Hilbert Transform and its Geophysical Applications

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Possible applications of Hilbert integral transform in geophysics are summarized. The main attention is devoted to seismology.

1. Introduction

This paper presents some properties and possible applications of the Hilbert integral transform. Similarly to the Fourier transform, the Hilbert transform technique is applicable in most of geophysical branches, namely in seismology, magnetics, gravimetry, ionospheric sounding, prospection geophysics, etc. There is a large variety of possible applications in pure theory and instrumental problems as well as in inverse problems and data processing. Hilbert transform, applied quite sporadic up to this time, can now be used routinely, due to the rapid development of computational facilities. Only some possible applications will be mentioned here.

2. Hilbert transform properties and applications

2.1. Definition

Hilbert transform of the function $f(t)$ will be denoted $g(t)$; $(g(t) = H\{f(t)\})$. It is given by the following formula (see, e.g., [1]):

$$g(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(s)}{s-t}\, ds\,, \qquad f(t) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{g(s)}{s-t}\, ds\,. \tag{1}$$

The integrals are the Cauchy principal values (V.P.):

$$\begin{aligned} g(t) &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{t-\varepsilon} \frac{f(s)}{s-t}\, ds + \int_{t+\varepsilon}^{\infty} \frac{f(s)}{s-t}\, ds \right) = \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{f(t+s) - f(t-s)}{s}\, ds\,. \end{aligned} \tag{2}$$

To ensure the existence of these integrals, 'Lipschitz conditions' must be satisfied by $f(t)$ [12]. We shall not deal with them here, however, most geophysical quantities satisfy them.

Hilbert transform and its inversion are given by the integrals (1) of convolution type. Therefore they may be written as follows:

$$g(t) = -\frac{1}{\pi t} \star f(t) , \qquad f(t) = \frac{1}{\pi t} \star g(t) . \tag{3}$$

The relation of the Hilbert transform to the Fourier transform is of great importance. Let us denote $F(\omega)$ the Fourier spectrum of $f(t)$. The spectrum of $-1/\pi t$ being i sign ω, the spectrum of the Hilbert transform may be easily written,

$$f(t) \supset F(\omega) \Rightarrow g(t) \supset i \operatorname{sign}(\omega) F(\omega) . \tag{4}$$

The following expression of the Hilbert transform by means of the Fourier integral will be used later:

$$g(t) = -\frac{1}{\pi} Im \int_0^\infty F(\omega) e^{i\omega t} d\omega . \tag{5}$$

2.2. Numerical evaluation

Two possibilities of Hilbert transform numerical evaluation are evident from the given formulas. The first one — the numerical evaluation of the convolution integral (2), i.e., in the time-domain. The second one — using the Fourier transform (5). Both these methods have been programmed for a computer. The time-domain calculations are prefered for a very detailed analysis of the Hilbert transform, especially in the vicinity of such points,where $f(t)$ or its derivative are discontinuous. However, there are many geophysically important functions such as seismograms, etc. that are smooth enough. In such cases, Hilbert transform calculations by means of the Fourier integral are quite satisfacfactory, rather more economical using the Fast Fourier Transform (*FFT*) algorithm. For example, time-domain calculation for 1000 points takes a few minutes on *IBM* 7040 computer while *FFT* calculation takes a few seconds of computer time only.

2.3. Examples

Let us present here some simple examples of the Hilbert transform pairs, see Figs. 1–4. All of them were calculated in the time-domain. Note that the function $f(t)$ non-zero in (a, b) possesses the Hilbert transform $g(t)$ that is non-zero in $(-\infty, +\infty)$, in general.

Roughly speaking, the Hilbert transform $g(\tau)$ represents certain integral measure of a rate of change of the function $f(t)$ in the neighbourhood of the point τ. In other words, increasing rate of change of $f(t)$ near $t = \tau$, the absolute value of $g(\tau)$ increases. Just at the discontinuity $t = \tau$ of $f(t)$, $g(\tau)$ reaches logarithmically infinity (Fig. 1). The Hilbert transform of a continuous function (even with certain points of first derivative discontinuity) is everywhere finite (Fig. 2). There are some examples of smooth functions and

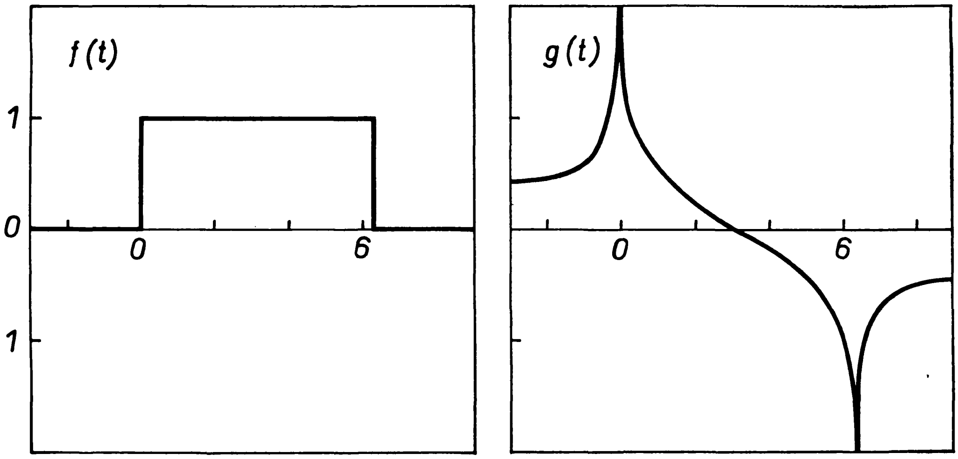


Fig. 1. Hilbert transform $g(t)$ of the rectangular function $f(t)$

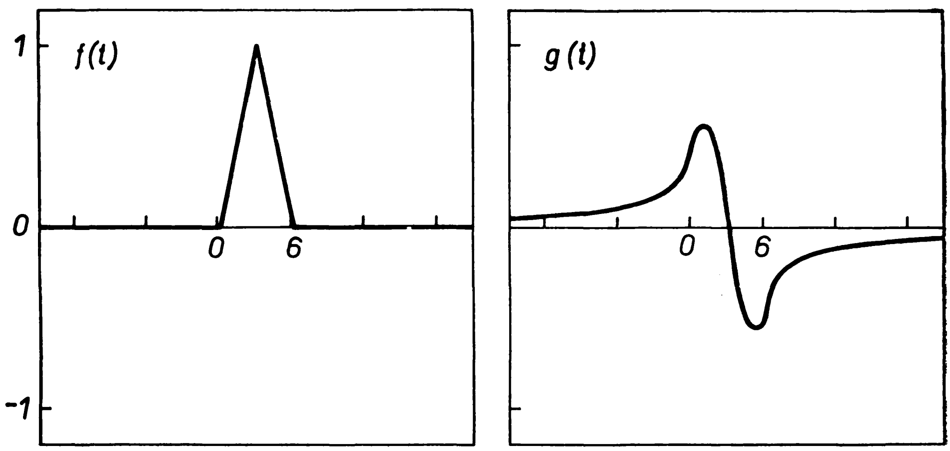


Fig. 2. Hilbert transform $g(t)$ of the triangular function $f(t)$

their Hilbert transforms in Fig. 3, 4. In such cases, zero points of $g(t)$ correspond to the minima and maxima of $f(t)$, minima and maxima of $g(t)$ correspond to the most rapid changes of $f(t)$.

2.4. Relation of the Hilbert transform to analytic functions of complex variable

A great number of geophysical applications consists in close relation of the Hilbert transform to analytic functions of complex variable. Let us consider the complex function $\varphi(z) = u(x, y) + i\, v(x, y)$ which is analytic in some region of a complex plane and decreasing to zero for $|z| \to \infty$. In geophysics, there are many examples of two-dimensional

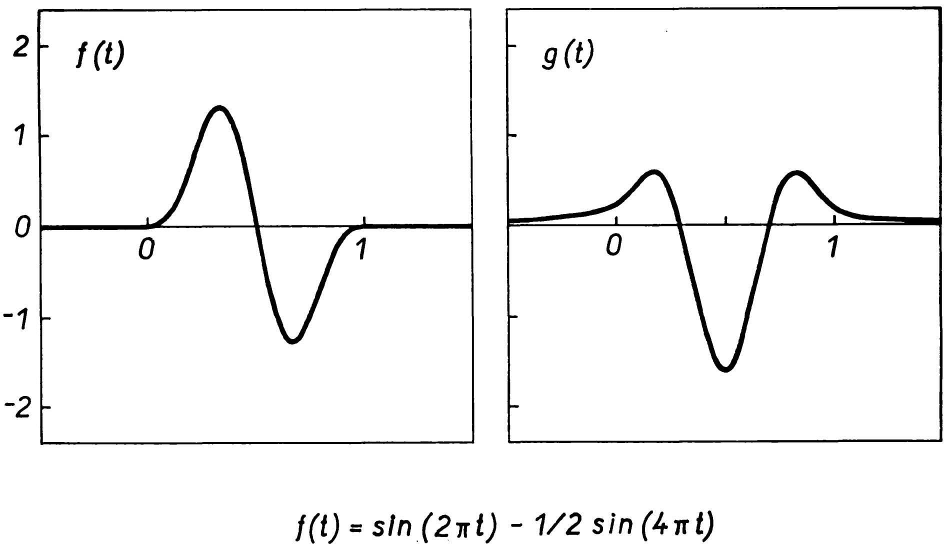


Fig. 3. Hilbert transform $g(t)$ of the function $f(t)$ given by

$f(t) = \sin 2\pi t — 0.5 \sin 4\pi t$ for $0 \leq t \leq 1$,
$f(t) = 0$ for $t < 0, t > 1$

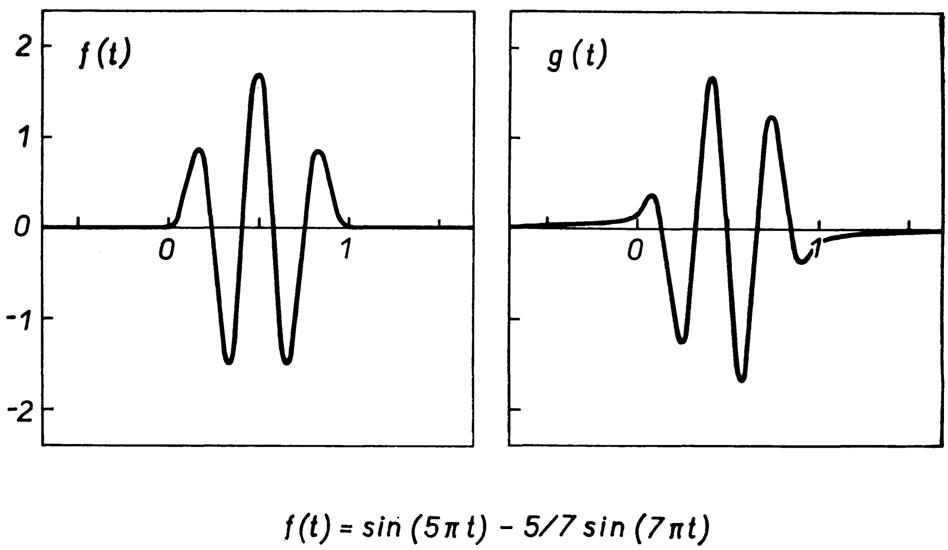


Fig. 4. Hilbert transform $g(t)$ of the function $f(t)$ given by

$f(t) = \sin 5\pi t — 5/7 \sin 7\pi t$ for $0 \leq t \leq 1$,
$f(t) = 0$ for $t < 0, t > 1$

fields having such "complex potential" [12]; namely a steady flow of an incompressible liquid, an electrostatic field, a stationary electromagnetic field outside sources and currents, two-dimensional gravitational field outside masses, etc.

It is well known that the imaginary part $v(x, y)$ of the analytic function $\varphi(z)$ may be derived from $u(x, y)$; in a special case of a function $\varphi(z)$ that is analytic in an upper half-plane, u and v along the real axis (i.e., $u(x, 0)$ and $v(x, 0)$) form the Hilbert transform pair. Consequently, the function analytic in the upper half-plane (and decreasing to zero for $|z| \to \infty$) may be reconstructed everywhere from the values of its real part along the real axis as well as from its imaginary part along the real axis.

It means, for example, that on the Earth's surface, the vertical and horizontal components of magnetic anomaly intensity are the Hilbert transform pair. A measurement of one component only is sufficient, the second one can be calculated from the first one. This applies of course to such medium, the properties of which depend on two coordinates only (three dimensional structures greatly elongated in one direction) [11].

2.5. Analytic signal

Two real functions of real variable, namely $f(t)$ and $g(t)$, forming the Hilbert transform pair can be considered as the real and imaginary part of an analytic function, so called analytic signal [1]

$$\hat{f}(t) = f(t) - ig(t) . \tag{6}$$

From the Fourier spectrum $F(\omega)$ of the function $f(t)$, spectrum of the analytic signal may be easily obtained:

$$\hat{f}(t) \supset \begin{cases} 2F(\omega) & \text{for } \omega > 0 , \\ F(\omega) & \text{for } \omega = 0 , \\ 0 & \text{for } \omega < 0 . \end{cases} \tag{7}$$

It contains no negative-frequency components.

A similar situation arises in the frequency-domain. Function $f(t)$ being zero for $t < 0$ possess a spectrum that is an "analytic signal in the frequency-domain". In other words, real and imaginary part of this spectrum are the Hilbert transform pair.

$$Im\, F(\omega) = - H\{Re\, F(\omega)\} . \tag{8}$$

This property is of great importance, especially in the theory of causal systems (see later).

2.6. Other properties of the Hilbert transform

Let us note some other properties of the Hilbert transform.

$$\int_{-\infty}^{+\infty} f^2(t)\, dt = \int_{-\infty}^{+\infty} g^2(t)\, dt , \tag{9}$$

$$\int_{-\infty}^{+\infty} f(t)\, g(t)\, dt = 0 , \tag{10}$$

$$\frac{dg}{dt} = H\left\{\frac{df}{dt}\right\} , \tag{11}$$

$$g(t + a) = H\{f(t + a)\}, \tag{12}$$

$$g(at) = H\{f(at)\}. \tag{13}$$

Formula (9) will be used later in the signal energy calculation (see 2.10.2). Formula (10) expresses an orthogonality property of the function and its Hilbert transform. Formula (11) is very important. It states that a derivative of the Hilbert transform equals to the Hilbert transform of the derivative. For example, neither the Fourier, nor the Laplace transform possess such a property. Consequently, the function $f(t)$ being the real solution of an ordinary homogeneous differential equation, another, linearly independent, solution is obtained automatically as the Hilbert transform of $f(t)$ (if it exists, of course). Simple examples of such Hilbert transform pairs that form two linearly independent solutions of a differential equation of second order are ($\exp(i\omega t)$, $\exp(-i\omega t)$), ($\sin t$, $\cos t$). Formulas (12) and (13) express the shifting and similarity properties of the Hilbert transform.

2.7. Relation of the Hilbert transform to singular integral equations

Let us note the close resemblence of the Hilbert transform to the so-called singular integral of Cauchy type [8],

$$p(z) = \frac{1}{2\pi i} \int_L \frac{q(\tau)}{\tau - z}\, d\tau, \qquad z \to \xi \in L. \tag{14}$$

From the properties of singular integrals of Cauchy type, many properties of the Hilbert transform follow. The Hilbert transform inversion serves as a typical example. Remember the definition formulas for the Hilbert transform and its inversion, i.e., (1). It is evident that the inversion formula represents the solution of a simple singular integral equation, namely the equation that defines the Hilbert transform. Even more general singular integral equations may be solved by the Hilbert transform technique, e.g., the following:

$$F(z) = a\varphi(z) + \frac{b}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z}\, d\tau, \qquad z \to \xi \in L. \tag{15}$$

Such equations appear quite frequently in the plane elasticity as well as in the inverse problems of two-dimensional potential theory. F usually represents certain measurable physical quantity, say a "generalized" intensity due to the "generalized" density distribution φ that is to be found. The inverse problem for a two-dimensional magnetic anomaly due to the surface distribution of electric currents is a good example [11]. In this case, the solution may be obtained immediately, applying the inverse Hilbert transform to a certain quantity that characterizes the anomalous magnetic field along the Earth's surface.

Up to now, only the physical quantities conjugated (by the Hilbert transform) along the Earth's surface — that is along the real axis — have been discussed. Practically, there is no loos of generality. In fact, the Hilbert transform properties for a circular region are very similar to that of half-plane. Another generalization may be simply reached using the conformal transformation of regions outside sources to the half-plane or the circular region.

2.8. Hilbert transform applications in the wave propagation problems

2.8.1. Construction of waveforms of elementary waves

The Hilbert transform has played a useful role in radio engineering for a long time. In elastic wave propagation, however, its applications have not been so common until recently. In the theory of seismic wave propagation, the Hilbert transform can be used mainly to construct the waveforms of individual waves in asymptotic methods (such as the ray methods). The wave field which is generally very complicated may be often decomposed into individual "elementary" waves. In simpler situations, such as in reflection seismics, these elementary waves may be described by a simple formula $U(t) = Af(t-\tau)$, where $f(t)$ is the time-function of the source, A and τ are real. Here A and τ denote the amplitude and the arrival time of the wave under consideration. They can be determined by well-known methods. The above formula may be used only in situations where the waveform does not change as a signal progresses. In many important applications, however, this assumption cannot be used; the waveform changes with time increase. Then the above formula for $U(t)$ must be generalized. The initial and boundary conditions require A and τ complex. When A is complex and τ real, U may be written as a real part of the analytic signal; i.e., as a combination of the function f and its Hilbert transform g,

$$U(t) = |A| \, . \, f(t - \tau \, ; \, \arg A) \, ,$$
$$f(t; x) = f(t) \cos x + g(t) \sin x \, . \tag{16}$$

The simple examples are the super-critically reflected waves and refracted waves beyond the caustic.

The situation becomes a little more complicated for τ complex. In this case, U may be written as a combination of the real and imaginary part of $g(t - \tau)$. In this special case, however, g given by (1) is complex due to the complex τ. The resulting formulas are straight-forward. The examples are the signals propagating in absorbing media, the inhomogeneous waves, the screened waves, etc. It should be noted that in the frequency-domain the complex τ generates an exponential damping multiplier exp $(-\alpha\omega)$.

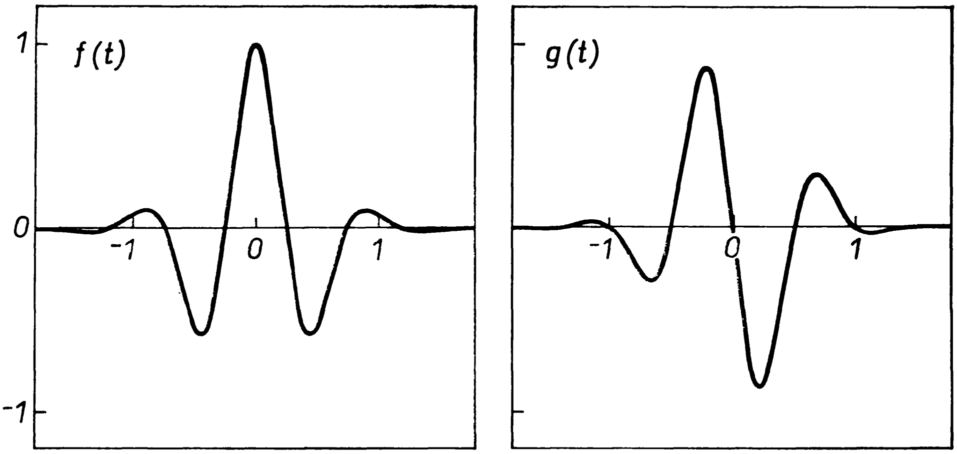
2.8.2. Construction of ray-theoretical seismograms

The above mentioned properties of the Hilbert transform can be used to construct ray-theoretical seismograms. Under the ray-theoretical seismograms we understand the theoretical seismograms constructed on the principle of the ray theory.

The procedure is as follows: (i) the wave field is decomposed into individual waves; (ii) the elementary seismograms of these individual waves are computed; (iii) the elementary seismograms are summed up to give the desired theoretical seismogram,

$$W(t) = \sum_{(i)} u\,(t) \, . \tag{17}$$

Each elementary wave $u\,(t)$ is described by three parameters, namely the amplitude $|A|$, phase $\chi = \arg A$, arrival time τ (see (16)). These parameters can be computed without serious difficulties using the ray theory, even for very general media. The values of $|A|$,



$$f(t) = e^{-(\frac{\pi}{2} t)^2} \cos(2\pi t)$$

Fig. 5. Hilbert transform $g(t)$ of the function $f(t)$ given by (18) (with $\gamma = 4, w = 2\pi, \nu = 0$)

χ, τ being known for each elementary wave, the computation of elementary seismogram is straightforward. Of course, we must also know (see (16)) the time function of the source, and we need to compute its Hilbert tranform. The computation of the theoretical seismograms can be rather faster when we use special time function of the source, namely:

$$f(t) = \exp(-\omega^2 t^2/\gamma^2) \cos(\omega t + \nu) . \qquad (18)$$

Choosing proper parameters, this three-parametric time function can simulate a large variety of real wavelets. In this case, the following approximate formula for the Hilbert transform holds:

$$g(t) \sim -\exp(-\omega^2 t^2/\gamma^2) \sin(\omega t + \nu) . \qquad (19)$$

Fig. 5 gives an example of $f(t)$ given by (18) (with $\gamma = 4$, $\omega = 2\pi$, $\nu = 0$) and $g(t)$. The difference between $g(t)$ found by the time-domain integration and by the approximate formula (19) is too small to be seen in the given scale; both the curves coincide very well.

Simple approximate formula for $f(t, \chi)$ is then obtained, using (19):

$$f(t; \chi) \sim \exp(-\omega^2 t^2/\gamma^2) \cos(\omega t + \nu + \chi) . \qquad (20)$$

Using this approximate expressions, the computation of elementary seismograms does not practically need more computation time than the computation of the time-function of the source.

In inhomogeneous media with curved interfaces an infinite number of elementary waves can arrive at the receiver within even a narrow time window. The problem of constructing ray-theoretical seismograms in inhomogenous media with curved interfaces has not been solved quite satisfactorily. The problem becomes considerably simpler in the medium composed of plane parallel homogeneous layers, when we do not take into account the *PS* and *SP* conversion. The number of waves which arrive at the receiver within a finite time window is always finite in this case. The situation is simplified by the fact that elementary waves can be grouped into the families of kinematically and dynamically analogous waves. This considerably decreases the number of waves which must be taken

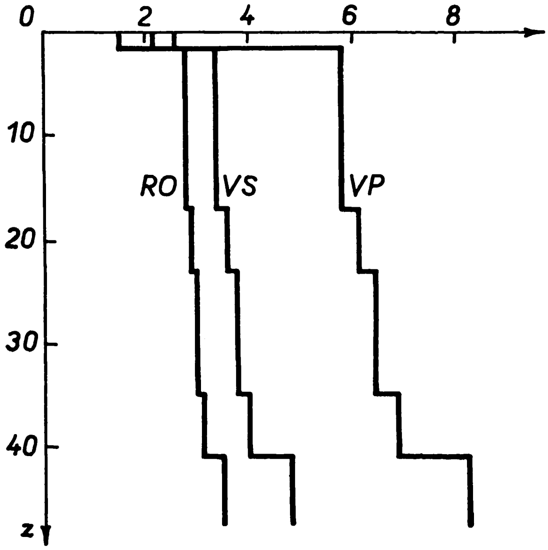


Fig. 6. Velocity — depth and density — depth graphs for computations of theoretical seismograms presented in Figs. 7–9

into account. A detailed analysis of this problem was given by Petrashen and Vavilova [9]. However even in this case the number of groups remains very high, mainly when the epicentral distance is large and/or thin layers are present in the medium and/or when we are interested in a long interval of the theoretical seismogram.

Examples of ray-theoretical seismograms are given in Figs. 7–9. The theoretical seismograms were constructed for a simple model of a five-layer Earth's crust, shown in Fig. 6. The theoretical seismograms are self-explanatory. It should be noticed that the phase denoted by arrows at the epicentral distances of 40 and 50 km correspond to the wave reflected from a sharp interface at a depth of 1.5 km. To simplify the picture, these phases are plotted 10-times weaker with respect to reality. The seismograms were calculated for three different time-functions of the source. All of them are given by the formula (18), where $\gamma = 4$, $\nu = 0$. However in Fig. 7, 8, and 9, three different values of the parameter ω were used, namely $\omega = 4\pi$, $\omega = 8\pi$, $\omega = 16\pi$, respectively. In other words, the prevailing frequency of the source time-function was approximately 2, 4, and 8 cps, respectively. All possible types of compressional waves arriving within 15 s time interval

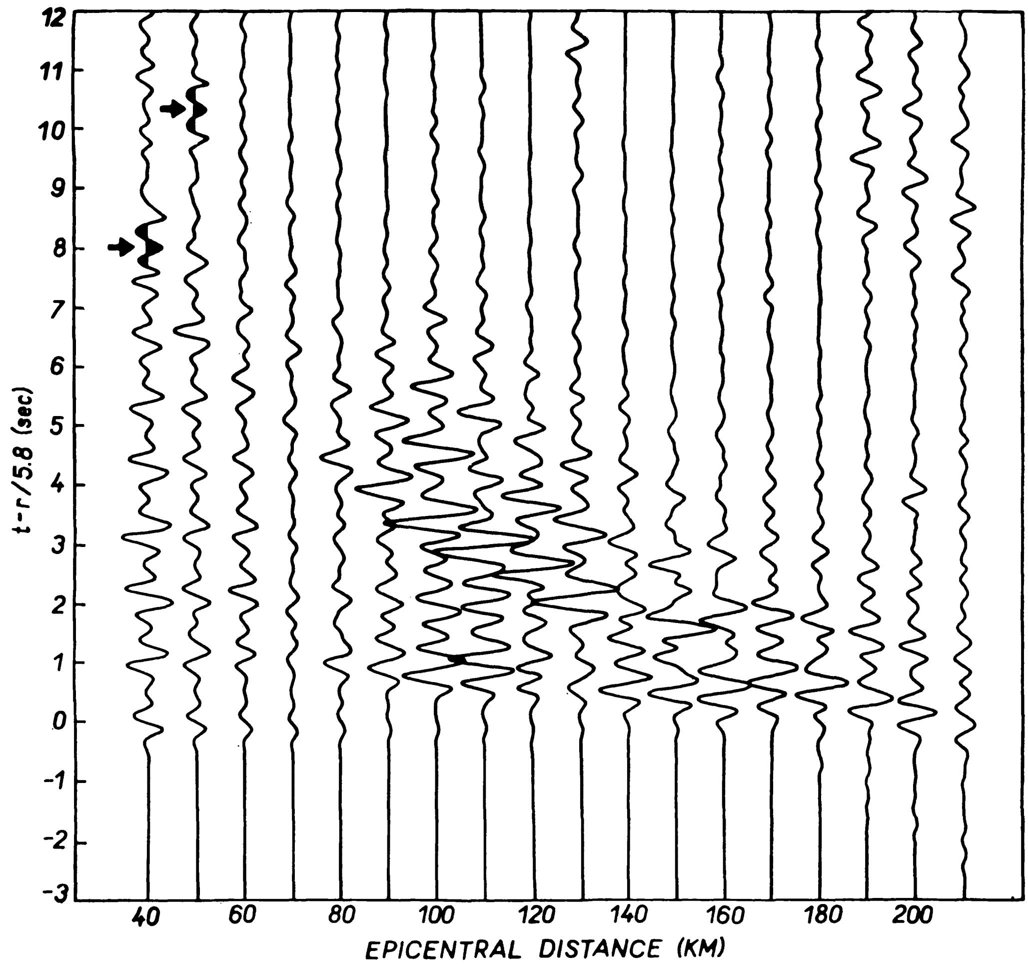


Fig. 7. Ray — theoretical seismogram for the model of the Earth's crust given in Fig. 6. Time — function of the source is given by (18), where $\gamma = 4$, $\nu = 0$, $\omega = 4\pi$. See details in text

are considered. Both the number of elementary waves and of kinematic groups increase quickly, increasing the epicentral distance. For the epicentral distance of 40 km, the number of kinematic groups is 215 and the number of elementary waves is 1522. For the epicentral distance of 200 km, however, the number of kinematic groups is 2495 and the number of elementary waves is 353 706.

We believe, the ray-theoretical seismograms may help in the interpretation of deep seismic sounding measurements of the Earth's crust and of the uppermost mantle as well as in refraction seismics.

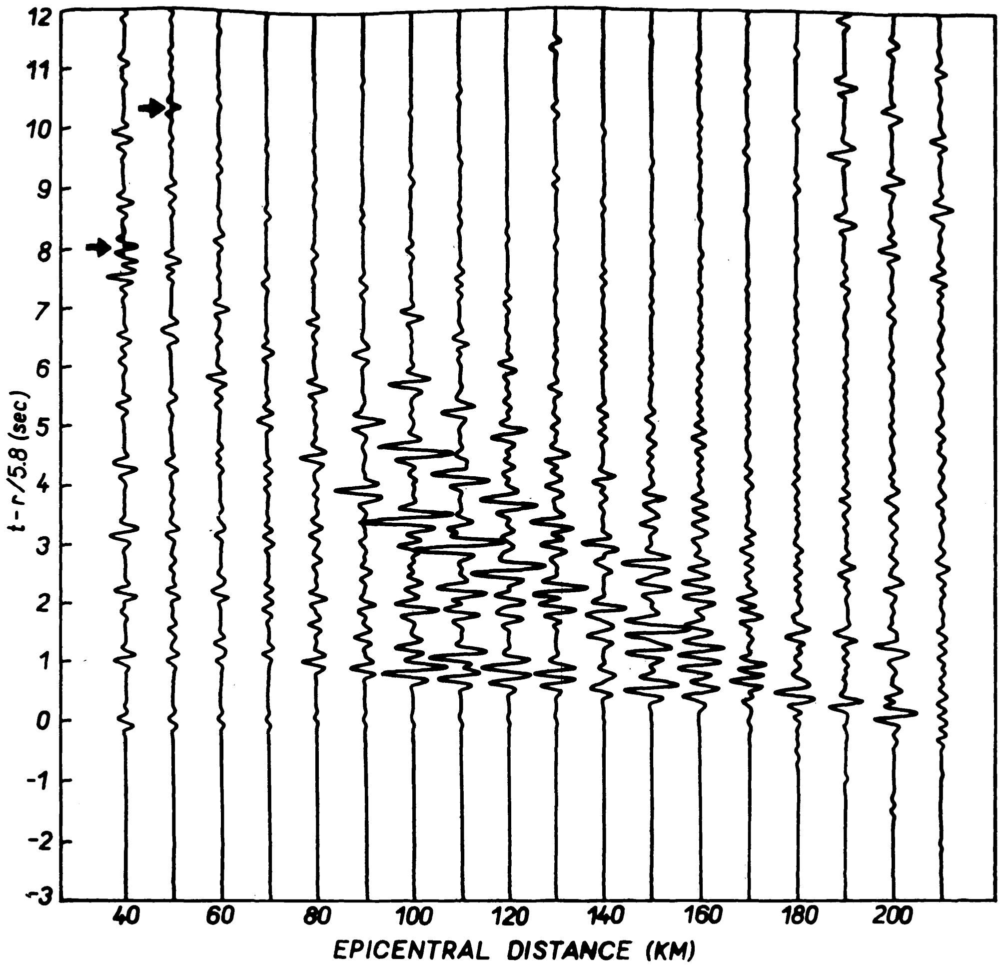


Fig. 8. Ray — theoretical seismogram for the model of the Earth's crust given in Fig. 6. Time — function of the source is given by (18), where $\gamma = 4, \nu = 0, \omega = 8\pi$. See details in text

2.9. Hilbert transform applications to causal systems

In geophysics, we meet "causal systems" everyday. Causal systems are such systems in which the effect does not precede its cause. In causal systems, the Hilbert transform plays an important role.

Consider the frequency responses of linear systems, e.g. seismographs. It follows from the causality of linear, minimum phase, systems that the real and imaginary responses are a Hilbert transform pair. A similar conclusion is also reached for the attenuation (logarithmic amplitude) and the phase response. Hence, when we know the amplitude response, we can compute the phase response as the Hilbert transform of the attenuation

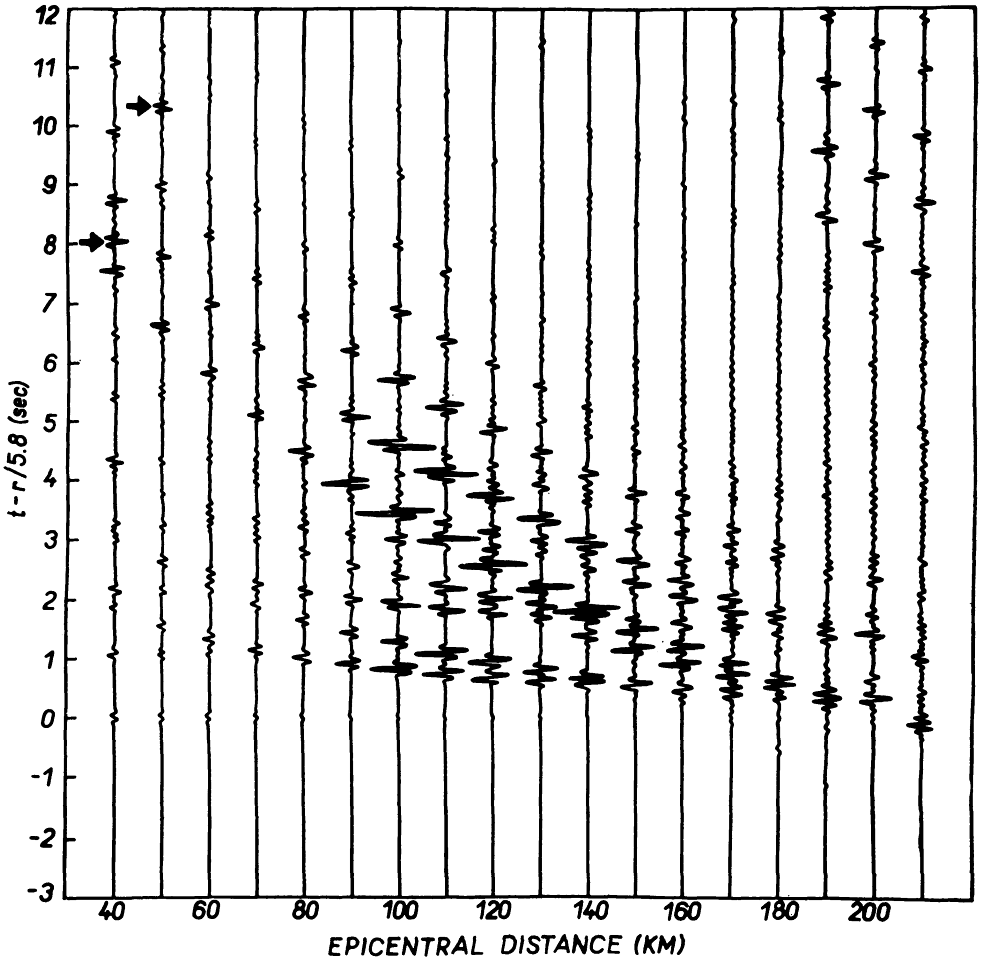


Fig. 9. Ray — theoretical seismogram for the model of the Earth's crust given in Fig. 6. Time — function of the source is given by (18), where $\gamma = 4$, $\nu = 0$, $\omega = 16\pi$. See details in text

response, and vice versa. The mentioned property of the Hilbert transform has probably most common applications in various branches of science and engineering [2]. It should be also mentioned that the analytic signals play a key role also in the study of nonlinear systems [3].

An example of a causal system of large geophysical importance is a viscoelastic Boltzmann solid (for details see [5]). It is generally accepted that the causal relations hold between strain and stress. We can say that the deformation at the present time is due to forces that acted in the past, and not in the future. This concept is often reffered to as the axiom of nonretroactivity. It follows from the causality that the real and imaginary parts of the spectrum of the creep function are the Hilbert transform pair. The same conclusion is obtained for the spectrum of the relaxation function. Therefore, it is not possible to generate the models of viscoelastic solid in an arbitrary way, the causality must be observed. It should be stressed that these conclusions are applicable to general viscoelastic media; the Maxwell, Voight and standard linear solid being of course included.

The resulting relations are rather important in the wave-absorption studies. The frequency dependence of the phase velocity (the dispersion of velocity) can be computed from the known frequency dependence of absorption, and vice versa. The corresponding formulas are known as "dispersion relations" or "absorption-dispersion relations". These relations have been broadly used in many branches of physics.

2.10. Hilbert transform applications to data processing

2.10.1. Envelopes, instantaneous frequency

Let us briefly discuss Hilbert transform applications to data processing [1].

From the Hilbert transform of a signal, e.g., of a seismogram, its envelope may be simply derived as the modulus of the analytic signal,

$$\begin{aligned} \hat{f}(t) &= A(t) \exp (i\Phi(t)) \,, \\ A(t) &= \sqrt{(f^2(t) + g^2(t))} \,, \\ \Phi(t) &= \tan^{-1} (-g(t)/f(t)) \,. \end{aligned} \tag{21}$$

The phase $\Phi(t)$ of the analytic signal can be numerically differentiated to give instantaneous frequency at an arbitrary moment:

$$\omega(t) = \frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} \,. \tag{22}$$

2.10.2. Calculation of energy of signal

How to use the envelopes in data processing? The first possibility comes from the following formula which expresses the energy E of the signal $f(t)$ having the spectrum $F(\omega)$:

$$E = 2 \int_0^{\infty} |F(\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{+\infty} f^2(t) \, dt = \pi \int_{-\infty}^{+\infty} A^2(t) \, dt \,. \tag{23}$$

It is evident that the signal energy may be calculated in the frequency-domain as well as in the time-domain. In the time-domain, there are two formulas, namely that one using the function $f(t)$ itself and another one using its envelope $A(t)$. It is possible that the latter will be less influenced by random noise than the former.

It is of great practical importance to calculate and to compare the energy of various parts of the record, e.g., the energy of a principal impulse and that of the coda following it. In this way, so-called "complexity" of record could be described quantitatively using Hilbert transform technique. The notion of complexity is frequently used in the analysis of source properties (e.g., of nuclear explosions, multiple events, etc.) as well as in the analysis of medium properties and building responses, etc.

2.10.3. Group correlation of seismic waves

As B. J. Gelchinskiy showed, the envelopes may also be used in the group correlation of seismic waves (besides other well known methods). The figures in his paper [6] indicate that quite clear separation of the individual wave groups may be observed on the envelopes even in the case of complex interference patterns on the records; these groups may be correlated.

This important property of envelopes will be applicable probably to the detailed nter pretation of teleseismic events, i.e., to the phase detection and separation.

2.10.4. Hilbert transform applications to the dispersion measurements

Let us now pay attention to the Hilbert transform applications in the signal dispersion measurements. Two methods described in detail in [4] will be discussed.

The first method of group velocity determination consists in the following procedure. As well as in the classical "hand" method, the group velocity U is calculated using the simple formula $U = \Delta/t$, where Δ is the epicentral distance, t is the travel time. The group velocity corresponds to the instantaneous frequency at the time t. Whereas in the classical "hand" method the instantaneous frequency (or period) is determined directly from the record, here it may be calculated by the numerical differentiation of the phase curve (see (22)). From this point of view, the described method is the numerical and continuous analogy of the classical one. Of course, it possesses much of the well known disadvantages of the classical method (e.g., it is not applicable to the records containing several modes; it becomes incorrect where the wave-train form varies abruptly, etc.). The example of this approach to the *PL*-waves dispersion measurements is given in [10].

Many of the difficulties of the method described above can be removed using the method of multiple band-pass filtration [4]. Note that the method was further developed as the so-called frequency-time analysis or *FTAN* method in [7]. The basic idea is as follows: Narrow band-pass filter is applied to the seismogram consecutively at various frequencies. Using Hilbert transform technique, the envelopes are constructed and plotted (in numerical form) against the time and frequency. It can be shown that the envelope maxima determine the group velocity curve. The examples in the cited papers [4, 7] show that this method gives satisfactory results (dispersion curves) even if several modes are present in the record. In such a way, individual modes can be separated.

It is to be emphasized that not only the dispersive signals such as surface waves, *PL* waves or magnetosphere oscillations can be processed by this technique. Also the power time-frequency variation of body wave signals can be studied by the same method. The examples of the analysis of the Earth's core seismic phases *PKP* are presented in [7]. In the paper [13], a similar method is applied in the interpretation of strong earthquake ground motion.

3. Conclusion

There is a great number of important geophysical applications of the Hilbert integral transform, in theory as well as in instrumental problems and in data processing. Some of them have been discussed in this paper.

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