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A Method of BAZLEY-FOX Type for the Eigenvalues of the LAPLACE Operator

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For determining lower bounds to eigenvalues of the Laplacian in bounded domains of the Euklidian space $R_m (m \geqq 2)$ with boundary conditions of the first kind there will be given a method of intermediate problems. The method leads to an eigenvalue problem for matrices.

1. Let be $G \subset R_m$ $(m \geqq 2)$ a bounded domain with a piecewise smooth boundary Γ. We consider the eigenvalue problem

$$-\Delta u = -\sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} = \lambda u(x) \ (x = (x_1, \ldots, x_m) \in G), \ u(x) = 0 \ (x \in \Gamma). \quad (1')$$

With the selfadjoint extension A of the negative Laplacian in the Hilbert space $H = L_2(G)$ we describe (1′) by

$$Au = \lambda u \,. \quad (1)$$

The eigenvalues of A (each according to its multiplicity) let by designed by

$$0 < \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n \leqq \ldots \,. \quad (2)$$

In comparison with other methods [6], [7], [1], [2], [5], [4] our device will be applicable not only for special domains G, do not need special series of functions from the range of definition $D(A)$ and will lead to a finite matrix eigenvalue problem.

2. First we construct a bounded domain $G_0 \supset G$ with the boundary Γ_0. The only demand is, that the eigenvalue problem (1′) for the domain G_0 is solvable (we can take as G_0, for instance, a sphere or a cube of dimension m). Let be A_0 the corresponding selfadjoint operator in $H_0 = L_2(G_0)$, its eigenvalues

$$0 < \lambda_1^0 \leqq \lambda_2^0 \leqq \ldots \leqq \lambda_n^0 \leqq \ldots \quad (3)$$

and its orthonormed in H_0 eigenfunctions

$$u_1^0, u_2^0, \ldots, u_n^0, \ldots \,. \quad (4)$$

We have (see, for instance, [3])

$$\lambda_i^0 \leqq \lambda_i \ (i = 1, 2, \ldots) \,. \quad (5)$$

Instead of the operator A we now consider the operator

$$A^{(k)} = A_0 + (1 + k\Theta(x))\, I = A_0 + A', \quad I \leqq A' \leqq (1 + k)\, I \tag{6}$$

in H_0. Here are $k = \text{const} > 0$, I the identical operator and

$$\Theta(x) = \begin{cases} 1 & \\ & \text{for } x \in \\ 0 & \end{cases} \begin{matrix} G_0 - G \\ \\ G\,. \end{matrix}$$

Theorem 1. For any $k > 0$ the operators $A^{(k)}$ are symmetric and positive definite in H_0 and have a point spectrum only. If $\lambda_i^{(k)}$ denote the eigenvalues of $A^{(k)}$ $(i = 1, 2, \ldots)$, then hold

$$\lambda_i^{(k)} \leqq \lambda_i^{(k')} \leqq \lambda_i + 1 \quad (k \leqq k',\ i = 1, 2, \ldots),$$
$$\lim_{k \to \infty} \lambda_i^{(k)} = \lambda_i + 1 \quad (i = 1, 2, \ldots)\,. \tag{7}$$

Now we construct, as in the work [1], from (6) the intermediate operators

$$A_n^{(k)} = A_0 + A' P_n\,, \tag{8}$$

where P_n are the orthogonal projectors in the energetic Hilbert space H_A, (see, for instance, [5]) onto the span of linearly independent elements $p_1, p_2, \ldots, p_n \in H_{A'}$. If we choose $A' p_i = u_i^0$, that is

$$p_i(x) = u_i^0(x)\,(1 + k\Theta(x))^{-1}\,, \tag{9}$$

then holds

Theorem 2. The eigenvalues $\lambda_{i,n}^{(k)}$ of $A_n^{(k)}$ from (8) with fulfilling (9) are

(i) the eigenvalues of the symmetric matrix

$$\Lambda_n^0 + (S_n^{(k)})^{-1}\,, \tag{10}$$

where

$$\Lambda_n^0 = (\lambda_i^0 \delta_{ij})_{i,j=1}^n\,, \quad S_n^{(k)} = (s_{ij}^{(k)})_{i,j=1}^n\,,$$
$$s_{ij}^{(k)} = \frac{1}{1+k}\,\delta_{ij} + \frac{k}{1+k}\,(u_i^0, u_j^0)_{L_2(G)}\,,$$

(ii) the values $\lambda_{n+1}^0, \lambda_{n+2}^0, \ldots$.

These eigenvalues $\lambda_{i,n}^{(k)}$ are lower bounds to $\lambda_i + 1$ $(i = 1, 2, \ldots)$. Since for fixed i and n, $\lambda_{i,n}^{(k)}$ increases with k, this parameter k may be choosen as large as possible, especially $k \to \infty$.

In relation to the convergence of the method, we have

Theorem 3. For any $i = 1, 2, \ldots$ and any $\varepsilon > 0$ there exist a $k_0(\varepsilon)$ and a $n_0(k, \varepsilon)$ such that

$$0 \leqq \lambda_i + 1 - \lambda_{i,n}^{(k)} < \varepsilon \quad \text{for} \quad k > k_0(\varepsilon) \quad \text{and} \quad n > n_0(k, \varepsilon)\,.$$

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