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T-quasigroups
Part II.

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In this paper we continue the investigation of T – quasigroups. The definition and some basic properties of T – quasigroups can be found in our paper "T – quasigroups. Part I.", which appeared in [1]. All the notation is the same as in the paper mentioned above and we use it, as well as the results, without stating it explicitly. Thus we begin here from Theorem 19 and Lemma 17.

4° – Commutative, unipotent and idempotent T – quasigroups, α_n, β_n – quasigroups

Lemma 17. A T – quasigroup Q is commutative if and only if for any (and then for each) of its T – forms $(Q(+), \varphi, \psi, g)$ is $\varphi = \psi$.

Proof: 1. Be $(Q(+), \varphi, \varphi, g)$ a T – form of Q. Then for every $x, y \in Q$, $xy = \varphi(x) + \varphi(y) + g = yx$. Hence Q is commutative.

2. Let Q be commutative and $(Q(+), \varphi, \psi, g)$ be an arbitrary T – form of Q. Then $x \cdot O = \varphi(x) + g = O \cdot x = \psi(x) + g$, so that $\varphi(x) = \psi(x)$ for every $x \in Q$.

Theorem 19. Let Q be a T – quasigroup and $a \in Q$ such that for every $x \in Q$, $ax = xa$. Then Q is commutative.

Proof: Be $(Q(+), \varphi, \psi, g)$ any T – form of Q. Then for every $x \in Q$,

$$xa = \varphi(x) + \psi(a) + g = ax = \varphi(a) + \psi(x) + g.$$

If we put $x = O$, we get $\varphi(a) = \psi(a)$. Hence $\varphi = \psi$. By Lemma 17, Q is commutative.

Lemma 18: A T – quasigroup Q is unipotent if and only if for any (and then for each) of its T – forms $(Q(+), \varphi, \psi, g)$ is $\varphi = -\psi$.

Proof: 1. Be $(Q(+), -\psi, \psi, g)$ a T – form of Q. Then for every $x, y \in Q$,

$$xx = -\psi(x) + \psi(x) + g = g = yy.$$

2. Let Q be unipotent and $(Q(+), \varphi, \psi, g)$ be one of its arbitrary T – forms. Then for every $x \in Q$, $xx = OO = g$, so that $\varphi(x) + \psi(x) = O$, and therefore $\varphi = -\psi$.

Lemma 19: Let Q be a commutative and unipotent T – quasigroup and $Q(+)$ be an arbitrary T – group of Q. Then every non – zero element of $Q(+)$ has the order 2.

Corollary: Every finite commutative and unipotent T – quasigroup has a 2 – power order.

Proof: Be $(Q(+), \varphi, \psi, g)$ an arbitrary T – form of Q. By Lemmas 17, 18 we have $\varphi = \psi$ and $\varphi = -\psi$. Hence for every $x \in Q$, $\psi(x) = -\psi(x)$. But ψ is an automorphism of $Q(+)$. Thus $x = -x$.

Definition 6: Let n be a positive integer, $n \geq 2$. A quasigroup Q is called γ_n – quasigroup if Q is simultaneously an α_n and β_n – quasigroup.

Definition 7: A quasigroup Q is called K – quasigroup if there exists a commutative quasigroup C and a unipotent quasigroup U such that Q is isomorphic to $C \times U$.

Theorem 20: Every commutative T – quasigroup is a γ_n – quasigroup for every integer $n \geq 2$. Every unipotent T – quasigroup is a γ_n – quasigroup for every even $n \geq 2$.

Corollary: Every K – T – quasigroup is a γ_n – quasigroup for every even $n \geq 2$.

Proof: The theorem follows from Lemmas 17, 18 and from Theorems 10, 11.

Lemma 20: Let Q be a β_n – quasigroup, $n \geq 2$. Then Q is a β_{kn} – quasigroup for every $k = 1,2,\ldots$ If, moreover, Q is an α_m – quasigroup for some $m \geq 2$, Q is α_{m+kn} – quasigroup for every $k = 1, 2 \ldots.$

Proof: Be $(Q(+), \varphi, \psi, g)$ an arbitrary T – form of Q. Then $\varphi^n = \psi^n$. Hence $\varphi^{nk} = \psi^{nk}$ for every $k = 1, 2, \ldots$, and hence, Q is a β_{nk} – quasigroup. If Q is an α_m – quasigroup, $\varphi\psi^{m-1} = \psi\varphi^{m-1}$. Hence $\varphi\psi^{m+nk-1} = \psi\varphi^{m+nk-1}$. Thus Q is an α_{m+nk} – quasigroup.

Theorem 21: Every γ_2 – quasigroup is a γ_n – quasigroup for every even $n \geq 2$.

Proof: This theorem follows directly from Lemma 20.

Lemma 21: Let Q be a T – quasigroup and let there be a number $n \geq 2$ such that at least one of the following conditions holds:

(i) Q is a β_n and β_{n+1} – quasigroup.

(ii) Q is an α_n and α_{n+1} – quasigroup.

(iii) Q is an α_{n+1} and β_n – quasigroup.

Then Q is commutative.

Proof: Be $(Q(+), \varphi, \psi, g)$ any T – form of Q. If (i) holds, then $\varphi^n = \psi^n$ and $\varphi^{n+1} = \psi^{n+1}$. Hence $\varphi = \psi$ and Q is commutative. If (ii) holds, $\varphi\psi^{n-1} = \psi\varphi^{n-1}$ and $\varphi\psi^n = \psi\varphi^n$. Therefore $\varphi\psi^n = \varphi\psi^{n-1}\psi = \psi\varphi^{n-1}\psi = \psi\varphi^n = \psi\varphi^{n-1}\varphi$. Hence $\varphi = \psi$. Finally, if (iii) holds, $\varphi^n = \psi^n$ and $\varphi\psi^n = \psi\varphi^n$. Hence $\varphi = \psi$.

Theorem 22: Let Q be a β_2 – quasigroup and let there be an odd number $n \geq 2$ such that Q is an α_n or β_n – quasigroup. Then Q is commutative.

Proof: The theorem follows from Lemmas 20, 21.

Lemma 22: Let $Q(+)$ be an Abelian group and φ, ψ two its automorphisms such that the mapping $x \to x + x$ is a permutation of the set Q, $\varphi\psi = \psi\varphi$ and $\varphi^2 = \psi^2$. Then there exist two subgroups $C(+)$ and $U(+)$ of $Q(+)$ such that $\varphi \mid C = \psi \mid C$ is an automorphism of $C(+)$, $\varphi \mid U = - \psi \mid U$ is an automorphism of $U(+)$ and $Q(+) = C(+) \dotplus U(+)$.

Proof: Be C the set of all $x \in Q$ such that $\varphi(x) = \psi(x)$ and U the set of all $y \in Q$ such that $\varphi(y) = - \psi(y)$. It is easy to show that C and U are subgroups of $Q(+)$. Since $Q(+)$ has no non - zero element of the order 2, $U(+) \cap C(+) = O$. Be $y \in Q(+)$ an arbitrary element. As the mapping 2φ is a permutation of the set Q, there is $x \in Q$ such that $y = \varphi(x) + \varphi(x)$. Put $a = \varphi(x) + \psi(x)$, $b = \varphi(x) - \psi(x)$. Evidently, $y = a + b$. But

$$\begin{aligned} &\varphi(a) = \varphi(\varphi(x) + \psi(x)) = \varphi^2(x) + \varphi\psi(x) = \\ &= \psi^2(x) + \psi\varphi(x) = \psi(\varphi(x) + \psi(x)) = \psi(a), \\ &\varphi(b) = \varphi(\varphi(x) - \psi(x)) = \varphi^2(x) - \varphi\psi(x) = \\ &= \psi^2(x) - \psi\varphi(x) = \psi(\varphi(x) - \varphi(x)) = - \psi(b). \end{aligned}$$

Hence $a \in C(+)$, $b \in U(+)$. Therefore $Q(+) = C(+) \dotplus U(+)$. Evidently $\varphi \mid C = \psi \mid C$ and $\varphi \mid U = - \psi \mid U$. Let, further, $x \in C(+)$. Then $\varphi^2(x) = \varphi\psi(x) = \psi\varphi(x)$ and $\psi\varphi^{-1}(x) = \varphi^{-1}\psi(x) = \varphi\varphi^{-1}(x)$. Hence $\varphi(x), \varphi^{-1}(x) \in C(+)$. Similarly, if $x \in U(+)$, $\varphi(x), \varphi^{-1}(x) \in U(+)$. Thus $\varphi \mid C$ is an automorphism of $C(+)$ and $\varphi \mid U$ is an automorphism of $U(+)$.

Theorem 23: Let Q be a γ_2 - quasigroup and $Q(+)$ its arbitrary T - group. Let the mapping $x \to x + x$ be a permutation of the set Q. Then Q is a K - quasigroup.

Corollary: Every finite γ_2 - quasigroup of odd order is a K - quasigroup.

Proof: Let $(Q(+), \varphi, \psi, g)$ be the corresponding T - form to the T - group $Q(+)$. By Theorems 10, 11, $\varphi^2 = \psi^2$, $\varphi\psi = \psi\varphi$. Hence, by Lemma 21, there exist two subgroups $C(+)$ and $U(+)$ of $Q(+)$ such that $\varphi|C = \psi|C$ and $\varphi|U = -\psi|U$ are automorphisms of $C(+)$ and $U(+)$ respectively and $Q(+) = C(+) \dotplus U(+)$. Define the mapping σ, $\sigma : Q \to C \times U$, as follows: $\sigma(x) = (c, u)$, where $c \in C$, $u \in U$ such that $x = c + u$. The mapping σ is an isomorphism of $Q(+)$ onto $C(+) \times U(+)$. Denote $\varphi|C = \varphi_1$, $\varphi|U = \varphi_2$ and put $\eta = \varphi_1 \times \varphi_2$, $\varrho = \varphi_1 \times (-\varphi_2)$. Then $\sigma\varphi(x) = \sigma\varphi(c + u) = \sigma(\varphi(c) + \varphi(u)) = (\varphi_1(c), \varphi_2(u)) = \eta(c,u) = \eta\sigma(x)$, $\sigma\psi(x) = \varrho\sigma(x)$.

Let $C(\circ)$ be the T - quasigroup of the T - form $(C(+), \varphi_1, \varphi_1, g_1)$ and $U(\cdot)$ the T - quasigroup of the T - form $(U(+), \varphi_2, -\varphi_2, g_2)$, where $\sigma(g) = (g_1, g_2)$. Evidently $C(\circ)$ is a commutative quasigroup and $U(\cdot)$ is a unipotent quasigroup. Further, for every $x, y \in Q$, $\sigma(x) = (a, b)$, $\sigma(y) = (v,z)$, we have

$$\begin{aligned} &\sigma(xy) = \sigma(\varphi(x) + \psi(y) + g) = \sigma\varphi(x) + \sigma\psi(y) + \sigma(g) = \\ &= \eta\sigma(x) + \varrho\sigma(y) + \sigma(g) = \eta(a,b) + \varrho(v,z) + (g_1, g_2) = \\ &= (\varphi_1(a), \varphi_2(b)) + (\varphi_1(v), - \varphi_2(z)) + (g_1, g_2) = \\ &= (\varphi_1(a) + \varphi_1(v) + g_1, \varphi_2(b) - \varphi_2(z) + g_2) = (a \circ v, b \cdot z). \end{aligned}$$

Hence σ is an isomorphism of Q onto $C(\circ) \times U(\cdot)$ and hence, Q is a K – quasigroup.

Theorem 24: Let $n \geq 2$ be a positive integer and Q be a γ_n – quasigroup. Let at least one of the following condition hold:

(i) The mapping $x \to xx$ is one – to – one.

(ii) The mapping $x \to xx$ is onto Q.

(iii) For every $x \in Q$, $e(x) = f(x)$.

(iv) For every $x \in Q$ there are $u, v \in Q$ such that $uv = vu = x$.

Then Q is commutative.

Corollary: Every idempotent γ_n – quasigroup is commutative.

Proof: Be $(Q(+), \varphi, \psi, g)$ a T – form of Q. We have $\varphi^n = \psi^n$, $\varphi\psi^{n-1} = \psi\varphi^{n-1}$. Hence

$$\begin{aligned}\varphi^{n-1}\psi\varphi^{n-1} &= \varphi^{n-1}\varphi\psi^{n-1} = \varphi^n\psi^{n-1} = \psi^n\psi^{n-1} = \\ &= \psi^{n-1}\psi^n = \psi^{n-1}\varphi^n = \psi^{n-1}\varphi\varphi^{n-1}.\end{aligned}$$

Therefore $\varphi^{n-1}\psi = \psi^{n-1}\varphi$. Evidently (ii) implies (iv) and (iii) implies (iv). Let (iv) hold and $x \in Q$ be an arbitrary element. There are $u, v \in Q$ such that $uv = vu = x + g$. That is, $\varphi(u) + \psi(v) = \varphi(v) + \psi(u) = x$.

Further,

$$\begin{aligned}\varphi^{n-1}(x) &= \varphi^{n-1}(\varphi(v) + \psi(u)) = \varphi^n(v) + \varphi^{n-1}\psi(u) = \\ &= \psi^n(v) + \psi^{n-1}\varphi(u) = \psi^{n-1}(\psi(v) + \varphi(u)) = \psi^{n-1}(x).\end{aligned}$$

We have proved that $\varphi^{n-1} = \psi^{n-1}$. Hence $\varphi = \psi$ and Q is commutative. Finally, let (i) hold. Since $x \to xx$ is one – to – one, the mapping $\xi = \varphi + \psi$ is one – to – one. Let $x \in Q$ be an arbitrary element. Put $a = \varphi^{n-1}(x) - \psi^{n-1}(x)$.

We have

$$\xi(a) = \varphi^n(x) - \varphi\psi^{n-1}(x) + \psi\varphi^{n-1}(x) - \psi^n(x) = O.$$

As $\xi(O) = O$, $a = O$. Hence $\varphi^{n-1}(x) = \psi^{n-1}(x)$. We have proved that $\varphi^{n-1} = \psi^{n-1}$. Hence Q is commutative.

Definition 8: A quasigroup Q is called anticommutative if for every $x, y \in Q$, $xy = yx$ implies $x = y$.

Lemma 23: A T – quasigroup Q is anticommutative if and only if for any (and then for each) of its T – forms $(Q(+), \varphi, \psi, g)$ and for every $x \in Q$, $x \neq O$, is $\varphi(x) \neq \psi(x)$.

Proof: 1. Let Q have such a T – form $(Q(+), \varphi, \psi, g)$. Let $x, y \in Q$ and $xy = yx$. That is, $\varphi(x) + \psi(y) + g = \psi(x) + \varphi(y) + g$. Hence $\varphi(x - y) = \psi(x - y)$ and hence $x = y$.

2. Let Q be anticommutative and $(Q(+), \varphi, \psi, g)$ be an arbitrary T – form of Q. Let $x \in Q$ be an arbitrary element such that $\varphi(x) = \psi(x)$. Then $O \cdot x = \psi(x) + g = \varphi(x) + g = x \cdot O$. Hence $x = O$.

Theorem 25: Let $n \geq 2$ be a positive integer and Q be an anticommutative γ_n – quasigroup. Then Q is unipotent.

Proof: Be $(Q(+), \varphi, \psi, g)$ any T – form of Q. Then $\varphi^n = \psi^n$, $\varphi\psi^{n-1} = \psi\varphi^{n-1}$.

Further,

$$O \cdot (\varphi^{n-1}(x) + \psi^{n-1}(x)) = \psi\varphi^{n-1}(x) + \psi^{n}(x) + g =$$
$$= \varphi\psi^{n-1}(x) + \varphi^{n}(x) + g = \varphi(\psi^{n-1}(x) + \varphi^{n-1}(x)) + g =$$
$$= (\varphi^{n-1}(x) + \psi^{n-1}(x)) \cdot O.$$

Hence $O = \varphi^{n-1}(x) + \psi^{n-1}(x)$ and hence, $\varphi = - \psi$. Thus, by Lemma 18, Q is unipotent.

Lemma 24: A T - quasigroup Q is idempotent if and only if for every its T - form $(Q(+), \varphi, \psi, g)$, g is the zero in $Q(+)$. In this case $\varphi + \psi = 1$.

Proof: 1. Be Q idempotent and $(Q(+), \varphi, \psi, g)$ be any T - form of Q. Then $O = O \cdot O = g$ and $x = xx = \varphi(x) + \psi(x)$ for every $x \in Q$.

2. Let a be an arbitrary element of Q. There is a T - form $(Q(\circ), \eta, \varrho, h)$ of Q such that a is the zero in $Q(\circ)$. Then $a = h = aa$. Hence Q is idempotent.

Theorem 25: Every idempotent T - quasigroup is Abelian.

Proof: Be Q an idempotent T - quasigroup and $(Q(+), \varphi, \psi, g)$ its T - form. Then, by Lemma 22, $\varphi + \psi = 1$. Therefore

$$\varphi\psi = \varphi(1 - \varphi) = \varphi - \varphi^2 = (1 - \varphi)\varphi = \psi\varphi.$$

By Theorem 12, Q is Abelian.

Theorem 26: Every idempotent β_2 — quasigroup is commutative.

Proof: Be Q such a quasigroup. Then for every $x, y \in Q$,

$$xy = xy \cdot xy = yy \cdot xx = yx.$$

Theorem 27: Let Q be a T - quasigroup. Then the following conditions are equivalent:

(i) The mapping $x \to xx$ is an endomorphism of Q.

(ii) The mapping $x \to e(x)$ is an endomorphism of Q.

(iii) The mapping $x \to f(x)$ is an endomorphism of Q.

(iv) Q is Abelian.

Proof: Evidently (iv) implies (i), (ii) and (iii). Now we prove that (i) implies (iv). Be $(Q(+), \varphi, \psi, g)$ any T - form of Q. Then

$$xy \cdot xy = \varphi^2(x) + \varphi\psi(y) + \psi\varphi(x) + \psi^2(y) + \varphi(g) + g + \psi(g) = xx \cdot yy =$$
$$= \varphi^2(x) + \varphi\psi(x) + \psi\varphi(y) + \psi^2(y) + \varphi(g) + \psi(g) + g.$$

Thus $\varphi\psi(x) + \psi\varphi(y) = \varphi\psi(y) + \psi\varphi(x)$. Hence $\varphi\psi = \psi\varphi$. Similarly we can prove that (ii) implies (iv) and (iii) implies (iv).

Theorem 28: Be Q a T - quasigroup and $x \in Q$. Then the following conditions are equivalent:

(i) L_x is an automorphism of Q.

(ii) R_x is an automorphism of Q.

(iii) x is idempotent and $L_xR_x = R_xL_x$.

(iv) Q is Abelian and x is idempotent.

Proof: (i) implies (iii). We have $L_x(xx) = x(xx) = L_x(x) \cdot L_x(x) = xx \cdot xx$. Hence $x = xx$. Further, $L_xR_x(y) = x \cdot yx = L_x(yx) = L_x(y) \cdot L_x(x) = R_xL_x(y)$.
Similarly we can prove that (ii) implies (iii).

(iii) implies (iv). Be $(Q(+), \varphi, \psi, g)$a T — form of Q such that x is the zero in $Q(+)$. By Lemma 5, $\varphi = R_{e(x)}, \psi = L_{f(x)}$.
Since x is idempotent, $e(x) = f(x) = x$. But $L_x R_x = R_x L_x$. Hence $\varphi\psi = \psi\varphi$.
(iv) implies (i) and (ii). For every $y, z \in Q$ we have $L_x(yz) = x \cdot yz = xx \cdot yz = = xy \cdot xz = L_x(y) \cdot L_x(z)$, $R_x(yz) = yz \cdot x = yz \cdot xx = R_x(y) \cdot R_x(z)$.
Thus L_x, R_x are automorphisms of Q.

Theorem 29: Be Q a T - quasigroup. Then the following conditions are equivalent:
(i) The mapping $x \to xx$ is an antiendomorphism of Q.
(ii) Q is a β_2 - quasigroup.
Proof: The proof is similar to that of Theorem 27.

Lemma 25: A T - quasigroup Q has at least one idempotent element if and only if there is a T - form $(Q(+), \varphi, \psi, g)$ of Q such that $g = O$.
Proof: Be $a \in Q$ an idempotent element. There is a T - form $(Q(+), \varphi, \psi, g)$ of Q such that $a = O$. But $g = O \cdot O = a \cdot a = a = O$. On the contrary, if $(Q(+), \varphi, \psi, O)$ is a T - form of Q then $O \cdot O = O$.

Lemma 26: Let Q be a T - quasigroup and $Q(+)$ its T - group. Let the group $Q(+)$ be cyclic. Then Q is Abelian.
Proof: Be $(Q(+), \varphi, \psi, g)$ a T - form of Q. Since the group $Q(+)$ is cyclic, there are numbers n, m such that $\varphi(x) = nx$, $\psi(x) = mx$ for every $x \in Q$. From this follows $\varphi\psi = \psi\varphi$.

Theorem 30: Every finite T - quasigroup of prime order is Abelian and commutative or anticommutative.
Proof: Be Q a T — quasigroup of prime order p and $(Q(+), \varphi, \psi, g)$ be any of its T - forms. The group $Q(+)$ is, evidently, cyclic. Hence, by Lemma 26, Q is Abelian. Let Q not be anticommutative. Then, by Lemma 23, there is $x \in Q(+)$ such that $x \neq O$ and $\varphi(x) = \psi(x)$. There are numbers n, m such that $\varphi(y) = ny$ and $\psi(y) = my$ for every $y \in Q(+)$. Hence $(n - m)x = O$. But x has the order p. Therefore $n = m \ (mod p)$. Hence $\varphi = \psi$ and Q is commutative.

Example 1: Let $Q(+)$ be an Abelian group having four elements 0,1,2,3 and let each of them have the order 2. The permutations

$$\varphi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{pmatrix}, \psi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{pmatrix}$$

are automorphisms of the group $Q(+)$. Moreover, $\varphi^2 = \psi^2 = 1, \varphi\psi \neq \psi\varphi, \eta^2 \neq 1$. Let $Q(\circ)$ be the T - quasigroup of the T - form $(Q(+), \varphi, \psi, O)$ and $Q(\cdot)$ be the T - quasigroup of the T - form $(Q(+), \eta, 1, O)$. Then $Q(\circ)$ is a β_2 - quasigroup and is not an Abelian quasigroup and $Q(\cdot)$ is an Abelian quasigroup and is not a β_2 — quasigroup.

Example 2: Be $Q(+)$ the cyclic group of the order 8. Put $\varphi(x) = 7x$ and $\psi(x) = = 3x$ for every $x \in Q$. Then φ, ψ are automorphisms of $Q(+)$. Be $Q(\cdot)$ the T - quasigroup of the T - form $(Q(+), \varphi, \psi, O)$. Since $\varphi^2(x) = 49x = 9x = \psi^2(x)$ and

$\varphi\psi(x) = \psi\varphi(x)$, $Q(\cdot)$ is a γ_2 - quasigroup. But $\varphi \neq \psi$, $\varphi \neq -\psi$ and the group $Q(+)$ is directly indecomposable. Hence $Q(\cdot)$ is not a K - quasigroup.

Example 3: Let $G_i(+) = \{g_i\}^+$ be cyclic groups of the order 2, for $i = 1,2,3,4$, and $Q(+) = \sum_{i=1}^{4} G_i(+)$. Be φ, ψ two automorphisms of the group $Q(+)$ such that $\varphi(g_1) = g_2$, $\varphi(g_2) = g_3$, $\varphi(g_3) = g_4$, $\varphi(g_4) = g_1$ and $\psi(g_1) = g_2$, $\psi(g_2) = g_1$, $\psi(g_3) = g_3$, $\psi(g_4) = g_4$. Let $Q(\cdot)$ be the T - quasigroup having the T - form $(Q(+), \varphi, \psi, O)$. Then, for every $n \geq 2$, $Q(\cdot)$ is not an α_n - quasigroup. Moreover, $Q(\cdot)$ is finite.

Example 4: Let $G_i(+) = \{g_i\}^+$, $i = 1, 2, 3, 4$, be cyclic groups of the order 3. Put $Q(+) = \sum_{i=1}^{4} G_i(+)$. Be φ, ψ two automorphisms of the group $Q(+)$ such that $\varphi(g_1) = g_1$, $\varphi(g_2) = g_3$, $\varphi(g_3) = g_4$, $\varphi(g_4) = g_2$ and $\psi(g_1) = g_3$, $\psi(g_2) = g_1$, $\psi(g_3) = g_2$, $\psi(g_4) = g_4$. Then $\varphi^3 = \psi^3$ and $\varphi\psi^2 = \psi\varphi^2$. Let $Q(\cdot)$ be the T - quasigroup having the T - form $(Q(+), \varphi, \psi, O)$. Then $Q(\cdot)$ is a γ_3 - quasigroup. Since $\varphi \neq \psi$, $Q(\cdot)$ is not commutative, and hence, $Q(\cdot)$ is not a K - quasigroup. Moreover, $Q(\cdot)$ has an odd order.

5° - The characteristic group and the multiplicative group

Lemma 27: Let Q_1, Q_2 be two T - quasigroups and $(Q_i(+), \varphi_i, \psi_i, g_i)$, $i = 1,2$, their arbitrary T - forms censecutively. Be $\eta : Q_1 \to Q_2$ a homomorphism. Put for every $x \in Q_1$, $\xi(x) = \eta(x) - \eta(O)$. Then $\xi : Q_1(+) \to Q_2(+)$ is a group homomorphism and $\xi\varphi_1 = \varphi_2\xi$, $\xi\psi_1 = \psi_2\xi$. Moreover, ξ is one - to - one (onto Q_2) if and only if η is one - to - one (onto Q_2).

Proof: Since $\eta : Q_1 \to Q_2$ is a homomorphism, we have for every $a, b \in Q_1$

$$\eta(\varphi_1(a) + \psi_1(b) + g_1) = \varphi_2\eta(a) + \psi_2\eta(b) + g_2. \tag{15}$$

For $b = \psi_1^{-1}(-g_1)$ we get

$$\eta\varphi_1(a) = \varphi_2\eta(a) + \psi_2\eta\psi_1^{-1}(-g_1) + g_2 = \varphi_2\eta(a) + g_3, \tag{16}$$

where $g_3 = \psi_2\eta\psi_1^{-1}(-g_1) + g_2$.

Similarly,

$$\eta\psi_1(b) = \psi_2\eta(b) + \varphi_2\eta\varphi_1^{-1}(-g_1) + g_2 = \psi_2\eta(b) + g_4, \tag{17}$$

where $g_4 = \varphi_2\eta\varphi_1^{-1}(-g_1) + g_2$.

If $a = \varphi_1^{-1}(-g_1)$, $b = \psi_1^{-1}(-g_1)$ then from (15) follows

$$\eta(-g_1) = g_3 + g_4 - g_2. \tag{18}$$

From (15), (16), (17) and (18) follows

$$\eta(a + b) = \eta(\varphi_1\varphi_1^{-1}(a) + \psi_1\psi_1^{-1}(b - g_1) + g_1) = \eta(a) + \eta(b - g_1) - \eta(-g_1). \tag{19}$$

Hence $\eta(b - g_1) = \eta(b) + \eta(-2g_1) - \eta(-g_1)$.

After substituting into (19) we get

$$\eta(a + b) = \eta(a) + \eta(b) + \eta(-2g_1) - 2\eta(-g_1). \tag{20}$$

Define the mapping $\xi : Q_1 \to Q_2$ as follows:

$$\xi(a) = \eta(a) + \eta(-2g_1) - 2\eta(-g_1) \quad \text{for every } a \in Q_1.$$

In view of (20), the mapping ξ is a group homomorphism of $Q_1(+)$ into $Q_2(+)$. Now substitute $a = O$, $b = g_1$ into (20). Then

$$\eta(-g_1) = \eta(O) + \eta(-g_1) + \eta(-2g_1) - 2\eta(-g_1),$$

hence $\eta(O) = 2\eta(-g_1) - \eta(-2g_1)$.
Thus $\xi(a) = \eta(a) - \eta(O)$ for every $a \in Q_1$.
Further, according to (16),

$$\xi\varphi_1(a) = \eta\varphi_1(a) - \eta(O) = \varphi_2\eta(a) + g_3 - \eta(O).$$

But $\varphi_2\eta(O) = \eta(O) - g_3$ and therefore

$$\xi\varphi_1(a) = \varphi_2\eta(a) - \varphi_2\eta(O) = \varphi_2(\eta(a) - \eta(O)) = \varphi_2\xi(a).$$

Similarly we can prove $\xi\psi_1 = \psi_2\xi$. The last part of the proof is evident.

Lemma 28: Let Q_1, Q_2 be two T – quasigroups and $(Q_1(+), \varphi_1, \psi_1, g_1)$ be an arbitrary T – form of Q_1. Be $\eta : Q_1 \to Q_2$ a homomorphism. Then there exists a T – form $(Q_2(+), \varphi_2, \psi_2, g_2)$ of the quasigroup Q_2 such that $\eta : Q_1(+) \to Q_2(+)$ is a group homomorphism, $\eta\varphi_1 = \varphi_2\eta$, $\eta\psi_1 = \psi_2\eta$ and $\eta(g_1) = g_2$.

Proof: There exists a T – form $(Q_2(+), \varphi_2, \psi_2, g_2)$ of Q_2 such that the element $\eta(O)$ is the zero in $Q_2(+)$. The mapping ξ, $\xi(x) = \eta(x) - \eta(O)$, is, by Lemma 27, a group homomorphism of $Q_1(+)$ into $Q_2(+)$ and $\xi\varphi_1 = \varphi_2\xi$, $\xi\psi_1 = \psi_2\xi$. But $\xi(x) = \eta(x) - \eta(O) = \eta(x)$. Hence $\xi = \eta$. Finally, $\eta(g_1) = \eta(O \cdot O) = \eta(O) \cdot \eta(O) = O \cdot O = g_2$.

Lemma 29: Let Q_1, Q_2 be two T – quasigroups and $(Q_2(+), \varphi_2, \psi_2, g_2)$ be an arbitrary T – form of Q_2. Be $\eta : Q_1 \to Q_2$ a homomorphism onto Q_2. Then there exists a T – form $(Q_1(+), \varphi_1, \psi_1, g_1)$ of the quasigroup Q_1 such that $\eta : Q_1(+) \to Q_2(+)$ is a group homomorphism, $\eta\varphi_1 = \varphi_2\eta$, $\eta\psi_1 = \psi_2\eta$ and $\eta(g_1) = g_2$.

Proof: There is an element a in Q_1 such that $\eta(a) = O$. Select a T – form $(Q_1(+), \varphi_1, \psi_1, g_1)$ such that the element a is the zero in $Q_1(+)$. Now we shall use Lemma 27.

Lemma 30: Let Q_i, $i = 1,2$, be two T – quasigroups and $(Q_i(+), \varphi_i, \psi_i, g_i)$ their arbitrary T – forms. Be $\eta : Q_1 \to Q_2$ a homomorphism. Then η is simultaneously a group homomorphism of $Q_1(+)$ into $Q_2(+)$ if and only if $\eta(O) = O$. In this case $\eta\varphi_1 = \varphi_2\eta$, $\eta\psi_1 = \psi_2\eta$ and $\eta(g_1) = g_2$.

Proof: The lemma is an easy consequence of Lemma 27.

Theorem 31: Let Q be a T – quasigroup and $(Q(+), \varphi, \psi, g)$, $(Q(\circ), \varrho, \tau, h)$ be two of its T – forms. Then there is an isomorphism $\xi : Q(+) \to Q(\circ)$ such that $\xi\varphi = \varrho\xi$, $\xi\psi = \tau\xi$.

Proof: By Lemma 27, the mapping ξ, $\xi(x) = x \circ O$, is such an isomorphism.

Definition 9: Let Q be a T – quasigroup and $T = (Q(+), \varphi, \psi, g)$ be its arbitrary T – form. Denote $A(Q, T)$ (or only $A(Q)$), the group generated by the elements

φ, ψ in the group S_Q (that is the same as in the group *Aut* $Q(+)$). The group $A(Q, T)$ is called characteristic group of the quasigroup Q (corresponding to the T - form T).

Theorem 32: Let Q be a T - quasigroup, $T = (Q(+), \varphi, \psi, g)$ and $S = (Q(\circ), \varrho, \tau, h)$ be two arbitrary T - forms of Q. Then there is an isomorphism $\mathfrak{a}$, $\mathfrak{a} : A(Q,T) \to A(Q, S)$ such that $\mathfrak{a}(\varphi) = \varrho$, $\mathfrak{a}(\psi) = \tau$. Moreover, $\mathfrak{a}$ is a restriction of an inner automorphism of the group S_Q.

Proof: According to Theorem 31, there is a permutation ξ of the set Q such that $\xi\varphi = \varrho\xi$, $\xi\psi = \tau\xi$. Put $\mathfrak{r}(\sigma) = \xi\sigma\xi^{-1}$ for every $\sigma \in S_Q$. Then $\mathfrak{r}$ is an inner automorphism of the group S_Q. Since $\mathfrak{r}(\varphi) = \varrho$ and $\mathfrak{r}(\psi) = \tau$, it is $\mathfrak{r}(A(Q,T)) = A(Q,S)$. Now it is sufficient to define $\mathfrak{a} = \mathfrak{r}|A(Q,T)$.

Theorem 33: Let Q, P be two T - quasigroups and $\eta : Q \to P$ an epimorphism. Be further $T = (Q(+), \varphi, \psi, g)$ any T - form of Q and $S = (P(+), \varphi_1 \psi_1, g_1)$ a T - form of P corresponding to η in the sense of Lemma 28. Then there exists an epimorphism $\mathfrak{a} : A(Q,T) \to A(P,S)$ such that $\eta\alpha(a) = \mathfrak{a}(\alpha)\eta(a)$ for every $\alpha \in A(Q,T)$ and $a \in Q$. If η is one - to - one, $\mathfrak{a}$ is one - to - one.

Proof: By Lemma 28, $\eta : Q(+) \to P(+)$ is an epimorphism and $\eta\varphi = \varphi_1\eta$, $\eta\psi = \psi_1\eta$. Be a, $b \in Q$. If $\eta(a) = \eta(b)$ then $\varphi_1\eta(a) = \varphi_1\eta(b)$, and hence, $\eta\varphi(a) = \eta\varphi(b)$. If, on the contrary, $\eta\varphi(a) = \eta\varphi(b)$ then $\varphi_1\eta(a) = \varphi_1\eta(b)$, therefore $\eta(a) = \eta(b)$. Similarly we can prove that $\eta(a) = \eta(b)$ if and only if $\eta\psi(a) = \eta\psi(b)$. From this it follows easily that for every $\alpha \in A(Q,T)$ and for every a, $b \in Q$ is $\eta(a) = \eta(b)$ if and only if $\eta\alpha(a) = \eta\alpha(b)$. Now we can define a mapping $\mathfrak{a}$, $\mathfrak{a} : A(Q,T) \to S_P$ as follows:

For every $\alpha \in A(Q,T)$, $\mathfrak{a}(\alpha)$ is a permutation of the set P such that for every $p \in P$, $\mathfrak{a}(\alpha)(p) = \eta\alpha(a)$, where $a \in Q$ such that $\eta(a) = p$. Evidently, $\mathfrak{a}$ is a homomorphism and $\mathfrak{a}(\varphi) = \varphi_1$, $\mathfrak{a}(\psi) = \psi_1$. Therefore $\mathfrak{a}$ is an epimorphism of $A(Q, T)$ onto $A(P, S)$. Let further $\mathfrak{a}$ not be one — to — one. Then there are α, $\beta \in A(Q,T)$ such that $\alpha \neq \beta$ and $\mathfrak{a}(\alpha) = \mathfrak{a}(\beta)$. Hence there is $a \in Q$ such that $\alpha(a) \neq \beta(a)$ and $\mathfrak{a}(\alpha)\eta(a) = \mathfrak{a}(\beta)\eta(a)$. Thus $\eta\alpha(a) = \eta\beta(a)$ and hence, η is not one - to - one.

Theorem 34: Be P a subquasigroup of a T - quasigroup Q and $T = (Q(+), \varphi, \psi, g)$ a P - canonic T - form of Q. Than there is an epimorphism $\mathfrak{a} : A(Q,T) \to A(P,T)$.

Proof: It is sufficient to define $\mathfrak{a}(\alpha) = \alpha|P$ for every $\alpha \in A(Q,T)$.

Theorem 35: Let H be a group having two generators. Then there is a T - quasigroup Q such that $A(Q) \cong H$ and Q has one element generator set.

Proof: Put $Q(+) = \sum\limits_{h \in H} H_h(+)$, where $H_h(+) = \{a_h\}^+$ is an infinite cyclic Abelian group. Be g, k two generators of the group H and φ, ψ two automorphisms of the group $Q(+)$ such that $\varphi(a_h) = a_{gh}$, $\psi(a_h) = a_{kh}$ for every $h \in H$. Be $Q(\cdot)$ the T - quasigroup of the T - form $(Q(+), \varphi, \psi, a_j)$, where j is the unit of the group H. Define the mapping $\mathfrak{a} : H \to Aut\ Q(+)$ as follows: For every $d \in H$, $\mathfrak{a}(d)$ is the automorphism of the group $Q(+)$ such that $\alpha(d)(a_h) = a_{dh}$ for every $h \in H$. Since $\alpha(g) = \varphi$, $\alpha(k) = \psi$ and $\alpha(h)(a_j) = a_h$ for every $h \in H$, α is an isomorphism of H onto $A(Q)$. Be $P(\cdot)$ the subquasigroup in $Q(\cdot)$ generated by the element O. By Lemma 11, the T-form $(Q(+), \varphi, \psi, a_j)$ is P-canonic. Hence $P(+)$ is a subgroup in $Q(+)$ and for every $\alpha \in A(Q(\cdot))$ the element $\alpha(a_j)$ is in $P(\cdot)$. Be $h \in H$ an arbitrary element. Then $\alpha(h)(a_j) \in P(\cdot)$. But $\alpha(h)(a_j) = a_h$. Hence $a_h \in P(+)$ for every $h \in H$, and hence, $P(+) = Q(+)$. Therefore $P(\cdot) = Q(\cdot)$.

Theorem 36: Be Q a T-quasigroup. Then $A(Q)$ is Abelian if and only if Q is Abelian.

Proof: This theorem follows directly from Theorem 12.

Lemma 31: Let Q be a T-quasigroup and $(Q(+), \varphi, \psi, g)$ its arbitrary T-form. Then for every $a,x \in Q$,

$$\begin{aligned} &L_a(x) = \varphi(a) + g + \psi(x),\ R_a(x) = \psi(a) + g + \varphi(x), \\ &L_a^{-1}(x) = -\psi^{-1}\varphi(a) - \psi^{-1}(g) + \psi^{-1}(x),\ R_a^{-1}(x) = \\ &= -\varphi^{-1}\psi(a) - \varphi^{-1}(g) + \varphi^{-1}(x). \end{aligned} \tag{21}$$

Proof: The lemma is obvious.

Theorem 37: Be Q a T-quasigroup and $(Q(+), \varphi, \psi, g)$ its arbitrary T-form. Then the multiplicative group G_Q of Q is generated in S_Q by all translations of $Q(+)$ and by permutations φ, ψ. Moreover, the multiplicative group $G_{Q(+)}$ of $Q(+)$ is a normal subgroup in G_Q and $G_Q/G_{Q(+)} \cong A(Q)$.

Proof: Denote H the group generated in S_Q by all translations of $Q(+)$ and by φ, ψ. The group G_Q is generated by all permutations R_x, L_y, $x,y \in Q$. In view of Lemma 31, $R_x = R^+_{\psi(x)+g}\varphi$, $L_y = L^+_{\varphi(y)+g}\psi$. Thus R_x, $L_y \in H$ and $G_Q \subseteq H$. By Lemma 5, $\varphi = R_{e(o)}$, $\psi = L_{f(o)}$, so that $\varphi,\psi \in G_Q$. Be $a \in Q$ arbitrary. Put $b = = \varphi^{-1}(a - g)$. Then $L_b = L^+_{\varphi(b)+g}\psi = L^+_a\psi$. But $L_b\psi \in G_Q$. Hence $L^+_a \in G_Q$ and hence, $H \subseteq G_Q$. Be $\alpha \in G_{Q(+)}$ an arbitrary element. Since $Q(+)$ is an Abelian group, there is $a \in Q$ such that $\alpha = L^+_a$. Further, since φ, ψ are automorphisms of $Q(+)$, $\varphi\alpha\varphi^{-1} = \varphi L^+_a\varphi^{-1} = L^+_{\varphi(a)}\,\varphi\varphi^{-1} = L^+_{\varphi(a)}$, $\psi\,\alpha\,\psi^{-1} = L^+_{\psi(a)}$. Hence the group $G_{Q(+)}$ is a normal subgroup in G_Q. Since $G_{Q(+)} \cap A(Q) = 1$, $G_Q/G_{Q(+)} \cong A(Q)$.

Theorem 38: Let Q be a T-quasigroup. Then the group G_Q is solvable if and only if the group $A(Q)$ is solvable.

Corollary: The multiplicative group of every Abelian quasigroup is solvable.

Proof: The theorem and its corollary follow from Theorems 36, 37.

Lemma 32: Let Q be a T-quasigroup, $(Q(+), \varphi, \psi, g)$ its arbitrary T-form and $a \in Q$. Then the group I_a of all inner permutations corresponding to a is generated by permutations $R_{e(a)}$, $L_{f(a)}$, T^a, where $T^a(x) = \psi^{-1}\varphi(x - a) + a$ for every $x \in Q$. If Q is commutative, the group I_a is generated by permutation $L_{f(a)}$, thus being a cyclic group.

Proof: The group I_a is generated by permutations

$$L_u^{-1}L_yL_x,\ R_v^{-1}R_xR_y, L_z^{-1}R_x, \text{ where } x, y \in Q \text{ and}$$
$$u = R_a^{-1}(y \cdot xa),\ v = L_a^{-1}(ay \cdot x),\ z = R_a^{-1}L_a(x).$$

In view of (21) we have

$$R_a^{-1}(y \cdot xa) = y + \varphi^{-1}\psi\varphi(x) + \varphi^{-1}\psi^2(a) - \varphi^{-1}\psi(a) + \varphi^{-1}\psi(g).$$

Hence for every $t \in Q$,

$$L_u^{-1}L_yL_x(t) = \psi(t - a) + a = L_{f(a)}(t).$$

Similarly, $R_v^{-1}R_xR_y = R_{e(a)}$ and

$L_z^{-1}R_x(t) = \psi^{-1}\varphi\,(t - a) + a$. If Q is commutative, $\varphi = \psi$ and $L_{f(a)} = R_{e(a)}$.

Lemma 33: Let Q be a T - quasigroup and $(Q(+), \varphi, \psi, g)$ its T - form. Denote the left (right) inverse quasigroup of the quasigroup Q by $Q(\circ)$ $(Q(\bullet))$. Then $Q(\circ), Q(\bullet)$ are T - quasigroups and $(Q(+), \varphi^{-1}, -\varphi^{-1}\psi, -\varphi^{-1}(g))$, $(Q(+), -\psi^{-1}\varphi, \psi^{-1}, -\psi^{-1}(g))$ are their T - forms respectively.

Proof: Be $x, y \in Q$, $z = x \circ y$. Then $zy = x$, hence $\varphi(z) + \psi(y) + g = x$. Therefore $z = \varphi^{-1}(x) - \varphi^{-1}\psi(y) - \varphi^{-1}(g)$. Thus $(Q(+), \varphi^{-1}, -\varphi^{-1}\psi, -\varphi^{-1}(g))$ is a T - form of $Q(\circ)$. For the other case similarly.

Theorem 39: Let Q be a T - quasigroup. Then all parastrophic quasigroups of Q are T — quasigroups.

Proof: The theorem follows from Lemma 33.

Theorem 40: All parastrophic quasigroups of every Abelian quasigroup are Abelian quasigroups.

Proof: By Lemma 33 and Theorem 12.

6° - Congruences of T - quasigroups

Theorem 41: Let Q be a T - quasigroup and $(Q(+), \varphi, \psi, g)$ its T - form. Be η a normal congruence of the quasigroup Q. Then η is a congruence of the group $Q(+)$.

Proof: Be $a,b \in Q$, $a\eta b$. Then $a \cdot \psi^{-1}(-g)\eta b \cdot \psi^{-1}(-g)$, so that $\varphi(a)\eta\varphi(b)$. Similarly, $\psi(a)\eta\psi(b)$. Further, $a = \varphi^{-1}(a) \cdot \psi^{-1}(-g)$, $b = \varphi^{-1}(b) \cdot \psi^{-1}(-g)$. Since $a\eta b$ and η is normal, we have $\varphi^{-1}(a)\eta\varphi^{-1}(b)$. If $c \in Q$ is arbitrary, then $\varphi^{-1}(a) \cdot \psi^{-1}(c - g)\,\eta\varphi^{-1}(b) \cdot \psi^{-1}(c - g)$, so that $a + c\,\eta\, b + c$. Thus η is a congruence on $Q(+)$.

Theorem 42: Be Q a T - quasigroup and $(Q(+), \varphi, \psi, g)$ its T - form. Be η a congruence of the group $Q(+)$. Then η is a congruence on the quasigroup Q if and only if $\varphi|Ker\ \eta$, $\psi|Ker\ \eta$ are endomorphisms of the group $Ker\ \eta$. Further, η is a normal congruence on Q if and only if $\varphi|Ker\ \eta$, $\psi|Ker\ \eta$ are automorphisms of the group $Ker\ \eta$.

Proof: 1. Let η be a congruence on Q. If $a\eta O$ then $a \cdot \psi^{-1}(-g)\,\eta\, O \cdot \psi^{-1}(-g)$, so that $\varphi(a)\,\eta\, O$. Similarly, $\psi(a)\eta O$.

2. Let $\varphi|Ker\ \eta$, $\psi|Ker\ \eta$ be endomorphisms of $Ker\ \eta$. Be $a,b \in Q$, $a\eta b$. Then $(a - b)\,\eta\, O$, hence $\varphi(a - b)\,\eta\, O$, so that $\varphi(a)\,\eta\,\varphi(b)$. Similarly, $\psi(a)\,\eta\,\psi(b)$. Be

further $c,d \in Q$, $c\eta d$. Then $\psi(c)\eta\psi(d)$, hence $(\varphi(a) + \psi(c) + g)\ \eta\ (\varphi(b) + \psi(d) + g)$. Thus $ac\ \eta\ bd$.

3. Let η be a normal congruence on Q and $a \in Ker\ \eta$. Then, according to 1), $\varphi(a), \psi(a) \in Ker\ \eta$. Since $O = O \cdot \psi^{-1}(-g)$ and η is normal, $b\ \eta\ O$, where $a = b \cdot \psi^{-1}(-g)$. It is $\varphi(b) = a$, so that $b = \varphi^{-1}(a) \in Ker\ \eta$. Similarly, $\psi^{-1}(a) \in Ker\ \eta$, hence $\varphi|Ker\ \eta$ and $\psi|Ker\ \eta$ are automorphisms of $Ker\ \eta$.

4. Let $\varphi|Ker\ \eta$, $\psi|Ker\ \eta$ be automorphisms of $Ker\ \eta$. In view of 2), η is a congruence on Q. Be $ab\ \eta\ cd$, $a\ \eta\ c$. Then $(\varphi(a) + \psi(b) + g)\ \eta\ (\varphi(c) + \psi(d) + g)$, hence $\psi(b)\ \eta\ \psi(d)$. Therefore $\psi(b - d)\ \eta\ O$, and hence $\psi^{-1}\psi(b - d)\ \eta\ O$. Thus $b\ \eta\ d$. The other part is quite similar.

Theorem 43: Be P a subquasigroup of a T – quasigroup Q. Then P is a normal subquasigroup in Q.

Proof: Let $(Q(+), \varphi, \psi, g)$ be a P – canonic T – form of Q. Then $P(+)$ is a subgroup in $Q(+)$. Be η congruence on $Q(+)$ such that $Ker\ \eta = P(+)$. Since $\varphi|P$, $\psi|P$ are automorphisms of $P(+)$, η is, by Theorem 42, a normal congruence on Q. But one class of η is just the subquasigroup P.

Example 5: From Theorem 42 follows that a congruence on a T – quasigroup need not be normal, that is, a homomorphic image of a T – quasigroup into a groupoid need not be a quasigroup. Be $Q(+)$ an Abelian group, $P(+)$ its subgroup and φ, ψ two automorphisms of $Q(+)$ such that $\varphi|P$, $\psi|P$ are endomorphisms but not automorphisms of the group $P(+)$. Be η the congruence on $Q(+)$ such that $Ker\ \eta = P(+)$. Put $Q(\cdot) = Q(+)^{(\varphi,\psi,1)}$. Then $Q(\cdot)$ is a T – quasigroup and η is a congruence on $Q(\cdot)$ but η is not normal on $Q(\cdot)$. We can take for instance the additive group of rational numbers like $Q(+)$, the additive group of integer numbers like $P(+)$ and set $\varphi(x) = 2x = \psi(x)$ for every $x \in Q(+)$. The quasigroup $Q(\cdot)$ will be in this case additionally commutative, hence Abelian.

Theorem 44: Be Q a T – quasigroup and $(Q(+), \varphi, \psi, g)$ its T – form. Let the permutations φ, ψ have a finite order in the group S_Q. Then every congruence on the quasigroup Q is normal.

Proof: Be η a congruence on Q and $\varphi^n = \psi^m = 1$, n, m convenient positive integers. Let $a\ \eta\ b$. It is $a \cdot \psi^{-1}(-g)\ \eta\ b \cdot \psi^{-1}(-g)$, hence $\varphi(a)\ \eta\ \varphi(b)$. Thus $\varphi^{n-1}(a)\ \eta\ \varphi^{n-1}(b)$. But $\varphi^{n-1} = \varphi^{-1}$, so that $\varphi^{-1}(a)\ \eta\ \varphi^{-1}(b)$. Be $c \in Q$ arbitrary. Then $\varphi^{-1}(a) \cdot \psi^{-1}(c - g)\ \eta\ \varphi^{-1}(b) \cdot \psi^{-1}(c - g)$, hence $a + c\ \eta\ b + c$. Thus η is a congruence on the group $Q(+)$. By Theorem 42, $\varphi|Ker\ \eta$, $\psi|Ker\ \eta$ are endomorphisms of $Ker\ \eta$. Since $\varphi^n = \psi^m = 1$, $\varphi|Ker\ \eta$ and $\psi|Ker\ \eta$ are automorphisms of $Ker\ \eta$. By Theorem 42, η is normal on Q.

Theorem 45: Be Q a T – quasigroup and K, H its subquasigroups. Let both K, H have at least one idempotent and let there be a congruence η on the quasigroup Q such that both K, H are classes of η. Then there exists an automorphism σ of the quasigroup Q such that $\sigma(K) = H$.

Proof: At first we shall prove that η is a normal congruence on Q. By Theorem 43, K is a normal subquasigroup in Q. Hence, there is a normal congruence ϱ such that K is one of its classes. Be $a, b \in Q$, $a\varrho b$ and $c \in K$ arbitrary. There is $x \in Q$ such that $cx = a$ and $y \in Q$ such that $yx = b$. Then $cx\ \varrho\ yx$, hence $c\ \varrho\ y$, so that $y \in K$. Thus $c\ \eta\ y$ and hence, $cx\ \eta\ yx$, which means $a\ \eta\ b$. Be, on the contrary, $a\ \eta\ b$. There is $x \in Q$ such that $xa \in K$. It is $xa\ \eta\ xb$, so that $xb \in K$. Therefore $xa\ \varrho\ xb$, hence $a\ \varrho\ b$. Thus we have proved that $\eta = \varrho$.

Be O an idempotent in H, e an idempotent in K. Be $(Q(+), \varphi, \psi, g)$ a H – canonic and $(Q(\circ), \varphi_1, \psi_1, g_1)$ a K – canonic T – form of Q such that O is the zero in $Q(+)$ and e is the zero in $Q(\circ)$. Then $O = g$, $e = g_1$. According to Lemma 27, the mapping σ, $\sigma(x) = x - e$ for every $x \in Q$, is an isomorphism of the group $Q(\circ)$ onto $Q(+)$ and $\varphi\sigma = \sigma\varphi_1$, $\psi\sigma = \sigma\psi_1$. But $\sigma(e) = e - e = O$. Hence, σ is an automorphism of the quasigroup Q. By Theorem 41, η is a congruence on the group $Q(+)$. Hence, for every $a,b \in Q$ is $a\ \eta\ b$ if and only if $\sigma(a)\ \eta\ \sigma(b)$. Therefore $\sigma(K) = H$.

Theorem 46: Be Q a T – quasigroup and $(Q(+), \varphi, \psi, g)$ its arbitrary T – form. Let the group $Q(+)$ have a set of generators X, $card\ X = \alpha$. Then there is a set Y of generators of Q and a set Z of generators of the multiplicative group G_Q such that $card\ Y \leq \alpha + 1$, $card\ Z \leq \alpha + 2$.

Proof: Be $X = \langle a_i \rangle$, $i \in \mathrm{I}$, a set of generators of the group $Q(+)$, $card\ \mathrm{I} = \alpha$. Be P a subquasigroup of Q generated by the set X and by the element O. In view of Lemma 11, $(Q(+), \varphi, \psi, g)$ is a P – canonic T – form, so that $P(+)$ is a subgroup in $Q(+)$. But $X \subseteq P(+)$, hence $Q(+) = P(+)$. Further, the set $\langle L^+_{a_i} \rangle$, $i \in \mathrm{I}$, is a set of generators of the group $G_{Q(+)}$. According to Theorem 37, the group G_Q is generated by the permutations $L^+_{a_i}$, $i \in \mathrm{I}$, and φ, ψ.

7° – Direct products

Remark 1: Be Q_i, $i \in \mathrm{I}$, a system of quasigroups, each of them having at least one idempotent element. Be e_i, $i \in \mathrm{I}$, a collection of idempotent elements, $e_i \in Q_i$. Denote $P^{e_i}_{i\in I}$ the set of all $\langle x_i \rangle \in \prod\limits_{i\in I} Q_i$ such that only for a finite number of indices $j \in \mathrm{I}$ is $x_j \neq e_j$. Then $P^{e_i}_{i\in I}$ is a subquasigroup in $\prod\limits_{i\in I} Q_i$. For $k \in \mathrm{I}$ define a mapping $\varphi_k : Q_k \to$ $\to P^{e_i}_{i\in I}$, $\varphi_k(x) = \langle x_j \rangle$, where $x_k = x$ and $x_j = e_j$ for every other $j \in \mathrm{I}$. Then φ_k is a monomorphism. If g_i, $i \in \mathrm{I}$, is any other collection of idempotents, $g_i \in Q_i$, then $P^{e_i}_{i\in I} \cong P^{g_i}_{i\in I}$. We shall denote the subquasigroup $P^{e_i}_{i\in I}$ by the symbol $\overset{*}{\prod\limits_{i\in I}} Q_i^{e_i}$.

Definition 10: We shall say that a quasigroup Q is an inner direct product of a system Q_i, $i \in \mathrm{I}$, of its subquasigroups, if:

1. $card\ \mathrm{I} \geq 2$.
2. There is $a \in Q$ such that for every $j \in \mathrm{I}$, $Q_j \cap \{\bigcup\limits_{i\neq j} Q_i\} = a$.
3. $\{\bigcup\limits_{i\in I} Q_i\} = Q$.

Remark 2: Let a T – quasigroup Q be an inner direct product of a system Q_i, $i \in \mathrm{I}$, of its subquasigroups. Be a the corresponding element. Evidently, $a \in Q_i$ for every $i \in \mathrm{I}$ and a is idempotent. For every $i \in \mathrm{I}$ there is a T – form $(Q_i(\circ), \varphi_i, \psi_i, a)$ of

the quasigroup Q_i such that a is the zero in $Q_i(\circ)$. Be further $(Q(+), \varphi, \psi, a)$ a T – form of Q such that a is the zero in $Q(+)$. By Lemma 11, $(Q(+), \varphi, \psi, a)$ is Q_i – canonic for every $i \in$ I. Thus $(Q_i(+), \varphi|Q_i, \psi|Q_i, a)$ is a T – form of Q_i. But the groups $Q_i(\circ)$ and $Q_i(+)$ have the same zero. Hence, by Lemma 9, $Q_i(\circ) = Q_i(+)$, $\varphi_i = \varphi|Q_i$, $\psi_i = \psi|Q_i$.
Now it is easy to show that $Q(+) = \sum_{i\in I} Q_i(+)$, $\varphi = \sum_{i\in I} \varphi_i$, $\psi = \sum_{i\ni I} \psi_i$.

From this we can deduce that $Q \cong \overset{\bullet}{\Pi}_{i\in I} Q_i^a$.

8° – The T – quasigroups of some classes

Definition 11: A quasigroup Q is called S – special (T – special), if for every $a,b \in Q$ the mapping $S_{a,b} = L_b^{-1}L_a^{-1}L_{ab}$ $(T_{a,b} = R_a^{-1}R_b^{-1}R_{ab})$ is its automorphism. A quasigroup Q is called special, if it is simultaneously S – and T – special.

Theorem 47: Let Q be a T – quasigroup. Then the following conditions are equivalent:
(i) Q is S – special.
(ii) For every $x \in Q$ the element $e(x)$ is idempotent and Q is Abelian.
(iii) There are an idempotent T – quasigroup K and a T – quasigroup P which is a right loop, such that $Q \cong K \times P$.

Proof: (i) implies (ii) and (iii). Be $x \in Q$. Then $S_{e(x),x} = L_{e(x)}^{-1}L_x^{-1}L_{x\cdot e(x)} = L_{e(x)}^{-1}$ is an automorphism of Q. Hence $L_{e(x)}$ is an automorphism of Q, and hence, by Theorem 28, $e(x)$ is idempotent and Q is Abelian. Denote K the set of all $e(x)$, $x \in Q$ arbitrary. Be $x, y \in Q$ and $u, v \in Q$ such that $ux = y$ and $xv = y$.
Since Q is Abelian, we have

$$xy \cdot (e(x) \cdot e(y)) = (x \cdot e(x)) \cdot (y \cdot e(y)) = xy, y \cdot e(y) =$$
$$= y = xv = (x \cdot e(x)) \cdot (v \cdot e(v)) = xv \cdot (e(x) \cdot e(v)) = y \cdot (e(x) \cdot e(v)).$$

Hence $e(x) \cdot e(y) = e(xy)$, $e(x) \cdot e(v) = e(y)$.
Similarly, $e(u) \cdot e(x) = e(y)$. Thus we have proved that K is a subquasigroup in Q, obviously idempotent. Now let us choose a fixed element $a \in K$. Be P the set of all $x \in Q$ such that $xa = x$; P is nonempty, since $a = e(u)$ for any $u \in Q$. Be $x,y \in P$ arbitrary. Then

$$xy \cdot a = xy \cdot aa = xa \cdot ya = xy.$$

If further $u \in Q$ such that $xu = y$ then

$$xu \cdot a = xa \cdot ua = x \cdot ua = ya = y.$$

Thus $ua = u$, so that $u \in P$. Similarly, if $vx = y$ then $v \in P$. Therefore P is a subquasigroup in Q and P is a right loop with right unit a. If $x \in K \cap P$ then $x = xx = xa$, hence $K \cap P = a$. Be x an arbitrary element of Q, $u \in Q$ such that $au = e(x)$ and $y \in Q$ such that $yu = x$. Then $u \in K$ and

$$yu \cdot e(x) = x \cdot e(x) = x = yu \cdot au = ya \cdot uu = ya \cdot u.$$

Therefore $ya = y$ and $y \in P$. As $x = yu$, it is $x \in \{K \cup P\}$. Thus $\{K \cup P\} = Q$. Now, according to Remark 2, we have $Q \cong K \times P$.
(iii) implies (ii). This part of the proof is evident (by Theorems 15, 25).
(ii) implies (i). The quasigroup Q has at least one idempotent element, hence Q has some T – form $(Q(+), \varphi, \psi, O)$.
Let $a,b \in Q$. We have $S_{a,b}(x) = \psi^{-1}(x) + g$, where $g = \psi^{-2}\varphi^2(a) - \psi^{-2}\varphi(a)$.
Hence $S_{a,b}(xy) = \psi^{-1}\varphi(x) + y + g$, $S_{a,b}(x) \cdot S_{a,b}(y) = \varphi\psi^{-1}(x) + y + \varphi(g) + \psi(g)$.
If we prove that $g = \varphi(g) + \psi(g)$, we prove that $S_{a,b}$ is an automorphism of Q. But for every $u \in Q$ the element $e(u) = \psi^{-1}(u - \varphi(u))$ is idempotent. Hence

$$\varphi\psi^{-1}(u) - \varphi^2\psi^{-1}(u) + u - \varphi(u) = \psi^{-1}(u) - \varphi\psi^{-1}(u).$$

Put $u = -\varphi\psi^{-1}(a)$. Hence we have

$$\varphi^3\psi^{-2}(a) - \varphi^2\psi^{-2}(a) + \varphi^2\psi^{-1}(a) - \varphi\psi^{-1}(a) = \varphi^2\psi^{-2}(a) - \varphi\psi^{-2}(a).$$

Therefore $g = \varphi(g) + \psi(g)$.

Theorem 48: Let Q be a T – quasigroup. Then the following conditions are equivalent:
(i) Q is T – special.
(ii) For every $x \in Q$ the element $f(x)$ is idempotent and Q is Abelian.
(iii) There are an idempotent T – quasigroup K and a T – quasigroup P, which is a left loop, such that $Q \cong K \times P$.

Proof: The proof is similar to that of Theorem 47.

Theorem 49: Let Q be a T – quasigroup. Then the following conditions are equivalent:
(i) Q is special.
(ii) Q is Abelian and for every $x \in Q$ the elements $e(x), f(x)$ are idempotent.
(iii) There are an idempotent T – quasigroup K and an Abelian group P such that $Q \cong K \times P$.

Proof: (i) implies (ii) and (iii). According to Theorems 47, 48, Q is Abelian and $e(x), f(x)$ are idempotent for every $x \in Q$. We have

$$\begin{aligned} x = x \cdot (e(x) \cdot e(x)) = (f(x) \cdot x)\,(e(x) \cdot e(x)) = \\ = (f(x) \cdot e(x))\,(x \cdot e(x)) = (f(x) \cdot e(x)) \,.\, x = f(x) \cdot x. \end{aligned}$$

Hence $f(x) \cdot e(x) = f(x)$, so that $e(x) = f(x)$.
Now we can proceed similarly as in the proof of Theorem 47, but P will be a loop, hence, by Theorem 14, an Abelian group.
The other part of the proof is evident.

Theorem 50: Let Q be a T – quasigroup. Then the following conditions are equivalent:
(i) Q is Abelian and for every $x \in Q$ the element xx is idempotent.
(ii) There are an idempotent T – quasigroup K and a unipotent T – quasigroup P such that $Q \cong K \times P$.

Proof: (i) implies (ii). Be $Q(*)$ the right inverse quasigroup of the quasigroup Q. Then, by Theorem 40, $Q(*)$ is an Abelian quasigroup. Since $xx \cdot xx = xx$, we have $xx * xx = xx, x * xx = x$. In view of Theorem 47, there are an idempotent T – quasigroup $K(*)$ and a T – quasigroup $P(*)$ being a right loop such that $Q(*) \cong K(*) \times \times P(*)$. Be K a right inverse quasigroup of $K(*)$, P right inverse of $P(*)$. Evidently, K is idempotent, P is unipotent and $Q \cong K \times P$.
(ii) implies (i). By Theorems 20, 25, Q is Abelian. The rest is evident.

Theorem 51: Let Q be a T – quasigroup. Then the following conditions are equivalent:
(i) For every $a, b, c \in Q$, $S_{a,b} = S_{a,c}$.
(ii) For every $a, b, c \in Q$, $T_{b,a} = T_{c,a}$
(iii) Q is Abelian.
Proof: Let $(Q(+), \varphi, \psi, g)$ be a T – form of Q and $x, y \in Q$. We have $S_{x,y}(z) = = \psi^{-1}(z) + \alpha(x, y)$, where $\alpha(x,y) = \psi^{-2}\varphi^2(x) - \psi^{-2}\varphi(x) + \psi^2\varphi\psi(y) - \psi^{-1}\varphi(y) + + \psi^{-2}\varphi(g) - \psi^{-1}(g)$.
Let (i) be valid. Then $\alpha(a, b) = \alpha(a, c)$ for every $a, b, c \in Q$. Hence $\psi^{-2}\varphi\psi(d) = = \psi^{-1}\varphi(d)$ for every $d \in Q$. Therefore $\varphi\psi(d) = \psi\varphi(d)$, hence Q is Abelian. Conversely, let Q be Abelian. Then $\varphi\psi = \psi\varphi$, henceforth we have

$$\alpha(x,y) = \psi^{-2}\varphi^2(x) - \psi^{-2}\varphi(x) + \psi^{-2}\varphi(g) - \psi^{-1}(g).$$

Thus $\alpha(a, b) = \alpha(a, c)$ for every $a, b, c \in Q$. Similarly we can prove that (ii) implies (iii) and (iii) implies (ii).

Theorem 52: Let Q be a T – quasigroup. Then Q is a left (right) IP – quasigroup if and only if for any (and then for each) of its T – forms $(Q(+), \varphi, \psi, g)$ is $\psi^2 = 1$ ($\varphi^2 = 1$).
Proof: Let Q be a left IP — quasigroup, $(Q(+), \varphi, \psi, g)$ its arbitrary T – form. There is a permutation α such that $\alpha(x)\,(xy) = y$ for every $x, y \in Q$. Hence,

$$\varphi\alpha(x) + \psi\varphi(x) + \psi^2(y) + \psi(g) + g = y.$$

From this we obtain $\psi^2 = 1$.
Let $(Q(+), \varphi, \psi\, g)$ be such a T – form that $\psi^2 = 1$. Put $\alpha(x) = -\varphi^{-1}\psi\varphi(x) + + \varphi^{-1}(-g - \psi(g))$.
Then

$$\alpha(x)\,(xy) = -\psi\varphi(x) - g - \psi(g) + \psi\varphi(x) + \psi^2(y) + \psi(g) + g = y.$$

Thus Q is a left IP – quasigroup. Similarly for the other case.

Theorem 53: Let Q be a T – quasigroup. Then the following conditions are equivalent:
(i) Q is a IP – quasigroup.
(ii) Q is a left IP – quasigroup and a β_2 – quasigroup.
(iii) Q is a right IP – quasigroup and a β_2 – quasigroup.
Proof: The theorem follows from Theorem 52.

9° – The free T – quasigroups

Construction of quasigroups F[x, x₀]

Let G be an arbitrary, but further on a fixed, free group freely generated by the elements η, ϱ. Be X an arbitrary non-empty set. Denote $B(X)$ the set of all ordered pairs (α, x), where $\alpha \in G$, $x \in X$. Be $F(+)$ a free Abelian group freely generated by the set $B(X)$. Define two permutations φ, ψ of the set $B(X)$ as follows: If $\alpha \in G$, $x \in X$ then $\varphi(\alpha, x) = (\eta\alpha, x)$ and $\psi(\alpha, x) = (\varrho\alpha, x)$. The permutations φ, ψ can be uniquely extended to automorphisms of the group $F(+)$; these automorphisms denote also φ, ψ. Further, let $j \in G$ be the unit of the group G. Finally, select an element $x_o \in X$. By the symbol $F[X, x_o]$ we shall denote the T – quasigroup having the T – form $T = (F(+), \varphi, \psi, (j, x_o))$. The T – form T we shall call a principal T – form of the T – quasigroup $F[X, x_o]$. It is evident that if $x_1, x_2 \in X$ are arbitrary then $F[X, x_1] \cong$ $\cong F[X, x_2]$.

Lemma 34: Let X be an arbitrary non-empty set and $x_o \in X$ an arbitrary element. Be $T = (F(+), \varphi, \psi, (j, x_o))$ the principal T – form of the T – quasigroup $F[X, x_o]$. Then there exists an isomorphism $\tau : G \to A(F[X, x_o], T)$ such that $\tau(\alpha)(\beta, x) = (\alpha\beta, x)$ for every $\alpha, \beta \in G$, $x \in X$. Hence $\tau(\eta) = \varphi$, $\tau(\varrho) = \psi$.

Proof: If $\alpha \in G$ then there is just one automorphism $\tau(\alpha)$ of the group $F(+)$ such that $\tau(\alpha)(\beta, x) = (\alpha\beta, x,)$, all $\beta \in G$, all $x \in X$. It is evident that τ is a monomorphism of G into *Aut* $F(+)$. Since $\tau(\eta) = \varphi$ and $\tau(\varrho) = \psi$, it must be $\tau(G) =$ $= A(F[X, x_o], T)$.

Lemma 35: Let X be an arbitrary non-empty set and $x_o \in X$ an arbitrary element. Be $T = (F(+), \varphi, \psi, (j, x_o))$ the principal T – form of $F[X, x_o]$. Be $C(X)$ the set consisting of all pairs (j, x), where $x \in X$, $x \neq x_o$, and of the element O (zero in $F(+)$). Then the set $C(X)$ is a set of generators of the quasigroup $F[X, x_o]$.

Proof: Denote P the subquasigroup in $F[X, x_o]$ that is generated by the set $C(X)$. Since $O \in P$, the T – form T is P – canonic. Hence $P(+)$ is a subgroup in $F(+)$ and $\xi(p) \in P$ for every ξ, p, $\xi \in A(F[X, x_o], T)$, $p \in P$. Further the element $O \cdot O = (j, x_o)$ is also in P. Be $\alpha \in G$ and $x \in X$ arbitrary elements. By Lemma 34, there is an isomorphism $\tau : G \to A(F[X, x_o,] T)$ such that $\tau(\alpha)(j, x) = (\alpha j, x) =$ $= (\alpha, x)$. Since $(j, x) \in P$, $(\alpha, x) \in P$. Hence $B(X) \subseteq P$. Therefore $P(+) = F(+)$, $P = F[X, x_o]$.

Theorem 54: Let X be an arbitrary non-empty set and $x_o \in X$ an arbitrary element. The quasigroup $F[X, x_o]$ is a free T – quasigroup freely generated by the set $C(X)$. Hence *rank* $F[X, x_o] =$ *card* X.

Proof: Be E a free T – quasigroup freely generated by the set $C(X)$. Be $S = (E(+), \alpha, \beta, g)$ a T – form of E such that the element $O \in C(X)$ is zero in $E(+)$. Since E is freely generated by $C(X)$, the identical mapping of the set $C(X)$ onto itself can be uniquely extended to a homomorphism σ, $\sigma : E \to F[X, x_o]$. In what follows we shall prove that σ is an isomorphism.

By Lemma 35, the set $C(X)$ is a set of generators of $F[X, x_o]$. Hence σ is an epimorphism. Be $T = (F(+), \varphi, \psi, (j, x_o))$ the principal T - form of $F[X, x_o]$. The element O is zero in both groups $F(+)$, $E(+)$ and $\sigma(O) = O$. Therefore, by Lemma 30, σ is an epimorphism of $E(+)$ onto $F(+)$, $\sigma\alpha = \varphi\sigma$, $\sigma\beta = \psi\sigma$ and $\sigma(g) = (j, x_o)$. Further, according to Theorem 33, there is an epimorphism $\mathfrak{a}$,
$\mathfrak{a} : A(E,S) \to A(F[X, x_o], T)$ such that $\mathfrak{a}(\gamma)(\sigma(h)) = \sigma\gamma(h)$, where $\gamma \in A(E, S)$ and $h \in E$ are arbitrary. Hence $\mathfrak{a}(\alpha) = \varphi$, $\mathfrak{a}(\beta) = \psi$. By Lemma 35, the group $A(F[X, x_o],T)$ is free and is freely generated by φ, ψ. Hence there is an epimorphism
$\mathfrak{b} : A(F[X, x_o], T) \to A(E, S)$ such that $\mathfrak{b}(\varphi) = \alpha$ and $\mathfrak{b}(\psi) = \beta$. But from this we can deduce that $\mathfrak{b}\mathfrak{a} = 1_{A(E,S)}$. Thus $\mathfrak{a}$ is one - to - one, and hence, $\mathfrak{a}$ is an isomorphism. Be $D \subseteq E$ the set consisting of all elements $\gamma(j, x)$ and $\gamma(g)$, all $\gamma \in A(E, S)$, all $x \in X$, $x \neq x_o$.
We have,

$$\sigma\gamma(j, x) = \mathfrak{a}(\gamma)\sigma(j, x) = \mathfrak{a}(\gamma)(j, x),$$
$$\sigma\gamma(g) = \mathfrak{a}(\gamma)\sigma(g) = \mathfrak{a}(\gamma)(j, x_o).$$

By Lemma 34, there is an isomorphism $\mathfrak{r}$, $\mathfrak{r} : G \to A(F[X, x_o], T)$ such that $\mathfrak{r}(\xi)(j, y) = (\xi j, y)$ for every $\xi \in G$ and $y \in X$.
Hence

$$\mathfrak{a}(\gamma)(j, x) = \mathfrak{r}\mathfrak{r}^{-1}\mathfrak{a}(\gamma)(j, x) = (\mathfrak{r}^{-1}\mathfrak{a}(\gamma) \cdot j, x) = (\mathfrak{r}^{-1}\mathfrak{a}(\gamma), x), \mathfrak{a}(\gamma)(j, x_o) = (\mathfrak{r}^{-1}\mathfrak{a}(\gamma), x_o).$$

Thus we have proved that $\sigma(D) \subseteq B(X)$. Be $b \in B(X)$ arbitrary. Hence there is $\xi \in G$ and $x \in X$ such that $b = (\xi, x)$. If $x \neq x_o$ then $\mathfrak{a}^{-1}\mathfrak{r}(\xi)(j, x) \in D$ and we have

$$\sigma\, \mathfrak{a}^{-1}\mathfrak{r}(\xi)(j, x) = \mathfrak{r}(\xi)(j, x) = (\xi, x) = b.$$

If $x = x_o$ then $\mathfrak{a}^{-1}\mathfrak{r}(\xi)(g) \in D$ and $\sigma\mathfrak{a}^{-1}\mathfrak{r}(\xi)(g) = (\xi, x_o) = b$. Therefore $\sigma(D) = B(X)$. Now we shall prove that the restriction $\sigma|D$ is one - to - one. Let $c, d \in D$ be such that $\sigma(c) = \sigma(d)$. Such cases can arise:
(i) $c = \gamma(j, x)$, $d = \delta(j, y)$, where $\gamma, \delta \in A(E, S)$, $x, y \in X$ and $x \neq x_o \neq y$.
(ii) $c = \gamma(j, x)$, $d = \delta(g)$, where $\gamma, \sigma \in A(E, S)$, $x \in X$, $x \neq x_o$.
(iii) $c = \gamma(g)$, $d = \sigma(g)$, where $\gamma, \delta \in A(E, S)$.
For (i): we have

$$\sigma\gamma(j, x) = \mathfrak{a}(\gamma)(j, x) = (\mathfrak{r}^{-1}\mathfrak{a}(\gamma), x) = \sigma\delta(j, y) = (\mathfrak{r}^{-1}\mathfrak{a}(\delta), y).$$

Hence $x = y$ and $\mathfrak{r}^{-1}\mathfrak{a}(\gamma) = \mathfrak{r}^{-1}\mathfrak{a}(\delta)$. But $\mathfrak{r}^{-1}\mathfrak{a}$ is one - to - one, therefore $\gamma = \delta$. Thus $c = d$.
For (ii): we have

$$\sigma\gamma(j, x) = (\mathfrak{r}^{-1}\mathfrak{a}(\gamma), x) = \sigma\delta(g) = (\mathfrak{r}^{-1}\mathfrak{a}(\delta), x_o).$$

Hence $x = x_o$, a contradiction.
For (iii) similarly as for (i).
Thus we have proved that $\sigma : D \to B(X)$ is a biunique mapping. Hence there is a mapping $\tau : B(X) \to D$ such that $\tau\sigma = 1_D$. Since $F(+)$ is a free Abelian group freely generated by the set $B(X)$, there is a homomorphism $\mu : F(+) \to E(+)$ such

that $\mu|B(X) = \tau$. Denote $H(+)$ the subgroup in $E(+)$ that is generated by the set D. Since for every $h \in H(+)$, $\gamma \in A(E, S)$ is $\gamma(h) \in H(+)$ and $g \in H(+)$, H is a subquasigroup in E. But $C(X) \subseteq H$. Hence $H = E$, $H(+) = E(+)$. Thus μ is an epimorphism and $\mu\sigma|D = 1_D$. Hence $\mu\sigma = 1_E$. Therefore σ is one – to – one. This completes the proof.

Theorem 55: Be F an arbitrary free T – quasigroup, $F(+)$ its T – group and $A(F)$ its characteristic group. Then $F(+)$ is a free Abelian group and $A(F)$ is a free group.

Proof: The theorem follows from Theorem 54.

Reference

[1] Kepka T., Němec P., Acta Univ. Carol., Math. et Phys., Vol. 12, No. 1, pp. 39—49.