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Direct Iterative Methods for Linear Systems Using Weak Splittings

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The splitting $A = M - N$ of a rectangular matrix A is called proper if the range and null spaces of A and M are equal. This idea was developed as a means of extending to the general case the usual splitting of a nonsingular matrix. For the linear system $Ax = b$ the iterative method $x^{(k+1)} = M^+Nx^{(k)} + M^+b$, where $A = M - N$ is a proper splitting, converges to the least squares solution of minimum norm, A^+b, if and only if $\varrho(M^+N) < 1$. Here A^+ and M^+ denote the usual Moore-Penrose pseudoinverses of A and M. The method avoids the use of the normal system $A^TAx = A^Tb$.

This paper extends these results in two ways: (1) by considering the least squares and the minimum norm solutions separately so that the pseudoinverses are easier to calculate, and (2) by weakening the conditions of a proper splitting to requiring only equality of the ranges of A and M when $Ax = b$ may be inconsistent and only equality of the null spaces of A and M when $Ax = b$ is consistent. In addition, convergence theorems are obtained in terms of matrices leaving positive cones invariant.

I. Introduction

Consider the rectangular system of linear equations

$$Ax = b \tag{1.1}$$

where A is a real $m \times n$ matrix and b is a real m-vector. In the special case where $m = n$ and A is nonsingular, iterative methods of the form $x^{(k+1)} = Gx^{(k)} + c$ are usually employed to obtain the solution whenever m is large and the matrix A is sparse. This iterative formula is obtained by splitting A into the form $A = M - N$ where M is itself nonsingular and then letting $G = M^{-1}N$ and $c = M^{-1}b$. The sequence $\{x^k\}$ then converges to the solution to (1.1) for every $x^{(0)}$, if and only if the spectral radius $\varrho(M^{-1}N)$, of $M^{-1}N$ is less than one. Conditions under which $\varrho(M^{-1}N) < 1$ have been described by Varga [10], Collatz [2], Fiedler and Pták [3], Ortega and Rheinboldt [6], Marek [5], Young [11], and others. In such studies the concept of matrix monotonicity plays a fundamental role.

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In the more general case where A may be singular and in particular rectangular, the system (1.1) may be under- or over-determined. Here one normally wishes to compute the solution $\tilde{x}$ of minimum Euclidean norm if (1.1) is underdetermined and some vector y that minimizes the Euclidean norm of $b - Ax$ when (1.1) is over-determined. In the first case $\tilde{x}$ is called the *minimum norm solution* to (1.1) and in the second case y is called a *least squares solution.* In the general case then, there is exactly one least squares solution of minimum norm. Such a vector y is called the *best least squares solution* to (1.1) and is given by $y = A^+b$ where A^+ is the *pseudo-inverse* of A; that is, A^+ satisfies $A = AA^+A$, $A^+ = A^+AA^+$, with AA^+ and A^+A symmetric. More generally Xb provides a least squares solution to (1.1) where $AX = AA^+$. Such $n \times m$ matrices are known as *least squares inverses* of A and are denoted by A_l^-. Moreover if (1.1) is consistent and $XA = A^+A$, then Xb is the solution of minimum norm. These matrices are called *minimum norm inverses* of A and are denoted by A_m^-. Of course, $A_l^- = A^+$ if A has full column rank, $A_m^- = A^+$ if A has full row rank and $A^+ = A^{-1}$ if A is square and nonsingular. However, if $0 < \text{rank}\, A < \min\{m, n\}$ then A_l^- and A_m^- are not unique. Very little use of these particular matrices has yet been made in computational methods for singular systems, although they are usually much easier to compute than A^+. Each of A^+, A_l^- and A_m^- are solutions to $A = AXA$. Such solutions are called *generalized inverses* (g-inverses) of A and are denoted by A^- [9].

In [8] and in a recent joint paper [1], a new method for iterating to the best least squares solution has been suggested. The method involves splitting the coefficient matrix A and avoids the use of the often ill-conditioned normal system $A^TAx = A^Tb$. The splitting $A = M - N$ is called a *proper splitting* of A provided that $\mathscr{R}(A) = \mathscr{R}(M)$ and $\mathscr{N}(A) = \mathscr{N}(M)$, that is, A and M have the same range and the same null space. (If A and M are square and nonsingular then the usual splitting is a proper splitting.) More recently [4], these ideas have been partially extended to operator equations $Tx = f$ where T is a bounded linear operator from a Banach to a Hilbert space.

In this paper these results are extended in two ways: (1) by considering the least squares and the minimum norm solutions separately so that the appropriate g-inverses are easier to calculate, and (2) by weakening the conditions of a proper splitting to requiring only equality of the ranges of A and M when (1.1) is over-determined and only equality of the null spaces of A and M when (1.1) is underdetermined.

The following notation will be used throughout the paper:

R^n denotes the n-dimensional real space and

$R^{m\times n}$ denotes the $m \times n$ real matrices.

For $K \subseteq R^n$, K will be called a positive cone if K is a pointed, solid, closed, convex cone.

For the sake of brevity the proofs of the results in the following sections are omitted.

2. Splittings

Let $A = M - N$ be a proper splitting of A so that $\mathscr{R}(A) = \mathscr{R}(M)$ and $\mathscr{N}(A) = \mathscr{N}(M)$ and let M^- denote any g-inverse of M. Then it can be shown that $A = M(I - M^-N)$, $I - M^-N$ is nonsingular, $A^- = (I - M^-N)^{-1}M^-$ is a g-inverse of A and A^-b is the unique solution to the system $x = M^-Nx + M^-b$ for any $b \in R^m$. In particular then, the iteration $x^{(k+1)} = M^-Nx^{(k)} + M^-b$ converges to A^-b for every $x^{(0)}$ if and only if $\varrho(M^-N) < 1$. These same facts hold with M^- replaced by a least squares g-inverse M_l^-, A^- by A_l^- and also with M^- replaced by a minimum norm g-inverse M_m^- and A^- by A_m^-. This then provides a method for iterating to least squares approximate solutions or accordingly to the minimum norm solution to (1.1), whenever $A = M - N$ in a proper splitting and $\varrho(M_l^-N) < 1$ or $\varrho(M_m^-N) < 1$, respectively.

However, except for special cases such as those that arise in a natural way in the numerical solution of partial differential equations by finite difference methods, proper splittings are not very easy to obtain where $\varrho(M^-N) < 1$. Thus one would naturally like to delete one of the requirements that $\mathscr{R}(A) = \mathscr{R}(M)$ and $\mathscr{N}(A) = \mathscr{N}(M)$.

3. Over-Determined Systems

The purpose of this section is to consider a method of iterating to a least squares solution to (1.1), by using a splitting $A = M - N$ with only the requirement that $\mathscr{R}(A) = \mathscr{R}(M)$. The first lemma establishes a condition under which $I - M_l^-N$ is nonsingular.

Lemma 3.1. Let $A = M - N$ in $R^{m\times n}$ with $\mathscr{R}(N) \subseteq \mathscr{R}(M)$ and let M_l^- be a least squares g-inverse of M. If $\mathscr{R}(M_l^-) \cap \mathscr{N}(A) = \{0\}$, then $\mathscr{R}(A) = \mathscr{R}(M)$ and $I - M_l^-N$ is nonsingular.

Lemma 3.2. Let $A = M - N$ in $R^{m\times n}$ satisfy the conditions of Lemma 3.1. Then

1. $A_l^- = (I - M_l^-N)^{-1}M_l^-$ is a least squares g-inverse of A and
2. the iteration $x^{(k+1)} = M_l^-Nx^{(k)} + M_l^-b$ converges to the least squares solution A_l^-b to (1.1) for any $x^{(0)} \in R^n$, if and only if $\varrho(M_l^-N) < 1$.

Notice that the least squares solution A_l^-b to (1.1), specified in the preceding lemma, depends upon the particular choice of M_l^-, and that M_l^- uniquely determines A_l^-. The following theorem gives a necessary and sufficient condition for the iteration to converge to A_l^-b.

Theorem 3.3. Let K be a positive cone in R^n and let $A = M - N$ in $R^{m\times n}$ satisfy the conditions of Lemma 3.1, such that $M_l^-NK \subseteq K$. Let $A_l^- = (I - M_l^-N)^{-1}M_l^-$. Then $\varrho(M_l^-N) < 1$ if and only if $A_l^-NK \subseteq K$.

4. Under-Determined Systems

Now consider the case where (1.1) is assumed to be consistent. Here we wish to obtain the solution $\tilde{x}$ to (1.1) having minimum Euclidean norm. For this purpose we split A into $A = M - N$ with $\mathcal{N}(A) = \mathcal{N}(M)$, iterate to a vector $v \in R^m$, and then compute $\tilde{x} = M_m^- v$. As pointed out in [7], this problem arises in important algorithms used in mathematical programming.

The following sequence of results parallel those given in Section III.

Lemma 4.1. Let $A = M - N$ in $R^{m \times n}$ with $\mathcal{N}(M) \subseteq \mathcal{N}(N)$ and let M_m^- be a minimum norm g-inverse of M. If $\mathcal{R}[(M_m^-)^T] \cap \mathcal{N}(A^T) = \{0\}$, then $\mathcal{N}(A) = \mathcal{N}(M)$ and $I - NM_m^-$ is nonsingular.

Lemma 4.2. Let $A = M - N$ in $R^{m \times n}$ satisfy the conditions of Lemma 3.1. Then

1. $A_m^- = M_m^-(I - NM_m^-)^{-1}$ is a minimum norm of g-inverse of A,
2. the iteration $v^{(k+1)} = NM_m^- v^{(k)} + b$ converges to a limit $v \in R^m$ for each $v^{(0)}$, if and only if $\varrho(NM_m^-) < 1$ and
3. $\tilde{x} = M_m^- v$ is then the minimum norm solution to (1.1) in R^n.

Theorem 4.3. Let K be a positive cone in R^m and let $A = M - N$ in $R^{m \times n}$ satisfy the conditions of Lemma 3.1, such that $NM_m^- K \subseteq K$. Let $A_m^- = M_m^-(I - NM_m^-)^{-1}$. Then $\varrho(NM_n^-) < 1$ if and only if $NA_m^- K \subseteq K$.

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