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Multivariate Approximation Theory with Λ-Splines

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Λ-splines have been introduced in one-dimensional spline theory by JEROME and PIERCE (1972). By considering tensor products of one-dimensional Λ-spline spaces one can deduce results about approximation of real-valued functions in several real variables which belong to L^2 or some spaces of Sobolev type by functions in finite dimensional subspaces. One obtains generalizations of some error estimates which have been given by SCHULTZ [2].

If B is a real Banach space, $f \in B$, $\bar{C}$ a chosen set of finite dimensional subspaces S, $\bar{C} \subset B$, then we define

$$E(f, S, B) = \inf_{y \in S} \|f - y\|_B \qquad \text{for all } S \in \bar{C}.$$

For each positive integer i, $1 \leq i \leq N$ we consider finite intervals $I_i \equiv [a_i, b_i]$ and the rectangular parallelepiped $H \equiv \mathop{\times}\limits_{i=1}^{N} I_i \subset R^N$. If L_M^2 is a finite dimensional subspace of $L^2(H)$ of dimension M and L_{i,M_i}^2 is a finite dimensional subspace of $L^2(I_i)$ of dimension M_i then we consider the orthogonal projections

$$P(L_M^2): \quad L^2(H) \to L_M^2$$
$$P(L_{i,M_i}^2): \; L^2(I_i) \to L_{i,M_i}^2, \qquad 1 \leq i \leq N.$$

If $f \in L^2(H)$, then $P(L_{i,M_i}^2)f$, $1 \leq i \leq N$, denotes the projection with respect to the i-th variable of f, with the other variables held fixed. If $L_M^2 \equiv \bigotimes_{i=1}^{N} L_{i,M_i}^2 \subset L^2(H)$, then $P(L_M^2) \equiv \prod_{i=1}^{N} P(L_{i,M_i}^2)$.

Let us now introduce Λ-splines and some of the results of JEROME and PIERCE [1]. Consider the $2n$-th order self-adjoint differential operator

$$\Lambda \equiv \sum_{j=0}^{n} (-1)^j D^j(c_j(x)\, D^j) \tag{1}$$

where $c_j(x) \in C^j[a, b]$ for $0 \leq j \leq n$, $c_n(x) > 0$ for $x \in (a, b)$, $c_n^{-1}(x) \in L^1[a, b]$. Let $\bar{H}$ denote the linear space of real-valued functions f, defined on $[a, b]$ such that $D^{n-1}f$ is absolutely continuous and $\sqrt{c_n} D^n f \in L^2[a, b]$. Let $M = \{\mu_i\}_{i=1}^{k}$ be a set of linear point-functionals which are linearly independent and continuous on $\bar{H}$ and let $\bar{r} = (r_1, r_2, \ldots, r_k)$ be a vector of real numbers. We introduce the bilinear form

$$B(u, v) \equiv \sum_{j=0}^{n} \int_a^b c_j(x)\, D^j u(x) \cdot D^j v(x)\, dx \qquad \text{for } u, v \in \bar{H}. \tag{2}$$

Then a function $s \in \bar{H}$ is called a Λ-spline interpolating $\bar{r}$ with respect to M if it solves the minimization problem

$$B(s, s) = \inf_{w \in U(\bar{r})} B(w, w) \tag{3}$$

where $U(\bar{r}) \equiv \{w \in \bar{H};\ \mu_j w = r_j,\ 1 \leq j \leq k\}$. The set M of linear functionals is said to generate a Hermite-Birkhoff (HB) interpolation problem if to each $\mu_i \in M$ there corresponds a pair (x_i, j_i) such that $\mu_i g = D^{j_i} g(x_i)$, where $a \leq x_i \leq b$ and $0 \leq j_i \leq n - 1$. Define the partition of $[a, b]$ $\Delta \equiv \{x_1;\ \mu_1 g = g(x_1), \mu_1 \in M\}$ and let h be the maximal length of intervals into which $[a, b]$ is divided by the points of Δ.

If $Sp(\Lambda, M)$ denotes the class of all Λ-splines s such that s satisfies (3) for some $\bar{r}$ and if $g \in \bar{H}$, then $\tilde{g} \in Sp(\Lambda, M)$ is called an $Sp(\Lambda, M)$-interpolate of $g(x)$ if $\mu g = \mu \tilde{g}$ for all $\mu \in M$.

According to [1] there holds

Theorem 1. Let $\Lambda g \in L^2[a, b]$ and let M generate an (HB) interpolation problem. If $\tilde{g} \in Sp(\Lambda, M)$ interpolates g and M contains the following derivative evaluations at the endpoints

$$\{\mu_j;\ \mu_j g \equiv D^j g(a),\ 0 \leq j \leq n - 1\},\ \{\mu_j;\ \mu_j g \equiv D^j g(b),\ 0 \leq j \leq n - 1\} \tag{4}$$

then, for h sufficiently small, there exists a positive constant K, independent of g and Δ, such that

$$\|D^j(g - \tilde{g})\|_{L^2[a,b]} \leq K h^{2n-j-1}\, \omega_1\left(\frac{1}{c_n}, nh\right) \|\Lambda g\|_{L^2[a,b]}, \quad 0 \leq j \leq n - 1 \tag{5}$$

where

$$\omega_1(f, \delta) \equiv \sup_{\substack{x, x+t \in [a,b] \\ 0 \leq t \leq \delta}} \int_x^{x+t} |f(t)|\, dt .$$

From (5) we obtain immediately

$$E(g, Sp(\Lambda, M), L^2[a, b]) \leq K h^{2n-1} \omega_1\left(\frac{1}{c_n}, nh\right) \|\Lambda g\|_{L^2[a,b]} . \tag{6}$$

For each I_i, $1 \leq i \leq N$ we consider now an operator Λ_i of the form (1). We define the partition $\Omega \equiv \mathop{\times}\limits_{i=1}^{N} \Delta_i$ of H where Δ_i is a partition of I_i. Let $\varrho \equiv \max\limits_{1 \leq i \leq N} h_i$. Then, combination of Theorem 2.1 of [2] and (6) yields

Theorem 2. Let $\Lambda_i g \in L^2(H)$. For each I_i, $1 \leq i \leq N$, M_i may generate a unique (HB) interpolation problem with the conditions of Thm.1. Then there exists a constant K, independent of g and Ω, and

$$E(g, \bigotimes_{i=1}^{N} Sp(\Lambda_i, M_i), L^2(H)) = \|g - P(\bigotimes_{i=1}^{N} Sp(\Lambda_i, M_i)\, g\|_{L^2(H)} \leq$$

$$\leq K\varrho^{2n-1} \sum_{i=1}^{N} \left[\omega_1\left(\frac{1}{c_{n,i}}, n\varrho\right) \|\Lambda_i g\|_{L^2(H)}\right]. \tag{7}$$

Several generalizations can easily be given. One can consider tensor products of the construction of Jerome and Pierce which does not involve condition (4). One can obtain estimates in the W_2^j-norm ($0 \leq j \leq n - 1$) and, as has been done by Schultz, one can consider the approximation on regular bounded open sets Ω such that $\bar{\Omega} \subset \operatorname{int} H$.

References

[1] Jerome, J., Pierce, J.: On Spline Functions Determined by Singular Self-adjoint Differential Operators. J. Approximation Theory 5, 15 (1972).

[2] Schultz, M. H.: L^2-multivariate Approximation Theory. Siam J. Numer. Anal. 6, 184 (1969).