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An Approach to Solvability in Orthomodular Lattices

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The solvability of generalized orthomodular lattices was defined by E. L. Marsden. In this paper we propose a slightly simplified construction.

If $\mathscr{L} = \langle L, \cup, \cap, \perp, 0, 1\rangle$ is an orthomodular lattice we let $\mathscr{L}^{(i+1)}$ denote the ideal of the lattice $\mathscr{L}$ generated by all elements of the form $[x, y] = (x \cup y) \cap \cap (x^{\perp} \cup y) \cap (x \cup y^{\perp}) \cap (x^{\perp} \cup y^{\perp})$ where $x, y \in L^{(i)}$ and $\mathscr{L}^{(0)} = \mathscr{L}$. We shall say that $\mathscr{L}$ is *solvable* if there exists an $n \geqq 0$ such that $L^{(n)} = \{0\}$.

The aim of this note is the proof of the following theorem which is an equivalent of Marsden's result [3], Th. 9:

Theorem. *An orthomodular lattice is solvable iff it is distributive.*

Let $\mathscr{L}$ denote a uniquely complemented lattice where the mapping $\varphi : a \mapsto a'$ is antitone. Since $(a')' = a$, it is easy to see that $(a' \cap b')' \geqq a \cup b$ and that $(a \cup b)' \leqq a' \cap b'$. From this we get $(a \cup b)' = a' \cap b'$ and $(a \cap b)' = a' \cup b'$. We summarize these facts in a useful variant of a known result ([1], Th. 17, p. 44):

Proposition. *A uniquely complemented lattice is Boolean iff the mapping $\varphi : a \mapsto a'$ is antitone.*

As an immediate consequence of this result we state (cf. [2]).

Corollary. *A uniquely complemented lattice satisfying the condition $x \cap y = 0 \Rightarrow x' \leqq y$ is Boolean.*

Proof. If $a \leqq b$, then $b' \cap a = 0$ and so we have $b' \leqq a'$.

We shall say for brevity that a lattice $\mathscr{L}$ satisfies the condition (P_i), $i \geqq 0$, if $\mathscr{L}$ is orthomodular and if every interval $[0; v]$, $v \in L^{(i)}$, is a uniquely complemented lattice.

Lemma. *If $\mathscr{L}$ satisfies (P_i), $i \geqq 1$, then $\mathscr{L}$ satisfies also (P_{i-1}).*

Proof (We shall use the identity (ii) of [1], p. 54, without making explicit references). Let $[0; u]$ be an interval of $\mathscr{L}^{(i-1)}$. Then $[0; u]$ is an orthomodular lattice where $a^+ = a^{\perp} \cap u$ is the orthocomplement of $a \in [0; u]$. Let $a^\star \in [0; u]$ be such that $a \cup a^\star = 1$ and $a \cap a^\star = 0$. We shall show that $a^\star = a^+$: Denote by $\langle p, q\rangle$ the element $(p \cup q) \cap (p^+ \cup q) \cap (p \cup q^+) \cap (p^+ \cup q^+)$. Since $\langle p, q\rangle \leqq [p, q]$, we have $\langle p, q\rangle \in L^{(i)}$ whenever $p, q \in [0; u]$. Now

$$
\begin{aligned}
f = \langle a^+ \cap (a \cup a^{\star +}),\ a^\star \cap (a \cup a^{\star +})\rangle = \\
= \{(a^+ \cap (a \cup a^{\star +})) \cup (a^\star \cap (a \cup a^{\star +}))\} \cap \\
\cap \{(a^+ \cap (a \cup a^{\star +})) \cup (a^{\star +} \cup (a^+ \cap a^\star))\} \cap \\
\cap \{(a \cup (a^+ \cap a^\star)) \cup (a^\star \cap (a \cup a^{\star +}))\} \cap \\
\cap \{(a \cup (a^+ \cap a^\star)) \cup (a^{\star +} \cup (a^+ \cap a^\star))\}
\end{aligned}
$$

and the last three members $\{\dots\}$ are equal to u. Thus $f = m_1 \cup m_2$ where $m_1 = a^+ \cap (a \cup a^{\star +})$, $m_2 = a^\star \cap (a \cup a^{\star +})$. Similarly we get $h = \langle n_1, n_2\rangle = n_1 \cup n_2$ where $n_1 = a \cap (a^+ \cup a^\star)$, $n_2 = a^{\star +} \cap (a^+ \cup a^\star)$. But $h, f \in L^{(t)}$ implies that also $m_1, m_2, n_1, n_2 \in L^{(t)}$. On the other hand

$$
\begin{aligned}
m_1 \cup n_1 = (a^+ \cap (a \cup a^{\star +})) \cup (a \cap (a^+ \cup a^\star)) = \\
= [(a^+ \cap (a \cup a^{\star +})) \cup a] \cap (a^+ \cup a^\star) = \\
= (a \cup a^+) \cap (a \cup a^{\star +}) \cap (a^+ \cup a^\star) = \\
= (a \cup a^{\star +}) \cap (a^+ \cup a^\star) = \xi \geqq m_2.
\end{aligned}
$$

Hence $\xi \geqq \eta^\perp$ where $\eta^\perp = m_2 \cup n_1$. Further,

$$
\begin{aligned}
(a \cup a^{\star +}) \cap (a^{\star +} \cup (a^+ \cap a^\star)) = a^{\star +} \\
(a^+ \cup a^\star) \cap (a^+ \cup (a \cap a^{\star +})) = a^+
\end{aligned}
$$

and this yields

$$
\begin{aligned}
\xi \cap \eta = (a \cup a^{\star +}) \cap (a^+ \cup a^\star) \cap (a^{\star +} \cup (a^+ \cap a^\star)) \cap (a^+ \cup (a \cap a^{\star +})) = \\
= a^{\star +} \cap a^+ = (a \cup a^\star)^+ = u^+ = 0.
\end{aligned}
$$

By orthomodularity, $\xi = \eta^\perp$ and thus $m_2 \cup n_1 = \xi$. We have also $m_1 \cap n_1 \leqq a^+ \cap a = 0$, $m_2 \cap n_1 \leqq a^\star \cap a = 0$. Since $[0; \xi]$ is uniquely complemented it follows that $m_1 = m_2 = m_1 \cap m_2 = (a^+ \cap a^\star) \cap (a^+ \cap a^\star)^+ = 0$. So we have $m_1 = a^+ \cap (a \cup a^{\star +}) = 0$, $a^+ \geqq (a \cup a^{\star +})^+$ and the orthomodularity implies that $a^+ = (a \cup a^{\star +})^+ = a^+ \cap a^\star \leqq a^\star$. From $m_2 = 0$ we obtain similarly $a^\star \leqq a^+$ and the lemma is proved.

Proof of Theorem. If $\mathscr{L}$ is orthomodular and $L^{(n)} = \{0\}$, then the condition (P_n) holds and, by Lemma, the condition (P_0) holds also. Since $1 \in L$, $\mathscr{L}$ is a uniquely complemented lattice where $k^\perp$ is the complement of $k \in L$ and since $\mathscr{L}$ is an ortholattice, the mapping $\varphi : k \mapsto k^\perp$ is antitone.

References

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