Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 15 (1974), No. 1-2, 155--156

Persistent URL: `http://dml.cz/dmlcz/142346`

1974 ACTA UNIVERSITATIS CAROLINAE - MATHEMATICA ET PHYSICA NO. 1-2 PAG. 155-156

A Remark on a Class of Universal Hill Functions

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Results closely related to the author's work [3] are surveyed in this paper. A more general class of universal hill functions and the approximation by them is considered.

Let $\omega(x)$ be an infinitely smooth rapidly decreasing function, $\Lambda(t)$ its Fourier transform, and $\Lambda(0) \neq 0$. In [3] the author is concerned with the approximation of the form $\sum c_k\omega((x/h - k)\eta(h))$ where $\eta(h)$ is a certain increasing function, $\eta(0) = 0$ (so-called Λ-admissible function, see Definition 4.1 [3]) in the one-dimensional Euclidean space R. Then for any $f \in W_2^\beta(R)$, $\varepsilon > 0$ and $\beta \geqq \alpha \geqq 0$, there exist coefficients c_k and a constant C independent of h such that

$$\left\| f(x) - \sum_{k=-\infty}^{\infty} c_k\omega\left(\left(\frac{x}{h} - k\right)\eta(h)\right)\right\|_\alpha \leqq C(\alpha, \beta, \varepsilon)\, h^{\beta-\alpha-\varepsilon}\|f\|_\beta . \qquad (1)$$

Therefore the approximation of this type is universal, i.e., for any approximated function f, we obtain the best possible order of approximation limited only by the smoothness of f. Analogously to the hill functions of Babuška [1] the function ω is called the universal hill function.

According to Babuška [1], and Fix and Strang [2] it is necessary for the Fourier transform of the hill function to have zeros at the points $2\pi j$ for all non-zero integers j. The multiplicity of these zeros determines the highest order of approximation attainable. The Fourier transform Λ of the universal hill function ω has — in general — no zeros at all. The quality of approximation is achieved only by the employment of the Λ-admissible function η.

According to the proof of Theorem 4.1 [3], the constant C in the error bound (1) is the sum of several constants. One of them is

$$C_1(\alpha) = \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} z^2(j, \alpha)\, |j|^{2\alpha}$$

(cf. (4.4) in [3]) where $z(j, \alpha)$ is a function satisfying the inequality

$$\left|\Lambda\left(\frac{x - 2\pi j}{\eta(h)}\right)\right| \leqq K(\alpha, \gamma)\, h^\gamma\, z(j, \alpha)$$

for all non-zero integers j, any $\gamma \geqq 0$, $0 < h < 1$ and $-\pi < x < \pi$ with some positive constant $K(\alpha, \gamma)$ (cf. (4.3) in [3]). It can be shown that if Λ has a zero at some of the points $2\pi j/\eta(h)$ then the constant C_1 (and finally also the constant $C(\alpha, \beta, \varepsilon)$) can be chosen less than in the general case. Moreover, the greater the multiplicity of the zero the less the constant C_1.

Apparently, the dependence of Λ (as well as of ω) on η is more complex in the class mentioned above, i.e. the class of universal hill functions the Fourier transform of which has zeros at some of the points $2\pi j/\eta(h)$. Let $\varphi(x, y)$ be a function of two real variables defined and continuous on a strip $R \times (0, y^\star)$ with some $y^\star > 0$. This function $\varphi(x, y)$ is supposed to be an infinitely smooth rapidly decreasing function of x for any fixed $y \in (0, y^\star)$. Let $\Phi(t, y)$ be the Fourier transform of $\varphi(x, y)$ with respect to x; y is considered to be a parameter. Further let $\Phi(0, y) = \lambda \neq 0$ independently of y. A Φ-admissible function η is defined analogously to Definition 4.1 [3]. Moreover, we require that $\eta(1) = y^\star$ and

$$\left| \Phi\left(\frac{x - 2\pi j}{\eta(h)}, \eta(h)\right) \right| \leqq K'(\alpha, \gamma)\, h^\gamma z(j, \alpha)\,.$$

The approximation by the function φ is universal since the statement of Theorem 4.1 [3] remains true, i.e., for any $f \in W_2^\beta(R)$, $\varepsilon > 0$ and $\beta \geqq \alpha \geqq 0$ there exist coefficients c_k' and a constant C' such that

$$\left\| f(x) - \sum_{k=-\infty}^{\infty} c_k' \varphi\left(\left(\frac{x}{h} - k\right) \eta(h), \eta(h)\right) \right\|_\alpha \leqq C'(\alpha, \beta, \varepsilon)\, h^{\beta-\alpha-\varepsilon} \|f\|_\beta\,.$$

Further, Theorems 4.2 and 4.3 [3] (concerning the choice of the function η and some computational aspects of the approximation by a universal hill function not having compact support) hold after obvious modifications.

Numerical experiments fully confirm the above statements. We solved a one-dimensional second order boundary value problem by the finite element method with the trial functions $\varphi((x/h - k)\, \eta(h), \eta(h))$. Putting $\Phi(\pm\, 2\pi j/\eta(h), \eta(h)) = 0$; $j = 1, \ldots, J$ for some $J > 0$ (with some of these zeros being possibly multiple) we obtained better results than in the case $\Phi(t, y) > 0$.

References

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