Leftrightarrow m \in n \vee m = n.$$

Instead of writing $(m \in n \vee m = n)$, we shall write sometimes $m \underline{\in} n$. Furthermore let us show that the set N is possible to order not only by the relation "$\in$", but by another way else.

Lemma 6. *For any* $m, n \in N$ *it holds: if* $m \in n$, *then* $n \notin m$.

Proof: Let there exist $m, m \in N$ such that $m \in n$ and $n \in m$ at the same time. Then (by Lemma 1) $m \in Tr(n)$, $n \in Tr(m)$ hold, which means $Tr(m) \subseteq Tr(n)$ and $Tr(n) \subseteq Tr(m)$. From this we obtain $Tr(m) = Tr(n)$. Since $n \in m$ there is $m \neq n$ (see the Consequence of Lemma 4), thus $Tr(m) = Tr(n)$ is a cycle in $Tr(\emptyset)$, which contradicts Lemma 2.

Lemma 7. *For any* $l, m, n \in N$ $[(l \in m) \wedge (m \in n)] \Rightarrow (l \in n)$ *holds.*

Proof. Let $l \in m$ and $m \in n$ hold, then $m \in Tr(l)$, $n \in Tr(m)$ and thus $Tr(m) \subseteq Tr(l)$, $Tr(n) \subseteq Tr(m)$, from which we are getting $Tr(n) \subseteq Tr(l)$; therefore $n \in Tr(l)$. According to Lemma 5, $n = l$ or $l \in n$ holds, where both the cases are mutually exclusive. If $n = l$, then $m \in n$ and $n \in m$ at the same time and we are coming to the contradiction with Lemma 6. Hence $l \in n$.

Note. The relation "$\in$" is thus antireflexive, antisymmetric and transitive relation and therefore it is the partial – ordering of N. From the previous Lemmas it follows, that

$$m \in n \Leftrightarrow [(n \in Tr(m)) \wedge (n \neq m)] \Leftrightarrow (n \in Tr(F(m)).$$

Let us show now, that it is possible to order the natural numbers also by the relation "$\subseteq$" or "$\subset$".

Lemma 8. *For any $m, n \in N$ $m \in Tr(n) \Leftrightarrow n \subseteq m$ holds.*

Proof: Let us prove first the implication $m \in Tr(n) \Rightarrow n \subseteq m$. Let us form the class

$$X = \{x; (x \in N) \wedge (\forall n)(n \in N) \Rightarrow [(x \in Tr(n)) \Rightarrow (n \subseteq x)]\}.$$

We shall show that $\emptyset \in X$. Clearly $\emptyset \in N$ and if $\emptyset \in Tr(n)$, then (according to Lemma 5) $n = \emptyset$ and thus $n \subseteq \emptyset$; if $\emptyset \notin Tr(n)$, there is obviously $\emptyset \in X$. Let $x \in X$; let us show that $F(x) \in X$, too. If $x \in X$, there is $x \in N$ and also $F(x) \in N$. If $F(x) \notin Tr(n)$, then $F(x) \in X$. Assume now that $F(x) \in Tr(n)$. We shall distinguish two cases: $x \in Tr(n)$ and $x \notin Tr(n)$. If $x \in Tr(n)$, then (from the definition of X) there is $n \subseteq x$ and therefore $n \subseteq x \cup \{x\} = F(x)$. Thus $F(x) \in X$. It remains the case of $x \notin Tr(n)$ and $F(x) \in Tr(n)$ at the same time. Let us show that $Tr(n) = Tr(F(x))$. If $F(x) \in Tr(n)$, there is $Tr(F(x)) \subseteq Tr(n)$. It remains to show that $Tr(n) \subseteq Tr(F(x))$. Clearly $x, n \in Tr(\emptyset)$. Since $x \notin Tr(n)$, Lemma 7 implies $n \in Tr(x) = \{x\} \cup Tr(F(x))$. If $n \in \{x\}$, there is $n = x$, which violates the condition $x \notin Tr(n)$. If $n \in Tr(F(x))$, there is $Tr(n) \subseteq Tr(F(x))$. Thus $Tr(n) = Tr(F(x))$. Then there are two possibilities: either $n = F(x)$, which leads immediately to the desired conclusion, since then $n \subseteq F(x)$ and thus $F(x) \in X$, or $n \neq F(x)$, which means that N contains a cycle; it contradicts Lemma 2. Thus the case of $x \notin Tr(n)$ and $F(x) \in Tr(n)$ is excluded. In the remaining cases there is $F(x) \in X$ and therefore X is a closed class, containing $\emptyset$ and hence $X = N$.

It is to show now that the opposite implication holds, i.e. $n \subseteq m \Rightarrow m \in Tr(n)$. Let us construct the class Y as follows:

$$Y = \{y; (y \in N) \wedge (\forall n)(n \in N) \wedge [(n \subseteq y) \Rightarrow (y \in Tr(n))]\}.$$

We are going to show that $Y =$ N. The empty set belongs to Y, since $\emptyset \in N$ and for every $n \in N$, $n \subseteq \emptyset$ if and only if $n = \emptyset$ holds, from where $\emptyset \in Tr(n)$. Let $y \in Y$. We shall prove that $F(y) \in Y$. If $y \in Y$, there is $y \in N$ and thus $F(y) \in N$. If $n \in N$ and $n \nsubseteq F(y)$, then $F(y) \in Y$ is obvious. Let us suppose that for $n \in N$, $n \subseteq F(y)$ holds. If in addition $n \subseteq y$, there is $y \in Tr(n)$ (from the definition of Y) and thus $F(y) \in Tr(n)$. Therefore $F(y) \in Y$. Assume that $n \subseteq F(y)$ and $n \nsubseteq y$ at the same time. If $n \subseteq F(y) = \{y\} \cup y$ and $n \nsubseteq y$, then $y \in n$. According to Lemma 3, $n \in Tr(F(y))$, which implies $F(y) \subseteq n$ (from the first part of the proof of Lemma 8). Altogether F$(y) = n \in Tr(n)$, thus $F(y) \in Y$. Hence Y is a closed subclass of N, containing $\emptyset$, and therefore $Y = N$.

Note. From the above Lemma and by Lemma 30, Part I, it follows that the relation R defined in N by:

$$\forall\, m, n \in N \quad m\, R\, n \Leftrightarrow m \subseteq n,$$

is the relation of the linear ordering in N.

Lemma 9. *For two arbitrary natural numbers m, n just one possibility holds:*

a) $m \in n$ b) $n \in m$ c) $n = m$.

Proof: Relation "$\in$" is a linear ordering of N (see the Note following Lemma 5); i.e. for arbitrary $m, n \in N$ $[(m \underline{\in} n) \vee (n \in m)]$ holds, which means $[(m \in n) \vee (n \in$

$\in m) \vee (m = n)]$. It is necessary to show, that all those possibilities are mutually exclusive. Assume, for instance, that $m \in n$, then $m \neq n$ (according to the Consequence of Lemma 4) and also $n \notin m$ by Lemma 6. Similarly for $n \in m$. Let thus $m = n$ and, for instance, $m \in n$ at the same time. Then $n \in n$, which contradicts Lemma 4; by the same way in the case $n = m$ and $n \in m$ at the same time.

Note. Lemma 8 gives us the possibility to express Lemma 9 by the following way: for every $m, n \in N$ just one possibility holds:

a) $m \subset n$, b) $n \subset m$, c) $n = m$.

Lemma 10. *The class N has the following properties:*

a) *N is the set, in which the transformation F is defined by: $F(x) = x \cup \{x\}$, for any $x \in N$,*

b) $\emptyset \in N$,

c) $x \in N \Rightarrow x \cup \{x\} \in N$,

d) $[(x, y \in N) \wedge (x \neq y)] \Rightarrow x \cup \{x\} \neq y \cup \{y\}$,

e) $x \in N \Rightarrow x \cup \{x\} \neq \emptyset$,

f) $[(M \subseteq N) \wedge (\emptyset \in M) \wedge [(x \in M) \Rightarrow (x \cup \{x\} \in M)]] \Rightarrow M \supseteq N$,

(thus N satisfies the Peanos' Axioms).

Proof: The assertion a) follows immediately from the definition of N and from the Axiom of Infinity. There is $\emptyset \in Tr^F(\emptyset) = N$ and thus b) holds. The assertion c) follows from the closure of the class N. Let $x, y \in N$, $x \neq y$ and suppose that $x \cup \{x\} = y \cup \{y\}$ at the same time. Then $x \in y$ and $y \in x$, which leads to the contradiction, since $x \in y$ implies $y \notin x$. For an arbitrary $x, x \in N$, there is $x \cup \{x\} \neq \neq \emptyset$, because $x \cup \{x\}$ contains at least one set. The class M from f) is, in the principle, a closed subclass of $Tr^F(\emptyset) = N$, containing $\emptyset$. According to Lemma 1, Part I, there is $M \supseteq N$.

Note. At the end let us show several simple Lemmas, describing some additional properties of the natural numbers.

Lemma 11. $(\forall m)\ (m \in N) \Rightarrow m \subset F(m)$.

Proof: There is $F(m) = m \cup \{m\}$, but $m \subseteq m \cup \{m\}$, i.e. $m \subseteq F(m)$. If $m = F(m)$, we obtain the contradiction with the Consequence of Lemma 2. Therefore $m \neq F(m)$ and thus $m \subset F(m)$.

Lemma 12. *Let $n \in N$. Then there is no $m \in N$, such that $n \in m$ and $m \in n \cup \{n\}$ at the same time.*

Proof: Assume there exist $m \in N$ and $n \in N$, such that $n \in m$ and $m \in n \cup \{n\}$ at the same time. If $m \in n$, we are getting the contradiction, since $n \in m$ cannot hold at the same time (see Lemma 9). Similarly we obtain the contradiction in the case, that $m \in \{n\}$, i.e. $m = n$.

Lemma 13. *If $m, n \in N$, then $m \cap n \in N$.*

Proof: Let us suppose $m, n \in N$ and, for instance, $m \subset n$. Let us prove, that $m \cap n = m$ (i.e. $m \cap n \in N$). Clearly $m \cap n \subseteq m$ holds. We shall prove the contrary i.e. $m \subseteq m \cap n$. Let us assume $z \in m$. Since $m \subset n$, there is $z \in n$ and thus

$z \in m \cap n$. Therefore $m \cap n = m$ holds. Analogically in the case $n \subset m$; for $n = m$ the assertion is obvious.

Lemma 14. *For any natural number n, there is $n \subseteq N$.*

Proof: Let us denote $X = \{x; (x \in N) \Rightarrow (x \subseteq N)\}$. There is $\emptyset \in N$ and $\emptyset \subseteq N$; thus $\emptyset \in X$. Let $x \in X$. If $x \in N$, there is $F(x) \in N$ (from the closure of N); for $x \notin N$, the implication is obvious. Since $F(x) = x \cup \{x\} \in N$ and $x \subseteq N$ (because $x \in N$), thus is also $x \cup \{x\} \subseteq N$ because for $x \in N$, there is $\{x\} \subseteq N$ and thus $F(x) = x \cup \{x\} \subseteq N$. Hence $X = N$.

Lemma 15. $(\forall\, m, a)\,[(m \in N) \wedge (a \in m)] \Rightarrow (a \in N)$.

Proof: Let us denote

$$X = \{x; (x \in N) \wedge (\forall\, a)\,(a \in X) \Rightarrow (a \in N)\}.$$

There is $\emptyset \in X$, since $\emptyset \in N$ and $a \notin \emptyset$ for any a holds. Let $x \in X$. Since $x \in N$, there is $F(x) \in N$. If $a \notin F(x)$, there is clearly $F(x) \in X$. If $a \in F(x)$, let us distinguish two cases. If $a \in x$, there is $a \in N$, according to the assumption $x \in X$. If $a \notin x$, then the expression $a \in x \cup \{x\}$ implies $a = x$. Thus $a \in N$. Therefore $X = N$ and the proof is complete.

Note. We do not consider the further development of the theory of natural numbers to be the subject of this article. Their further properties and the introduction of the operations on them, can be derived from the considerations mentioned here, by the usual way.

Appendix

Definition. *Let us consider $x \in A$ and F a mapping of A into A.*
The class $\{y; \varphi(y, x) \wedge \psi(y, x)\}$, where

$$\varphi(y, x) \equiv (\forall\, u)\,[[(\forall\, z)\,(z \in u) \Rightarrow (F(z) \in u)] \Rightarrow [(x \in u) \Rightarrow (y \in u)]]$$

$$\psi(y, x) \equiv (\exists\, v)\,[[(x \in v) \wedge (y \in v)] \wedge (\forall\, z)\,[(z \in v \wedge z \neq y) \Rightarrow (F(z) \in v)]]$$

is called a trajectory of x *and it is denoted* $Tr^F(x)$ *or briefly* $Tr(x)$.

Theorem. *Let F is a mapping of A into A, then*

$$Tr(x) = \{z; (\forall\, X)\,[[(X \subseteq A) \wedge [(\forall\, y)\,(y \in X) \Rightarrow (F(y) \in X)] \wedge (x \in X)] \Rightarrow (z \in X\,]\}$$

Proof: Let us show at first, that for an arbitrary subclass B of the class A, which is closed with respect to F and it contains x, there is $Tr(x) \subseteq B$ (where $Tr(x)$ is a class defined by the previous defiinition). Let us assume that B is a class of the properties, mentioned above and let $Tr(x) \nsubseteq B$. Then there exists y, such that $y \in Tr(x)$ and $y \notin B$ at the same time. Since $y \in Tr(x)$, $\psi(y, x)$ holds and thus the set v exists, such that $x, y \in v$ and for every z, $z \in v$ and $z \neq y$, there is $F(z) \in v$. Let us construct the class $B \cap v$. Clearly $B \cap v \subseteq v$ and since v is a set, $B \cap v$ is a set, too. Since $x \in v$ and $x \in B$, $x \in B \cap v$ holds and, according to our assumption ($y \notin B$), $B \cap v$ does not contain y. Let us show, that $B \cap v$ is closed. Suppose $z \in B \cap v$. The class B is closed and therefore $F(z) \in B$. Since $z \neq y$ ($z \in B$ and $y \notin B$), $F(z) \in v$

follows from the validity $\psi(y,x)$. Thus $B \cap v$ is a closed set, containing x. By $\varphi(y,x)$, y must belong to $B \cap v$. It contradicts the assumption $y \notin B$. Thus for any class $B \subseteq A$, which is a closed class, containing x, $Tr(x) \subseteq B$ holds.

To complete the proof of the Theorem we need to prove: if y belongs to every X, which is a closed class, containing x, then $y \in Tr(x)$ (where $Tr(x)$ is defined by the previous definition). For this purpose, we shall show that $Tr(x)$ is closed and that $x \in Tr(x)$. Let us show that $x \in Tr(x)$, i.e. $\varphi(x,x)$ and $\psi(x,x)$ hold. Clearly $\varphi(x,x)$ is satisfied. It remains to prove that $\psi(x,x)$ holds. For this purpose let us find a set v for which:

$$[(x \in v) \wedge [(\forall z)(z \in v \wedge z \neq x) \Rightarrow (F(z) \in v)]].$$

Let us put $v = \{x\}$. Then $x \in v$ and there does not exist such a z, for which $z \in v$ and $z \neq x$ at the same time. As the next step we shall show that $Tr(x)$ is closed. Assume $y \in Tr(x)$, which means $\varphi(y,x)$ holds and let $\varphi(F(y),x)$ does not hold. Then there exists a closed set w, such that $x \in w$ and $F(y) \notin w$. Since w is closed, $y \notin w$ holds (otherwise $F(y) \in w$, which leads to the contradiction). Thus the condition $\varphi(F(y),x)$ is satisfied. From the assumption, that $y \in Tr(x)$, $\psi(y,x)$ follows; i.e. a set v exists, such that $x, y \in v$ and for every z, for which $z \in v$, $z \neq y$, there is $F(z) \in v$. Let us create another set $v' = v \cup \{F(y)\}$ (v is the set due to the condition $\psi(y,x)$, in addition y is a set, since $y \in Tr(x)$ and $\{F(y)\}$ is a set by the Axiom of Couple; thus $v \cup \{F(y)\}$ is a set, too). Then $x \in v'$, $F(y) \in v'$ and for any z, where $z \in v'$ and $z \neq F(y)$ there is $z \in v$ (by the definition of v'). If $z \neq y$, then, from the validity of $\psi(y,x)$, $F(z) \in v$ follows and thus $F(z) \in v'$, If $z = y$, then $F(z) = F(y) \in v'$ is obvious. Thus $\psi(F(y),x)$ and $\varphi(F(y),x)$ hold simultaneously and therefore $F(y) \in$ $\in Tr(x)$, which implies the closure of $Tr(x)$. Hence, we have shown that $Tr(x)$ is a closed class, containing x. Since, according our assumption, y belongs to every closed subclass of A, containing x, y must belong to $Tr(x)$, too. Thus the proof of the Theorem is complete.