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Some Examples of Primitive Lattices

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Igošin [1] investigates characterizable primitive classes of lattices, i.e. primitive classes K such that any lattice belongs to K iff it does not contain a sublattice belonging to a given set of finite lattices. In the present paper we shall be concerned with primitive classes characterizable by means of a single lattice. We give some examples of them and prove that there are infinitely many such classes.

Given a lattice L, denote by $K(L)$ the class of all lattices that contain no sublattice isomorphic to L. We call L primitive if the class $K(L)$ is primitive.

THEOREM 1. Let L be an arbitrary lattice. The following holds:

(1) The class $K(L)$ is closed with respect to sublattices and isomorphic lattices.

(2) If L is subdirectly irreducible, then $K(L)$ is closed with respect to direct products.

(3) If L is finite and $K(L)$ is closed with respect to direct products, then L is subdirectly irreducible.

Proof. (1) is trivial. Let us prove (2). Suppose that there exists an isomorphism j of L onto a sublattice of the direct product $\underset{t\in T}{\mathrm{X}} A_t$ of a family of lattices $A_t \in K(L)$. For every $t \in T$ define a congruence Θ_t on L by $a\Theta_t b$ iff the t-th components of $j(a)$ and $j(b)$ are equal. As $\bigcap_{t\in T} \Theta_t$ is evidently the smallest congruence ι_L of L and L is subdirectly irreducible, there exists a $t \in T$ such that $\Theta_t = \iota_L$; this implies that there exists an isomorphism of L into A_t, a contradiction with $A_t \in K(L)$.

It remains to prove (3). Suppose that L is subdirectly reducible, so that L is isomorphic to a sublattice of the direct product of some lattices of cardinality smaller than Card(L). As all these lattices evidently belong to $K(L)$ and $K(L)$ is closed with respect to direct products and sublattices, we get $L \in K(L)$, a contradiction.

THEOREM 2. A lattice L is primitive iff it is non-trivial (i.e. of cardinality ≥ 2), finite, subdirectly irreducible and satisfies the following condition:

(H) Whenever there exists a homomorphism of some lattice A onto L, then A contains a sublattice isomorphic to L.

Proof. Let L be primitive. As $K(L)$ is non-empty, L is non-trivial. If L were infinite, then any finite lattice would evidently belong to the primitive class $K(L)$, so that by [2] any lattice would belong to $K(L)$; but $L \notin K(L)$. L is subdirectly irreducible by Theorem 1. Let us prove (H). As $L \notin K(L)$ and $K(L)$ is primitive, we have $A \notin K(L)$, too, so that A contains a sublattice isomorphic to L. To prove the converse implication, it is by Theorem 1 enough to show that $K(L)$ is closed with respect to homomorphic images; this follows from (H).

REMARK. It is easy to see that any primitive lattice is a sublattice of the free lattice. We do not know whether any non-trivial, finite and subdirectly irreducible sublattice of the free lattice is primitive.

THEOREM 3. Let L be a finite lattice. Define a lattice $L^\star$ in this way: L is a sublattice of $L^\star$; $L^\star \setminus L$ contains exactly three elements u, v, a; u is the smallest and v the greatest element of $L^\star$; a is incomparable with all elements of L. The following holds:
(1) If L is primitive, then $L^\star$ is primitive, too.
(2) If $K(L)$ is the class of all lattices satisfying a single equation $p(x_1, \dots, x_n) = q(x_1, \dots, x_n)$, then $K(L^\star)$ is the class of all lattices satisfying the equation

$$p^\star(x_1, \dots, x_{n+1}) = q^\star(x_1, \dots, x_{n+1}), \text{ where}$$
$$p^\star(x_1, \dots, x_{n+1}) = p(t_1, \dots, t_n),$$
$$q^\star(x_1, \dots, x_{n+1}) = q(t_1, \dots, t_n),$$
$$t_k = (x_k \wedge i) \vee o \quad (k = 1, 2, \dots, n),$$
$$o = (x_1 \wedge \dots \wedge x_n) \vee (x_{n+1} \wedge (x_1 \vee \dots \vee x_n)),$$
$$i = (x_1 \vee \dots \vee x_n) \wedge (x_{n+1} \vee (x_1 \wedge \dots \wedge x_n)).$$

Proof. (1) As L is subdirectly irreducible, the set of all its congruences that are different from the smallest congruence ι_L contains a smallest member Θ. The relation $\Theta \cup \iota_{L^\star}$ is evidently the smallest congruence of $L^\star$ different from $\iota_{L^\star}$. $L^\star$ is thus subdirectly irreducible. Let A be a lattice and f a homomorphism of A onto $L^\star$. By Theorem 2 it is sufficient to prove that A contains a sublattice isomorphic to $L^\star$. Denote by b the smallest and by c the greatest element of L. There exist $a', b', c' \in A$ such that $f(a') = a$, $f(b') = b$, $f(c') = c$. Put

$$v' = b' \vee a',$$
$$c'' = (c' \wedge v') \vee b',$$
$$u' = c'' \wedge a',$$
$$b'' = b' \vee u'.$$

We have evidently $u' < b'' < c'' < v'$, $c'' \wedge a' = u'$, $b'' \vee a' = v'$, $f(b'') = b$, $f(c'') = c$. It is easy to see that the interval $\{x \in A;\ b'' \leq x \leq c''\}$ is mapped by f onto L, so that it contains a sublattice L_0 isomorphic to L. The union $L_0 \cup \{a', u', v'\}$ is evidently a sublattice of A which is isomorphic to $L^\star$.

(2) Assume that the lattice $L^\star$ is a sublattice of a lattice S. The equation $p = q$ is not satisfied in L and so there exist elements $a_1, \dots, a_n$ of L such that $p(a_1, \dots, a_n) \neq q(a_1, \dots, a_n)$. Clearly $p^\star(a_1, \dots, a_n, a) = p(a_1, \dots, a_n) \neq q(a_1, \dots a_n) = q^\star(a_1, \dots, a_n, a)$. Thus the equation $p^\star = q^\star$ is not satisfied in S.

Conversely, let S be a lattice and let the equation $p^\star = q^\star$ be not satisfied in S. Let $a_1, \ldots, a_n, d$ be elements of S such that $p^\star(a_1, \ldots, a_n, d) \neq q^\star(a_1, \ldots, a_n, d)$. Put

$$\begin{aligned} r &= (a_1 \vee \cdots \vee a_n) \wedge (d \vee (a_1 \wedge \cdots \wedge a_n)), \\ s &= (a_1 \wedge \cdots \wedge a_n) \vee (d \wedge (a_1 \vee \cdots \vee a_n)), \\ h_k &= (a_k \wedge r) \vee s \qquad (k = 1, 2, \ldots, n). \end{aligned}$$

Since $p^\star(a_1, \ldots, a_n, d) = p(h_1, \ldots, h_n)$ and $q^\star(a_1, \ldots, a_n, d) = q(h_1, \ldots, h_n)$, the equation $p = q$ is not satisfied in the interval $[s, r]$. By the assumption, there exists

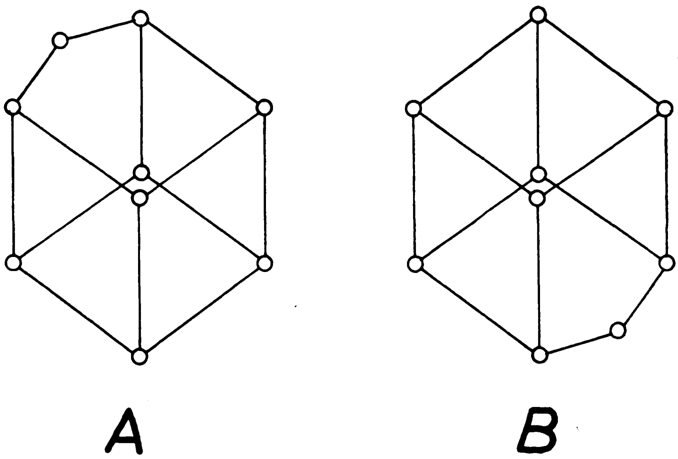


Fig. 1

a sublattice L' of $[s, r]$ isomorphic to L. It is easy to show that $d \wedge r = d \wedge s$ and $d \vee r = d \vee s$. From this it follows that the set $L' \cup \{d, d \wedge r, d \vee r\}$ forms a sublattice of S isomorphic to $L^\star$.

The two-element lattice is evidently primitive. There exist primitive lattices which can not be obtained from the two-element lattice by applications of Theorem 3. Some examples of ones will be now given.

H. Löwig [3] proved that the lattice A in Fig. 1 is primitive and that the class $K(A)$ is the class of all lattices satisfying the equation

$$(x \wedge y) \vee (x \wedge z) = x \wedge ((y \wedge z) \vee (z \wedge x) \vee (x \wedge y)).$$

Clearly, the dual of a primitive lattice is primitive, too. Thus, the lattice B in Fig. 1 is primitive and the class $K(B)$ is characterized by the dual equation.

The following two theorems give other examples of primitive lattices.

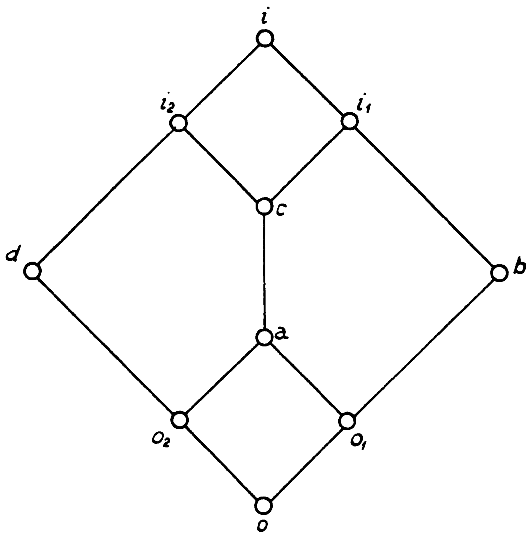
THEOREM 4. The lattice C in Fig. 2 is primitive and the class $K(C)$ can be characterized by one equation.

Proof. We shall find an equation $p = q$ and show that $K(C)$ is just the class of all lattices satisfying this equation. Let x_1, x_2, x_3, x_4 be variables. Put

$$\begin{aligned}
o' &= x_1 \wedge x_2 , & i' &= x_1 \vee x_2 , \\
a' &= (o' \vee x_3) \wedge i' , & c' &= (a' \vee x_4) \wedge i' , \\
o_1' &= c' \wedge x_1 , & o_2' &= c' \wedge x_2 , \\
i_1' &= a' \vee x_1 , & i_2' &= a' \vee x_2 , \\
o_1'' &= o_1' \wedge i_2' , & o_2'' &= o_2' \wedge i_1' , \\
o_1''' &= o_1'' \vee (x_1 \wedge i_2') , & o_2''' &= o_2'' \vee (x_2 \wedge i_1') , \\
i_1''' &= i_1' \wedge (x_1 \vee o_2''') , & i_2''' &= i_2' \wedge (x_2 \vee o_1''') , \\
p(x_1, x_2, x_3, x_4) &= o_1''' \vee o_2''' , & q(x_1, x_2, x_3, x_4) &= i_1''' \wedge i_2''' .
\end{aligned}$$

It is easy to see that in any lattice we have

$$\begin{aligned}
o' &\leq o_1''' \leq x_1 \leq i_1''' \leq i' , & o' &\leq o_2''' \leq x_2 \leq i_2''' \leq i' , \\
o_1''' &\leq p \leq q \leq i_2''' , & o_2''' &\leq p \leq q \leq i_1''' .
\end{aligned}$$



C

Fig. 2

Since $o_2''' \leq i_1'''$, $x_1 \leq i_1'''$ and $i_1''' \leq x_1 \vee o_2'''$, we have $x_1 \vee o_2''' = i_1'''$. Similarly we can get that $x_2 \vee o_1''' = i_2'''$, $x_1 \wedge i_2''' = o_1'''$ and $x_2 \wedge i_1''' = o_2'''$ holds in any lattice.

We are now able to finish the proof. If C is a sublattice of a lattice L, then the equation $p = q$ is not satisfied in L because

$$p(b, d, a, c) = a \neq c = q(b, d, a, c) .$$

Conversely, let L be a lattice such that the equation $p = q$ is not satisfied in L. Then there exist elements a_1, a_2, a_3, a_4 of L such that $p(a_1, a_2, a_3, a_4) \neq q(a_1, a_2, a_3, a_4)$. If in the expressions

$$o', i', o_1''', o_2''', i_1''', i_2''', p, q, x_1, x_2$$

we write a_1, a_2, a_3, a_4 instead of x_1, x_2, x_3, x_4, we get ten elements of L; it is easily verified that they form a sublattice of L isomorphic with C.

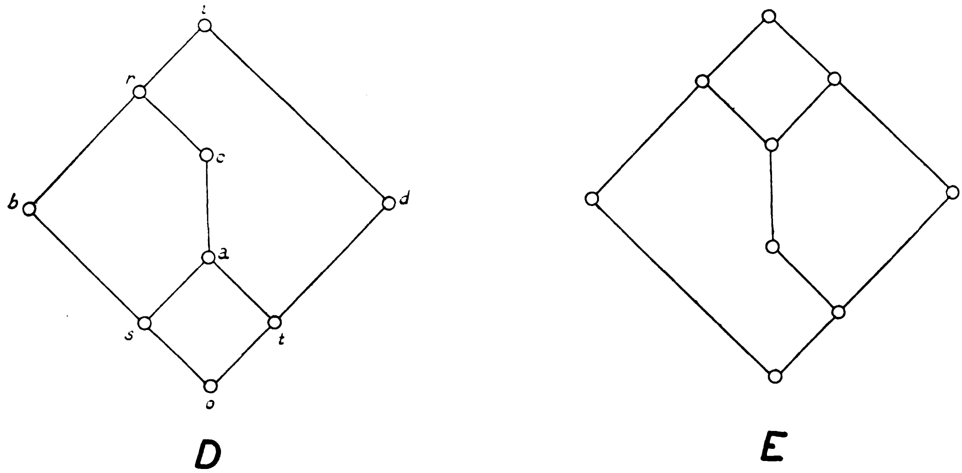


Fig. 3

THEOREM 5. The lattice D in Fig. 3 is primitive and the class $K(D)$ can be characterized by one equation. The same holds for the lattice E in Fig. 3, as it is dual to D.

Proof. Let x_1, x_2, x_3, x_4 be variables. Put

$$\begin{aligned} o' &= x_1 \wedge x_2 , & c' &= (x_3 \vee o') \wedge (x_1 \vee x_2) , \\ s' &= x_1 \wedge c' , & t' &= (x_4 \vee o') \wedge x_2 , \\ i' &= s' \vee x_2 , & b' &= x_1 \wedge i' , \\ t'' &= x_2 \wedge (b' \vee t') , & r' &= b' \vee t'' , \\ c'' &= ((c' \wedge i') \vee t'') \wedge r' , & s'' &= b' \wedge c'' , \\ a' &= s'' \vee t'' , & & \\ p(x_1, x_2, x_3, x_4) &= a' , & q(x_1, x_2, x_3, x_4) &= c'' . \end{aligned}$$

The following relations are evident:

$$\begin{aligned} & o' \leq s'' \leq a' \leq c'' \leq r' \leq i' , && s'' \leq b' \leq r' , \\ & o' \leq t'' \leq a' , && t'' \leq x_2 \leq i' . \end{aligned}$$

Since $t' \leq t''$ and $t'' \leq b' \vee t'$, we have $b' \vee t'' = b' \vee t'$ and hence $r' \wedge x_2 = t''$.

Since $s' \leq b' \leq x_1$ and $s' \leq c'$, we have $s' \leq c''$. Now it is easy to see that $s' \leq s''$ and hence $i' = s' \vee x_2 = s'' \vee x_2$.

We shall now prove that a lattice L belongs to $K(D)$ iff the equation $p = q$ is satisfied in L. First assume that $L \notin K(D)$. Then the equation $p = q$ is not satisfied in L because

$$p(b, d, c, t) = a \neq c = q(b, d, c, t)\,.$$

Conversely, let the equation $p = q$ be not satisfied in L. Then $p(a_1, a_2, a_3, a_4) \neq q(a_1, a_2, a_3, a_4)$ for some $a_1, a_2, a_3, a_4 \in L$. If in the expressions

$$o', i', s'', t'', a', c'', b', r', x_2$$

we write a_1, a_2, a_3, a_4 instead of x_1, x_2, x_3, x_4, we get nine elements of L; they constitute a sublattice of L isomorphic with D.

It follows from Theorem 3 that we have six infinite sequences of primitive lattices: we may start either from the two-element lattice or from one of the lattices A, B, C, D, E in Figures 1, 2, 3. If we take any member of these sequences, then the corresponding primitive class is characterized by one equation.

References

[1] Игошин В. И.: Характеризуемые многообразия решеток. „Упорядоченные множества и решетки", вып. 1, Издательство Саратовского университета, 1971, 22.

[2] Jónsson B.: Algebras whose congruence lattices are distributive. Math. Scand., 1967, 21, 110.

[3] Löwig H.: On the importance of the relation [(A, B), (A, C)] < (A,[(B, C), (C, A), (A, B)]) between three elements of a structure. Ann. Math., 1943, 44, 573.