Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 15 (1974), No. 1-2, 35--38

Persistent URL: http://dml.cz/dmlcz/142322

1974 ACTA UNIVERSITATIS CAROLINAE - MATHEMATICA ET PHYSICA NO. 1-2 PAG. 35-38

On a Class of Nonlinear Eigenvalue Problems

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An outline of the theory of overdamped nonlinear eigenvalue problems is given with particular reference to recent results on variational principles.

In the linear eigenvalue problem an operator function $T(\lambda) = \lambda I - G$ is considered, where λ is a complex parameter and G is a bounded operator in a Hilbert space X, say. One important problem is to characterize the spectrum, in particular the eigenvalues, i.e. those values λ with $T(\lambda)x = 0$ for some $x \in X$, $x \neq 0$. Research on problems with nonlinear parameters was stimulated by applications. The simplest examples are the quadratic polynomial

$$T(\lambda) = \lambda^2 C - \lambda B - A \tag{1}$$

and polynomials of higher order. Problems of this type where investigated by WIELANDT [22], CHARAZOV, MÜLLER [14], KREIN [8], LANGER [12], MONIEN [13] and others. In these papers the interest was concentrated on the completeness of eigenvectors and associated eigenvectors, the basic tool being the theory of nonselfadjoint operators.

Another approach was initiated by DUFFIN [2] who proved a minimax property for quadratic polynomials in a finite-dimensional space, which later on lead to a theory for general nonlinear problems. It is worth while to compare the two approaches. In (1) assume that A, B, C are symmetric matrices of order n with A, C positively definite. The standard "Frobenius" procedure and a following symmetrization transform (1) into a symmetric problem in X^2,

$$\begin{pmatrix} 0 & A^{\frac{1}{2}} C^{-\frac{1}{2}} \\ C^{-\frac{1}{2}} A^{\frac{1}{2}} & C^{-\frac{1}{2}} B C^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}. \tag{2}$$

Then the $2n$ eigenvalues can be obtained from the three standard variational principles for the Rayleigh quotient $r(x) = (Gx, x)/(x, x)$ of the symmetric matrix G,

$$\lambda_j = \max_{\substack{(x,x_k)=0 \\ k=1,\dots,j-1}} r(x) \qquad \text{(RAYLEIGH)} \tag{3}$$

$$\lambda_j = \min_{\substack{N \subset X \\ \dim N = j-1}} \max_{x \subset N^\perp} r(x) \qquad \text{(WEYL-FISCHER-COURANT)} \tag{4}$$

$$\lambda_j = \max_{\substack{M \subset X \\ \dim M =}} \min_{x \in M} r(x) \qquad \text{(RITZ-POINCARE)} \tag{5}$$

In Duffin's approach with the function T a functional p is associated, which is simply the root of the corresponding quadratic equation

$$p(x) = \frac{1}{2(Cx, x)} \{(Bx, x) + [(Bx, x)^2 + 4(Ax, x)(Cx, x)^{\frac{1}{2}}]\} . \tag{6}$$

The functional p has the properties

$$p(cx) = p(x) , \quad (T(p(x))x, x) = 0 , \quad (T'(p(x))x, x) > 0 . \tag{7}$$

Duffin proved that the range of p contains exactly n eigenvalues $\lambda_1 \geq \lambda_2 \geq \geq \dots \geq \lambda_n$ which can be characterized by the Courant principle (4) with r replaced by p. Furthermore $\lambda_n \leq p(x) \leq \lambda_1$ for all $x \neq 0$. The remaining n eigenvalues are associated with another functional (the other root of the quadratic equation). In fact in this approach we do need only the "overdamping condition"

$$(Bx, x)^2 + 4(Ax, x)(x, x) > 0 , \quad x \neq 0 . \tag{8}$$

Duffin's method has been adapted by ROGERS [15] to an arbitrary problem with a nonlinear function T and an associated Rayleigh functional with the properties (7) in R^n. Rogers obtained the existence of n eigenvalues in the range of p and their characterization by a Courant principle.

In the finite-dimensional case the Courant and the Ritz principle are trivially equivalent. The Rayleigh principle (3) can also be generalized [3],

$$\lambda_j = \max_{\substack{[x, x_k] = 0 \\ k = 1, \dots, j-1}} p(x) \tag{9}$$

where the inner product is defined by

$$[x, y] = \left(\frac{T(p(x)) - T(p(y))}{p(x) - p(y)} x, y \right) \quad \text{for } p(x) \neq p(y) \tag{10}$$

and $[x, y] = (T'(p(x))\, x, y)$ for $p(x) = p(y)$.

In [4] an existence theorem for eigenvalues via the Rayleigh principle has been proved, based on a compactness argument and successive orthogonalization with respect to [,]. The most restricting hypothesis was a representation $T(\lambda) = = r(\lambda)I - V(\lambda)$ with a scalar function r and compact V.

If we look again at the linear case we find that the validity of the variational principles is not restricted to compact operators. If the symmetric linear operator G has a finite or denumerable sequence of eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots$ above the continuous spectrum, then the Courant principle yields all of these eigenvalues. If the number of eigenvalues is finite, say N, then the minmax-values of order $N + 1$, $N + 2, \dots$ are equal to the supremum of the continuous spectrum.

In his dissertation WERNER [20] applied this idea to nonlinear problems: The range $W = \{p(x) : x \in X, x \neq 0\}$ of p (field of values of the eigenvalue problem) is divided into three disjoint parts

$$W_0 = \{\lambda \in W : \exists \{x_\nu\} , \quad \|x_\nu\| = 1, \quad x_\nu \rightharpoonup 0 , \quad p(x_\nu) \to \lambda\} ,$$
$$W_+ = \{\lambda \in W : \lambda > \sup W_0\} , \quad W_- = \{\lambda \in W : \lambda < \inf W_0\} . \tag{11}$$

The set W_0 generalizes the convex hull of the continuous spectrum in the linear case. The spectrum of T (with respect to p) is the set $\sigma = \{\lambda \in \overline{W}: 0 \in \sigma(T(\lambda))\}$. If $W_+ \neq \emptyset$ then $W_+ \cap \sigma$ consists of finitely or denumerably many eigenvalues (of finite multiplicity). They can be obtained by the Ritz principle.

Werner's proof does not imply a proof for the linear case, too. Quite in contrary the variational principle for the linear case in used throughout the proof. In [20] the Courant principle is derived from the Ritz method only with assistance of the Rayleigh principle and considerable effort. Hence it seems advisable to prove the Courant principle first and derive the Ritz method in a few steps as pointed out in [4]. The details of this approach will be published elsewhere [7].

The Ritz method can be applied in the following way. Choose a subspace M of dimension m, let Q be the orthogonal projection onto M. Define

$$\tilde{T}(\lambda) = QT(\lambda)\,Q, \quad p = p\,|_M\,.$$

Then $\tilde{T}, \tilde{p}$ form again a nonlinear eigenvalue problem with eigenvalues $\alpha_1 \geq \geq \alpha_2 \geq \ldots \geq \alpha_m$, and the inequalities $\lambda_1 \geq \alpha_1, \ldots, \lambda_m \geq \alpha_m$ provide Ritz's lower bounds.

Generalizations of Temple's and Weinstein's theorems on upper bounds have also been proved [6].

It should be well realized that to an operator function T there might exist several Rayleigh functionals p, each selecting an appropriate subset from all possible eigenvalues (exactly n eigenvalues if dim $X = n$, in this case the eigenvectors form a base of X).

Although the theory of this class of problems is well developed, its importance for practical applications should not be overemphasized. Let us assume a mechanical system with n degress of freedom performs oscillations described by a matrix eigenvalue problem

$$Ax = -\lambda^2 Cx \tag{12}$$

with A, C positively definite. The eigenvalues are on the imaginary axis, the problem (12) is symmetric. If the oscillations are damped the problem changes into

$$\lambda^2 Cx + \lambda\tau Bx + Ax = 0\,, \tag{13}$$

where B is positively definite and $\tau > 0$ is a parameter. For growing τ the eigenvalues move into the left halfplane and towards the real axis, until for sufficiently large τ an overdamping condition is satisfied and all eigenvalues are real. Then the results mentioned above can be applied. However, in many applications the intermediate problems (with damped oscillations and complex eigenvalues) will be of interest.

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