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On Embedding of Lattices in Simple Lattices

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B. Jonsson [2] has proved that every lattice is a sublattice of a subdirectly irreducible lattice. Here we strengthen this result.

In order to simplify the notation we shall omit the one-element classes in the expression $\mathfrak{z} = (N_\lambda)_{\lambda \in \Lambda}$ for a partition on a set $M \neq \emptyset$. *E. g.*, $\mathfrak{z} = (m, n), (p, q)$ means that the partition $\mathfrak{z}$ of the partition lattice $\mathfrak{z}(M)$ consists of two two – element classes (m, n), (p, q) and that the other classes of $\mathfrak{z}$ have only one element. If $\mathfrak{z}_1 \leqq \mathfrak{z}_2$, $\mathfrak{z}_3 \leqq \mathfrak{z}_4$, $\mathfrak{z}_1 \wedge z_4 = \mathfrak{z}_3$ and $\mathfrak{z}_1 \vee \mathfrak{z}_4 = \mathfrak{z}_2$, then we shall write $\mathfrak{z}_2/\mathfrak{z}_1 \searrow \mathfrak{z}_4/\mathfrak{z}_3$ resp. $\mathfrak{z}_4/\mathfrak{z}_1 \nearrow \mathfrak{z}_2/\mathfrak{z}_1$.

Theorem 1. *The partition lattice $\mathfrak{z}(M)$ is simple.*

Proof. It may be assumed without loss of generality that card $M \geqq 3$. We shall decompose the proof in the following steps (The statement (ii) is contained in [1]; for the sake of completeness we shall give here another proof.):

(i) If $\mathfrak{z}_0$, $\mathfrak{z}_1$ are two atoms of (M), then the intervals $[0, \mathfrak{z}_0]$, $[0, \mathfrak{z}_1]$ are projective. In fact, the transitivity of the projectivity makes it sufficient to show that for any two atoms $\mathfrak{z}'$, $\mathfrak{z}''$ of the form $\mathfrak{z}' = (m, n)$, $\mathfrak{z}'' = (n, p)$ the intervals $[0, \mathfrak{z}']$ $[0, \mathfrak{z}'']$ are projective. But the atoms $\mathfrak{z}'$, $\mathfrak{z}''$, $\mathfrak{z}''' = (m, p)$ generate in $\mathfrak{z}(M)$ a sublattice isomorphic with the five-element modular lattice M_5 having all prime intervals projective.

(ii) Any two prime intervals of $\mathfrak{z}(M)$ are projective. Indeed, if $\mathfrak{z}_1 -\!\!< \mathfrak{z}_2$, *i.e.*, if $\mathfrak{z}_2$ covers $\mathfrak{z}_1$, and $0 < \mathfrak{z}_1$, then there exists ([3], Satz 53) a relative complement $\mathfrak{z}_1'$ of $\mathfrak{z}_1$ in $[0, \mathfrak{z}_2]$. Since $\mathfrak{z}(M)$ is relatively atomic in the sense of Szász, there is a $\mathfrak{z}_{10}'$ such that $0 -\!\!< \mathfrak{z}_{10}' \leqq \mathfrak{z}_1'$ and $\mathfrak{z}_1 \vee \mathfrak{z}_{10}' = \mathfrak{z}_2$, $\mathfrak{z}_1 \wedge \mathfrak{z}_{10}' = 0$. Consequently, $\mathfrak{z}_2/\mathfrak{z}_1 \searrow \mathfrak{z}_{10}'/0$ which completes by /i/ the proof of /ii/.

/iii/ Let us now consider a maximal system $\mathfrak{y}$ consisting of disjoint two-element classes of the form (m, m_2), $m_1, m_2 \in M$ which may be interpreted as ordered pairs. Let $M_1 = \{m \mid \exists\, n \in M\, (m, n) \in \mathfrak{y}\}$, $M_2 = \{m \mid \exists\, n \in M\, (n, m) \in \mathfrak{y}\}$. Since card $(M \setminus (M_1 \cup M_2)) \leqq 1$, only two cases are possible:

/iv/ *Case I*: $M = M_1 \cup M_2$. Let $\mathfrak{a} = (M_1), (M_2)$, $\mathfrak{w}_1 = (M_1)$, $\mathfrak{w}_2 = (M_2)$, $\mathfrak{q} = 0$, $\mathfrak{w} = \mathfrak{y}$. Then we have (cf. Fig. 1)

$$I/\mathfrak{w} \searrow \mathfrak{w}_2/0 \nearrow \mathfrak{a}/\mathfrak{w}_1 \qquad (1)$$

and

$$I/\mathfrak{a} \searrow \mathfrak{w}/\mathfrak{q} \nearrow I/\mathfrak{w}_1 \qquad (2)$$

/v/ *Case II*: $M = M_1 \cup M_2 \cup \{q\}$. Let $(p_1, p_2) \in \mathfrak{y}$ and let $\mathfrak{w}$ denote the partition on M obtained from $\mathfrak{y}$ by replacing the class (p_1, p_2) by the class (p_1, q, p_2). Further, let $\mathfrak{w}_1 = (M_1 \cup \{q\})$, $\mathfrak{w}_2 = (M_2)$, $\mathfrak{a} = (M_1 \cup \{q\}), (M_2)$, $\mathfrak{q} = (p_1, q)$. It is easy to check the validity of (1) and (2) also for this case (cf. Fig. 2).

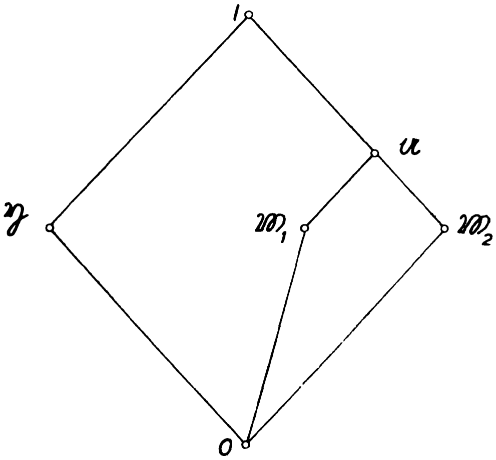

Fig. 1.

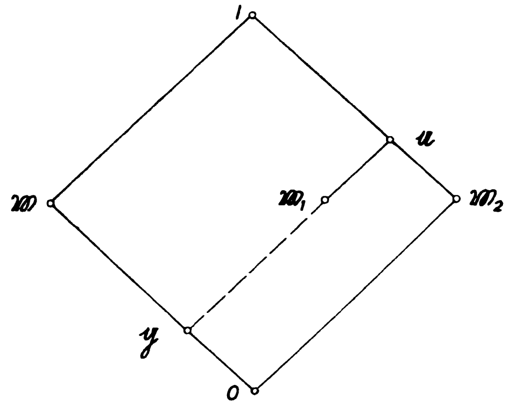

Fig. 2.

/vi/ If $\equiv$ denotes a congruence such that $\mathfrak{z}^\star \equiv \mathfrak{z}^+$, $\mathfrak{z}^\star \neq \mathfrak{z}^+$, then there is a $\mathfrak{z}^\# \in \mathfrak{z}(M)$ such that $\mathfrak{z}^\star \wedge \mathfrak{z}^+ \prec \mathfrak{z}^\# \leqq \mathfrak{z}^\star \vee \mathfrak{z}^+$ and $\mathfrak{z}^\# \equiv \mathfrak{z}^\star \wedge \mathfrak{z}^+$. Hence, by /ii/, $I \equiv \mathfrak{a}$, since $I \succ \mathfrak{a}$. Therefore $I \equiv \mathfrak{w}_1$ and $\mathfrak{w} \equiv \mathfrak{q}$, by (2). Thus $\mathfrak{a} \equiv \mathfrak{w}_1$ and by (1) we have $I \equiv \mathfrak{w}$. Finally, since $\mathfrak{q} \succeq 0$, necessarily also $\mathfrak{q} \equiv 0$. Summarizing $I \equiv \mathfrak{w} \equiv \mathfrak{q} \equiv 0$ we conclude that $\mathfrak{z}(M)$ is simple.

Remark. By the Whitman's theorem it is now clear that every lattice can be embedded in a simple lattice. This may be proved directly in the following way:

Theorem 2. *Every lattice is a sublattice of a simple lattice.*

Proof. Let L be an arbitrary lattice with extreme elements 0 and 1. For every $a \in L$ different from both 0 and 1 we construct a lattice $\alpha_a(L)$ as follows: $\alpha_a(L) = L \cup \{u, v\}$ where u, v are two different elements not belonging to L; $x < y$ in $\alpha_a(L)$ if and only if one of the following five cases takes place:

/i/ $x, y \in L$ & $x < y$ in L;
/ii/ $x = u$ & $y = 1$;
/iii/ $x = v$ & $y = 1$;
/iv/ $x \in L$ & $y = u$ & $x \leqq a$ in L;
/v/ $x = 0$ & $y = v$.

Evidently, L is a sublattice of $\alpha_a(L)$; 0 and 1 are the extreme elements of $\alpha_a(L)$;

whenever Θ is a congruence relation of $\alpha_a(L)$ and $a \Theta x$ for some $x \in L$ where $a < x$, then $a \Theta 1$. (Proof: We have $a \vee u \Theta x \vee u$, *i.e.* $u \Theta 1$; consequently, $u \wedge v \Theta 1 \wedge v$, *i.e.* $0 \Theta v$, which gives $a \vee 0 \Theta a \vee v$, *i.e.* $a \Theta 1$.)

We can construct quite similarly for every $a \in L$ different from both 0 and 1 a lattice $\beta_a(L)$ such that L is its sublattice, 0 and 1 are its extreme elements and whenever Θ is a congruence relation of $\beta_a(L)$ and $a \Theta x$ for some $x \in L$ satisfying $x < a$, then $a \Theta 0$.

Put, moreover, $\alpha_0(L) = \alpha_1(L) = \beta_0(L) = \beta_1(L) = L$. Arrange elements of L into a (possibly transfinite) sequence $a_0, a_1, a_2, \ldots, a_\omega, \ldots$; define lattices $L_0, L_1, L_2, \ldots, L_\omega, \ldots$ as follows: $L_0 = L$; $L_\gamma = \beta_{a_\gamma}(\alpha_{a_\gamma}(L_{\gamma-1}))$ if $\gamma - 1$ exists; $L_\gamma = \bigcup_{\delta<\gamma} L_\delta$ if γ is limit. Put $L^\star = \bigcup_\gamma L_\gamma$, so that $L^\star$ is a lattice, L is its sublattice, 0 and 1 are its extreme elements and whenever $x \Theta y$ where x and y are two different elements of L and Θ is a congruence relation of $L^\star$, then $0 \Theta 1$, *i.e.* Θ is the greatest congruence relation of $L^\star$.

The union of the increasing chain (of type ω) of lattices $L, L^\star, (L^\star)^\star, \ldots$ is a simple lattice containing L as a sublattice. As every lattice can be embedded into a lattice with extreme elements, the theorem is thus proved.

Added id proof. Theorem 1 has been proved by O. Ore in Duke math. J. 9 1942, p. 626, where, however, the resulting line of reasoning did not possess the strictly elementary and relatively self-contained character of the proof given in this paper.

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