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Finite Element Analysis of a System of Quasi-parabolic Partial Differential Equations

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Applying LAPLACE transform finite element method is generalized to solutions of a system of quasi-parabolic partial differential equations.

1. Introduction

We shall consider a system of partial differential equations of shallow viscoelastic shells

$$K_{ijkl}\left[\frac{h^3}{12}\, w_{,ijkl} + h(u_{i,j} + b_{ij}w)\, b_{kl}\right] = Lq\,,$$

$$K_{ijkl}\,(u_{k,jl} + b_{kl}w_{,j}) = 0,\ (i, j, k, l = 1{,}2)\,, \tag{1.1}$$

where

$$K_{ijkl} = \sum_{\nu=0}^{r} K^{(\nu)}_{ijkl} D^\nu, \quad L = \sum_{\nu=0}^{s} L_\nu D^\nu \tag{1.2}$$

are polynomials in $D = \frac{\partial}{\partial t}$, w is the displacement of the middle surface of the shell in x_3 direction and u_1, u_2 the displacements in x_1, x_2 directions, respectively, h — the thickness of the shell, b_{ij} — tensor of curvature, $K^{(\nu)}_{ijkl}$ — tensors of stiffnesses and $q(x, t)$ — the transverse loading of the shell. We use the usual indicial notation. Latin subscripts have the range of integers 1,2 and summation over repeated subscripts is implied. Subscripts preceded by a comma indicate differentiation with respect to corresponding Cartesian spatial coordinates.

In the case of real material it holds

$$K_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} \geq 0 \tag{1.3}$$

for arbitrary values of ε_{ij}, where equality occurs if and only if $\varepsilon_{ij} = 0$ for all i, j. Further the operators K_{ijkl} are symmetric

$$K_{ijkl} = K_{jikl} = K_{klij} \tag{1.4}$$

and polynomials (1.2) have real negative roots.

Simultaneously we consider the system of integrodifferential equations

$$\int_0^t G_{ijkl}(t-\tau)\,\frac{\partial}{\partial \tau}\left[\frac{h^3}{12}\,w,_{ijkl} + h(u_{i,j} + b_{ij}w)\,b_{kl}\right] \mathrm{d}\tau = Lq\,, \tag{1.5}$$

$$\int_0^t G_{ijkl}(t-\tau)\,\frac{\partial}{\partial \tau}\,(u_{k,jl} + b_{kl}w,_j)\,\mathrm{d}\tau = 0\,,$$

where G_{ijkl} is symmetric and it holds

$$G_{ijkl}(0)\,\varepsilon_{ij}\,\varepsilon_{kl} \geq 0\,. \tag{1.6}$$

We shall consider following boundary conditions

$$w = \frac{\partial w}{\partial n} = 0\,, \qquad u_1 = u_2 = 0 \qquad \text{on } \partial\Omega \tag{1.7}$$

or

$$w = K_{ijkl}\,w,_{ij}\,\nu_k^n \nu_l^n = 0\,, \qquad u_1 = u_2 = 0 \qquad \text{on } \partial\Omega\,,$$

where n denotes the outward normal and ν_i^n direction cosines of this normal.

From the physical point of view it is convenient to consider the initial conditions in the form

$$\left.\frac{\partial^\nu w}{\partial t^\nu}\right|_{t=0^-} = \left.\frac{\partial^\nu u_i}{\partial t^\nu}\right|_{t=0^-} = 0\,, \qquad (\nu = 0, 1, 2, \ldots, r-1)\,. \tag{1.8}$$

Initial values at $t = 0^+$ can be different from zero and are to be obtained from the solution.

2. Functional of the Generalized Potential Energy

We shall assume that $q(x, t)$ belongs to the class of slowly increasing functions $U(x, t)$, which fulfil in Ω for $t > 0$ and for each $\delta > 0$ the condition

$$|u(x, t)| < M(\delta)\,e^{\delta t}. \tag{2.1}$$

where $M(\delta)$ depends on u but does not depend on x.

Applying generalized Laplace transform [5] to equations (1.1) and (1.5) one obtains

$$\tilde{K}_{ijkl}\left[\frac{h^3}{12}\,\tilde{w},_{ijkl} + h(\tilde{u}_{i,j} + b_{ij}\tilde{w})\,b_{kl}\right] = \tilde{L}\,\tilde{q}\,,$$

$$\tilde{K}_{ijkl}\,(\tilde{u}_{k,jl} + b_{kl}\,\tilde{w},_j) = 0 \qquad (i = 1,2) \tag{2.2}$$

and

$$p\tilde{G}_{ijkl}\left[\frac{h^3}{12}\,w,_{ijkl} + h(\tilde{u}_{i,j} + b_{ij}\tilde{w})\,b_{kl}\right] = \tilde{L}\tilde{q}\,,$$

$$p\tilde{G}_{ijkl}(\tilde{u}_{k,jl} + b_{kl}\tilde{w},_j) = 0 \qquad (i = 1,2) \tag{2.3}$$

where tildas denote Laplace transforms.

These conditions can be written in the form

$$\tilde{K}_{\alpha\beta}\tilde{a}_\beta = \tilde{q}_\alpha \,, \tag{3.4}$$

where formulas for $\tilde{K}_{\alpha\beta}$ and $\tilde{q}_\alpha$ can be easily obtained from (2.4), (3.2), and (3.3). The solution of this system is given by the formula

$$\tilde{a}_\alpha(p) = \frac{\tilde{F}_{\beta\alpha}\tilde{q}_\beta}{|\tilde{K}_{\alpha\beta}|} \,. \tag{3.5}$$

As

$$\tilde{K}_{\alpha\beta} = \sum_{\nu=0}^{r} K_{\alpha\beta}^{(\nu)} p^\nu \tag{3.6}$$

is a p-matrix, $|\tilde{K}_{\alpha\beta}|$-the determinant is a polynomial in p of the degree $r(m+2n)$ and $\tilde{F}_{\alpha\beta}$-the adjoint matrix is a polynomial in p of the degree $r(m+2n-1)$. Thus $\tilde{a}_\alpha(p)$ are rational functions of p and inverse transform can be achieved by the method of decomposition into partial fractions. Denoting the roots of the determinental equation

$$\Delta(p) = |\tilde{K}_{\alpha\beta}| = 0 \tag{3.7}$$

by $-p_\gamma$ and assuming that they are distinct one obtains

$$\tilde{a}_\alpha = \sum_{\gamma=1}^{s} \frac{A_{\alpha\beta}(p_\gamma)}{p+p_\gamma}\,\tilde{q}_\beta \tag{3.8}$$

where $s = r(m+2n)$ and

$$A_{\alpha\beta}(p_\gamma) = \frac{\tilde{F}_{\beta\alpha}(-p_\gamma)}{\Delta^{(1)}(-p_\gamma)} \,; \quad \Delta^{(1)}(p) = \frac{\mathrm{d}\Delta(p)}{\mathrm{d}p} \,. \tag{3.9}$$

Then

$$\tilde{w}_m = \sum_{\gamma=1}^{s} \frac{A_{\alpha\beta}(p_\gamma)}{p+p_\gamma}\,\tilde{q}_\beta\varphi_\alpha \tag{3.10}$$

and the inverse transform is given by the convolutional product

$$w_m = \sum_{\gamma=1}^{s} \varphi_\alpha \int_0^t \tilde{q}_\beta A_{\alpha\beta}(p_\gamma)\, e^{-p_\gamma(t-\tau)}\,\mathrm{d}\tau \,. \tag{3.11}$$

When the loading is constant in time $q = q(x)\,H(t)$ and $L = I$

$$w_m = \sum_{\gamma=1}^{s} (q, \varphi_\beta)\,\varphi_\alpha A_{\alpha\beta}(p_\gamma)\,(1 - e^{-p_\gamma t}) \,. \tag{3.12}$$

In the case of quasistatic problems p as can be proved are real positive and w_n assumes the form of Dirichlet exponential series. Then the approximate numerical inverse transform can be applied [4]. Similar results can be obtained for u_1 and u_2, too.

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