R. U. Verma

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Inversion Integrals for the Integral Transforms Involving the Meijer's G-Function as Kernel

R. U. VERMA

Department of Mathematics, University of Cape Coast, Ghana
(Current Address: University of Tripoli, Tripoli, Libya)

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In the present work, two integral equations with Meijer's G-function in the kernel have been solved. The technique employed in finding the solutions of integral equations is somewhat different than the techniques used in [1], [2], [6]. Later some special cases are discussed.

I. Introduction

Some inversion integrals for integral equations involving either a Chebysheb or Legendre or Gegenbauer polynomial in the kernel are given [1], [2], [6]. The central theme of this paper is to find the solutions of two integral equations with the G-function as kernel. As a large variety of functions that occur frequently in the problems of analysis and mathematical physics are only specialised or limiting forms of the kernel used in the present integral equations, our problem may prove of general interest. Later some specialised cases are derived from the main results.

We recall the definition of Meijer's G-function [4, p. 207]:

$$G^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right]=G^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}(a_p)\\ (b_q)\end{matrix}\right]=$$

$$=(2xi)^{-1}\int_C\frac{\prod_1^m\Gamma(b_j+s)\prod_1^n\Gamma(1-a_j-s)}{\prod_{m+1}^q\Gamma(1-b_j-s)\prod_{n+1}^p\Gamma(a_j+s)}\,x^{-s}ds\,. \tag{1.1}$$

The following results will be needed in the development of the solutions.

A special case of [5, p. 212, (79)], taking $\mu=1$ and replacing α by zy, is expressed as:

$$\int_y^\infty G^{3,1}_{3,5}\left[z(xy)\,\middle|\,\begin{matrix}k-m-1/2-\nu/2,-k+m+1/2+\nu/2,-1\\ \nu/2-\lambda-m,\nu/2-\lambda+m,0,-\nu/2+\lambda+m,-\nu/2+\lambda-m\end{matrix}\right]dx=$$

$$=y\,G^{2,1}_{2,4}\left[zy^2\,\middle|\,\begin{matrix}k-m-1/2-\nu/2,-k+m+1/2+\nu/2\\ \nu/2-\lambda-m,\nu/2-\lambda+m,-\nu/2+\lambda+m,-\nu/2+\lambda-m\end{matrix}\right],$$

$$-R_e(k-m-1/2-\nu/2)>0\,. \tag{1.2}$$

Similarly, another special case of [5, p. 200, (97)], taking $\mu = 1$ and replacing α by zy, is given by the relation:

$$\int_0^y G_{3,5}^{2,2}\left[z(xy)\,\middle|\,\begin{matrix}-1, k-m-1/2-\nu/2, -k+m+1/2+\nu/2\\ \nu/2-\lambda-m, \nu/2-\lambda+m, -\nu/2+\lambda+m, -\nu/2+\lambda-m, 0\end{matrix}\right] dx$$

$$= y\, G_{2,4}^{2,1}\left[zy^2\,\middle|\,\begin{matrix}k-m-1/2-\nu/2, -k+m+1/2+\nu/2\\ \nu/2-\lambda-m, \nu/2-\lambda+m, -\nu/2+\lambda+m, -\nu/2+\lambda-m\end{matrix}\right],$$

$$R_e(\nu/2-\lambda-m) > -1, \qquad R_e(\nu/2-\lambda+m) > -1\,. \tag{1.3}$$

2. Theorem 1.

If the integral equations

$$\int_0^x G_{3,5}^{3,1}\left[z(xy)\,\middle|\,\begin{matrix}k-m-1/2-\nu/2, -k+m+1/2+\nu/2, -1\\ \nu/2-\lambda-m, \nu/2-\lambda+m, 0, -\nu/2+\lambda+m, -\nu/2+\lambda-m\end{matrix}\right]\cdot$$

$$\cdot\, f(y)\, dy = \Phi(x, z)\,,$$

$$-R_e(k-m-1/2-\nu/2) > 0\,, \tag{2.1}$$

and

$$\int_0^\infty \Phi(x, z)\, dx = \Psi(z) \tag{2.2}$$

exist, then the solution of (2.1) is given as

$$f(y) = y^{-1}\int_0^\infty G_{2,4}^{2,1}\left[zy^2\,\middle|\,\begin{matrix}k-m-1/2-\nu/2, -k+m+1/2+\nu/2\\ \nu/2-\lambda-m, \nu/2-\lambda+m, -\nu/2+\lambda+m, -\nu/2+\lambda-m\end{matrix}\right]\cdot$$

$$\cdot\,\Psi(z)\, dz\,. \tag{2.3}$$

Proof. Let us suppose that both $\Phi(x, z)$ and $\Psi(z)$ exist. Substituting the value of $\Phi(x, z)$ from (2.1) in (2.2), we obtain

$$\int_0^\infty \left\{\int_0^x G_{3,5}^{3,1}\left[z(xy)\,\middle|\,\begin{matrix}k-m-1/2-\nu/2, -k+m+1/2+\nu/2, -1\\ \nu/2-\lambda-m, \nu/2-\lambda+m, 0, -\nu/2+\lambda+m, -\nu/2+\lambda-m\end{matrix}\right]\cdot\right.$$

$$\left.\cdot\, f(y)\, dy\right\} dx = \Psi(z)\,.$$

By inverting the order of integration which permissible due to the convergence of the integrals involved, one can get

$$\int_0^\infty f(y)\left\{G_{3,5}^{3,1}\left[z(xy)\,\middle|\,\begin{matrix}k-m-1/2-\nu/2, -k+m+1/2+\nu/2, -1\\ \nu/2-\lambda-m, \nu/2-\lambda+m, 0, -\nu/2+\lambda+m, -\nu/2+\lambda-m\end{matrix}\right]\cdot\right.$$

$$\left.\cdot\, x\, \mathrm{d}x\right\} dy = \Psi(z)\,.$$

From (1.2), we arrive at

$$\int_0^\infty G_{2,4}^{2,1}\left[zy^2 \middle| \begin{matrix} k-m-1/2-\nu/2, -k+m+1/2+\nu/2 \\ \nu/2-\lambda-m, \nu/2-\lambda+m, -\nu/2+\lambda+m, -\nu/2+\lambda-m \end{matrix}\right] yf(y)\,dy = \\ = \Psi(z)\,.$$

This is now in the form of the generalised Hankel transform [7, (1.1)], and hence inverting by applying [5, p. 5, (1)], we can find the solution (2.3).

3. Applications

(i) With $\lambda = -m$, our theorem leads to

Corollary 1. If the integral equations

$$\int_0^x G_{3,5}^{3,1}\left[[z(xy) \middle| \begin{matrix} k-m-1/2-\nu/2, -k+m+1/2+\nu/2, -1 \\ \nu/2, \nu/2+2m, 0, -\nu/2, -\nu/2-2m \end{matrix}\right] f(y)\,dy = \\ = \Phi_1(x,z)\,, \qquad -R_e(k-m-1/2-\nu/2) > 0\,, \tag{3.1}$$

and

$$\int_0^\infty \Phi_1(x,z)\,dx = \Psi_1(z)\;, \tag{3.2}$$

exist, then the solution of (3.1) is given by

$$f(y) = y^{-1}\int_0^\infty G_{2,4}^{2,1}\left[zy^2 \middle| \begin{matrix} k-m-1/2-\nu/2, -k+m+1/2+\nu/2 \\ \nu/2, 2m+\nu/2, -\nu/2, -2m-\nu/2 \end{matrix}\right] \Psi_1(z)\,dz, \tag{3.3}$$

where the generalised Hankel kernel in (3.3) is introduced by Bhise [3].

(ii) When $\lambda = -m$, $k+m = 1/2$ and using [4, p. 216, (3)] the theorem leads to

Corollary 2. If the integral equations

$$\int_0^x G_{3,5}^{3,1}\left[z(xy) \middle| \begin{matrix} -2m-\nu/2, 2m+\nu/2, -1 \\ \nu/2, \nu/2+2m, 0, -\nu/2, -\nu/2-2m \end{matrix}\right] f(y)\,dy = \Phi_2(x,z)\,, \\ -R_e(-2m-\nu/2) > 0\,, \tag{3.4}$$

and

$$\int_0^\infty \Phi_2(x,z)\,dx = \Psi_2(z)\,, \tag{3.5}$$

exist, then the solution of (3.4) is expressed as

$$f(y) = y^{-1}\int_0^\infty J^\nu(2yz^{1/2})\,\Psi_2(z)\,dz\,, \tag{3.6}$$

where $J_\nu(x)$ is Bessel function.

4. Theorem 2.

If the integral equations

$$\int_x^\infty G_{3,5}^{2,2}\left[z(xy)\left|\begin{matrix} -1, k-m-1/2-\nu/2, -k+m+1/2+\nu/2 \\ \nu/2-\lambda-m, \nu/2-\lambda+m, -\nu/2+\lambda+m, -\nu/2+\lambda-m, 0\end{matrix}\right.\right] \cdot$$

$$\cdot f(y)\,dy = \Phi_3(x, z)\,,$$

$$R_e(\nu/2-\lambda-m) > -1\,, \qquad R_e(\nu/2-\lambda+m) > -1\,, \tag{4.1}$$

and

$$\int_0^\infty \Phi_3(x, z)\,dx = \Psi_3(z)\,, \tag{4.2}$$

exist, then the solution of (4.1) is expressed as:

$$f(y) = y^{-1}\int_0^\infty G_{2,4}^{2,1}\left[zy^2\left|\begin{matrix} k-m-1/2-\nu/2, -k+m+1/2+\nu/2 \\ \nu/2-\lambda-m, \nu/2-\lambda+m, -\nu/2+\lambda+m, -\nu/2+\lambda-m\end{matrix}\right.\right] \cdot$$

$$\cdot\ \Psi_3(z)\,dz\,. \tag{4.3}$$

The proof of this theorem easily follows by proceeding on the lines of Theorem. 1.

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