Acta Universitatis Carolinae. Mathematica et Physica

Ladislav Beran

Reflections and coreflections in generalized orthomodular lattices

*Acta Universitatis Carolinae. Mathematica et Physica*, Vol. 16 (1975), No. 2, 57--61

Persistent URL: http://dml.cz/dmlcz/142368

# Reflection and Coreflection in Generalized Orthomodular Lattices

L. BERAN
Department of Algebra, Charles University, Prague

*Received 4 March 1974*

The object of this paper is to show that the concept of reflection and coreflection can be used to advantage when investigating orthomodular lattices. In addition, the commutator sublattice of a generalized orthomodular lattice is considered, and some of its characteristic properties are presented.

In what follows, by an *allele* one means a quotient $b/a$ of a lattice $\mathscr{L}$ such that there exists a quotient $d/c$ of $\mathscr{L}$ which is projective with $b/a$ and which satisfies $b \leqq c$ or $a \geqq d$. In this case we write $b/a \S d/c$. The set of all the alleles of $\mathscr{L}$ is denoted by $\mathsf{A}(\mathscr{L})$. It is known [1] that the relation $\beta$ defined on a relatively complemented lattice $\mathscr{L}$ by

$$a \equiv b(\beta) \Leftrightarrow \{([m, n] \subset [a \wedge b, a \vee b] \,\&\, n/m \S q/p) \Rightarrow m = n\}$$

is a congruence relation of the lattice $\mathscr{L}$. Similarly, the relation $\gamma$ defined by

$$a \equiv b(\gamma) \Leftrightarrow \exists\, n \in \mathsf{N}\ \exists\, a_1, a_2, \ldots, a_n$$
$$a \wedge b = a_0 \leqq a_1 \leqq \ldots \leqq a_n = a \wedge b$$

and $a_{i+1}/a_i \in \mathsf{A}(\mathscr{L})$ for every $i = 0, 1, \ldots, n - 1$ is a congruence relation on such a lattice.

The *reflection* of $\mathscr{L}$, written **Ref** $\mathscr{L}$, is the lattice $\mathscr{L}/\beta$; the *coreflection* of $\mathscr{L}$, written **Coref** $\mathscr{L}$, is the lattice $\mathscr{L}/\gamma$.

The commutator of two elements and the commutator sublattice $\mathscr{G}'$ of a generalized orthomodular lattice $\mathscr{G}$ were defined by Marsden in [3]. The reader is referred to [1] for other definitions.

If $\mathscr{L}$ is a relatively complemented lattice and $\mathscr{L}$ is an ortholattice, $\mathscr{L} = (L, \vee, \wedge, ', 0, 1)$, then every congruence $\varrho$ of the lattice $\mathscr{L} = (L, \vee, \wedge)$ is also a congruence of the algebra $(L, \vee, \wedge, ')$. As usual, the operations on the quotient algebra are denoted by the same symbols and so we write, e.g., $\mathscr{L}/\varrho = (L/\varrho, \vee, \wedge, ')$.

**Theorem 1.** *Let* $\mathscr{L} = (L, \vee, \wedge, ', 0, 1)$ *be a relatively complemented ortholattice. Then* $\mathscr{L}$ *is on orthomodular lattice if and only if its reflection* **Ref** $\mathscr{L} = (L/\beta, \vee, \wedge, ', [0], [1])$ *is orthomodular.*

**Proof. 1.** $\mathscr{L}$ is orthomodular iff every two elements $s$, $t$ of $\mathscr{L}$ satisfy $s \vee t =$

$= s \vee (s' \wedge (s \wedge t))$. Hence, the orthomodularity of $\mathscr{L}$ implies the orthomodularity of $\mathscr{L}/\beta$.

2. Let $\mathscr{L}/\beta$ be orthomodular, let $s, t \in L$ and suppose that $s \geqq t' \,\&\, s \wedge t = 0$. Now $s/0 \nearrow 1/t \searrow t'/0$ and $[t', s] \subset [0, s]$. By Remark of [1] this means that $s/t'$ is projective with a quotient $v/u$ where $[u, v] \subset [0, t']$ and, hence, $s \equiv t'(\gamma)$. On the other hand, in the quotient algebra $\mathscr{L}/\beta$ we have $[s] \geqq [t]' \,\&\, [s] \wedge [t] = [0]$ and, by orthomodularity of $\mathscr{L}/\beta$, we see that $s \equiv t'(\beta)$. Therefore $s \equiv t'(\beta \cap \gamma)$ and so $s = t'$.

Recall a lattice $\mathscr{L}$ is called *semi-discrete* [2] if for every two comparable elements $a, b$ there exists a finite maximal chain connecting $a$ with $b$.

**Theorem 2.** *Let $\mathscr{L}$ be a relatively complemented lattice satisfying one of the following conditions:*

**(i)** *$\mathscr{L}$ is semi-discrete;*

**(ii)** *every interval in $\mathscr{L}$ satisfies the descending chain condition;*

**(iii)** *every interval in $\mathscr{L}$ satisfies the ascending chain condition.*

*Then $\mathscr{L}$ is isomorphic to the direct product of* **Ref** *$\mathscr{L}$ and* **Coref** *$\mathscr{L}$.*

**Proof.** Since $\mathscr{L}$ is supposed to be relatively complemented, $\beta\gamma = \gamma\beta$; moreover, $\beta \cap \gamma$ is the diagonal $\Delta_L$ of $L^2$. Thus it is sufficient to show that

$$(1) \ \forall\, a < b \ \exists\, a_0, a_1, \ldots, a_n, n \in \mathsf{N}\,, \quad a = a_0 \leqq a_1 \leqq \ldots \leqq a_n = b$$

such that

$$\forall\, i = 0, 1, \ldots n - 1 \qquad a_i \equiv a_{i+1}(\gamma) \qquad \text{or } a_i \equiv a_{i+1}(\beta)\,.$$

Now, if $\mathscr{L}$ is semi-discrete, then there are $a_0, a_1, \ldots, a_n$ such that

$$a = a_0 \lessdot a_1 \ldots \lessdot a_n = b$$

where $\lessdot$ denotes the covering relation. If $a_i \equiv a_{i+1}(\beta)$ does not hold, then $a_{i+1}/a_i \in \in \mathsf{A}\,(\mathscr{L})$ and so $a_{i+1} \equiv a_i(\gamma)$.

If $\mathscr{L}$ satisfies the condition (**ii**) and if $a < b$, then either $a \equiv b(\beta)$ or there exists an interval $[p, q] \subset [a, b]$ such that $p \neq q$ and $p \equiv q(\gamma)$. If $p = a$ and $b = q$, we are done. If this is not the case, let $q^+$ denote a relative complement of $q$ in $[p, b]$. By [1, Lemma 2.3 (ii)] there exist elements $a_0, a_1, \ldots, a_k$, such that

$$a_0 = q^+ < a_1 < \ldots < a_k = b$$

and such that

$$\forall\, i = 0, 1, \ldots, k \qquad a_{i+1} \equiv a_i(\gamma)$$

If $a \equiv q^+(\beta)$, then the chain

$$a \leqq q^+ = a_0 < a_1 < \ldots < a_k = b$$

has the property (1). If $a \equiv q^+(\beta)$ is not valid, then $a < q^+$ and we set ${}^{(1)}a = a$, ${}^{(1)}b = q^+$. Now, the same argument may be applied to the interval $[{}^{(1)}a, {}^{(1)}b]$ and so we get that either (1) is true or there exist elements $a_i'$ such that

$$b = a_k > \ldots > a_1 > q^+ = a_0 = {}^{(1)}b = a'_{k'} > \ldots > a'_1 > {}^{(1)}q^+ = a'_0$$

and such that $a'_{i+1} \equiv a'_i(\gamma)$ for every $i = 0, 1, \ldots, k'$. By hypothesis this process will stop in a finite number of steps. Consequently, (1) is true.

The final statement of the theorem follows by duality.

**Lemma 3.** *If $\mathscr{L}$ is a non-distributive simple relatively complemented lattice with* 0, *then $a \equiv 0(\gamma)$ for every $a$ of $\mathscr{L}$.*

**Proof.** By [1, Proposition 2.7] there exist elements $c < d$ such that $c \equiv d(\beta)$ does not hold. Thus there are elements $p \neq q$ such that $[p, q] \subset [c, d]$ and $q/p \in \mathsf{A}(\mathscr{L})$. So we have $p < q$ and $p \equiv q(\gamma)$ and therefore $\gamma \neq \Delta_L$. Since $\mathscr{L}$ is simple, $\gamma = L \times L$.

**Proposition 4.** *Let $(G, \vee, \wedge)$ be a simple lattice which is not distributive. Let $\mathscr{G} = (G, \vee, \wedge, a - x, 0)$ be a generalized orthomodular lattice.*

*Then $\mathscr{G} = \mathscr{G}'$.*

**Proof.** This follows easily by using Lemma 3 and [1, Proposition 3.1].

**Theorem 5.** *Let $\mathscr{H} = (H, \vee, \wedge, a \perp x, 0)$ and $\mathscr{G} = (G, \vee, \wedge, a \top x, 0)$ be generalized orthomodular lattices. Suppose $\varphi$ is an isomorphism* **(or a homomorphism)** *of the lattice $(H, \vee, \wedge)$ on the lattice $(G, \vee, \wedge)$* **(or into the lattice** $(G, \vee, \wedge)$**).** *Let $\mathscr{H}^{\perp}, \mathscr{G}^{\top}$ denote the commutator sublattice of $\mathscr{H}$ and $\mathscr{G}$, respectively.*

*Then*

$$\varphi(\mathscr{H}^{\perp}) = \mathscr{G}^{\top}$$

**(or** $\varphi(\mathscr{H}^{\perp}) \subset \mathscr{G}^{\top}$**).**

**Proof.** If $h \equiv 0(\gamma(H, \vee, \wedge))$, then

$$0 = h_0 \leqq h_1 \leqq \ldots \leqq h_m = h, m \in \mathsf{N}$$

where for every $i = 0, 1, \ldots, m-1$ we have $h_{i+1}/h_i \S K_i/H_i$. If $\varphi$ is a homomorphism, then from this we get

$$0 = \varphi(0) = \varphi(h_0) \leqq \varphi(h_1) \leqq \ldots \leqq \varphi(h_m) = \varphi(h)$$

and $\varphi(h_{i+1})/\varphi(h_i) \S \varphi(K_i)/\varphi(H_i)$. Therefore $\varphi(h) \equiv 0(\gamma(G, \vee, \wedge))$.

**Corollary 1.** *Let $\mathscr{G} = (G, \vee, \wedge)$ be a lattice and let $\top$ and $\perp$ be two "relative operations" defined on G in such a way that $(G, \vee, \wedge, a \top x, 0)$ and $(G, \vee, \wedge, a \perp x, 0)$ are generalized orthomodular lattices.*

*Then $\mathscr{G}^{\top} = \mathscr{G}^{\perp}$ where $\mathscr{G}^{\top}, \mathscr{G}^{\perp}$ denote the corresponding commutator sublattices.*

**Corollary 2.** *Suppose $f$ is an automorphism* **(or endomorphism)** *of a lattice $(G, \vee, \wedge)$. If $\mathscr{G} = (G, \vee, \wedge, a \top x, 0)$ is a generalized orthomodular lattice, then*

$$f(G^{\top}) = G^{\top}$$

**(or** $f(G^{\top}) \subset G^{\top}$).

The verification of the following technical lemma is straightforward and will therefore be omitted.

**Lemma 6.** *Suppose a lattice $(G, \vee, \wedge)$ is isomorphic with the direct product of lattices $\mathscr{H}, \mathscr{K}$. If $(G, \vee, \wedge, a - x, 0)$ is a generalized orthomodular lattice, then*

**(i)** *$\mathscr{H}$ and $\mathscr{K}$ determine also generalized orthomodular lattices;*

**(ii)**

$$(h, k) \leqq (a, b) \Rightarrow (a, b) - (h, k) = (a - h, b - k)$$

*for every* $(h, k), (a, b)$ *of the direct product* $\mathcal{H} \times \mathcal{K}$;

**(iii)**

$$\mathbf{com}_{[0, q \wedge g]}\,(q, g) = (\mathbf{com}_{[0, q_1 \vee g_1]}\,(q_1, g_1)\,,\ \mathbf{com}_{[0, q_2 \vee g_2]}\,(q_2, g_2))$$

where $q = (q_1, g_2)$, $g = (g_1, g_2)$.

**Proposition 7.** *Let* $\mathcal{G}$ *be a generalized orthomodular lattice and let* $\mathcal{G}$ *be isomorphic with the direct product* $\mathcal{H} \times \mathcal{K}$ *of two lattices* $\mathcal{H}, \mathcal{K}$.

*Then*

$$\mathcal{G}' \cong \mathcal{H}' \times \mathcal{K}' \text{ and } (\mathcal{H} \times \mathcal{K})' = \mathcal{H}' \times \mathcal{K}'$$

*where* $\mathcal{H}' \times \mathcal{K}'$ *denotes the direct product of the generalized orthomodular lattices* $\mathcal{H}, \mathcal{K}$.

**Proof.** In view of Theorem 5 it suffices to prove that $(\mathcal{H} \times \mathcal{K})' = \mathcal{H}' \times \mathcal{K}'$. Clearly, $(\mathcal{H} \times \mathcal{K})' \subset \mathcal{H}' \times \mathcal{K}'$. But if $t$ is of $\mathcal{H}' \times \mathcal{K}'$, then $t = (h', k')$ where

$$\bigvee_{i=1}^{m} \mathbf{com}_{[0, h_i \vee h_i^*]}\,(h_i, h_i^*) \geqq h' \in H'\,,$$

$$\bigvee_{j=1}^{n} \mathbf{com}_{[0, k_j \vee k_j^*]}\,(k_j, k_j^*) \geqq k' \in K'\,.$$

We may here assume that $m = n$. By Lemma 6 (**iii**) we get

$$(h', k')) \leqq \Big(\bigvee_{i=1}^{m} \mathbf{com}_{\dots}\,(h_i, h_i^*)\,,\ \bigvee_{i=1}^{m} \mathbf{com}_{\dots}\,(k_i\, k_i^*)\Big) =$$

$$= \bigvee_{i=1}^{m} \big(\mathbf{com}_{[0, h_i \vee h_i^*}\ (h_i, h_i^*), \mathbf{com}_{[0, k_i \vee k_i^*]}\ (k_i, k_i^*)\big) =$$

$$= \bigvee_{i=1}^{m} \mathbf{com}_{[0, q_i \vee g_i]}\,(q_i, g_i) \in (\mathcal{H} \times \mathcal{K})'$$

where $q_i = (h_i, k_i)$, $g_i = (h_i^*, k_i^*) \in H \times K$.

**Theorem 8.** *Let* $\mathcal{G}$ *be a generalized orthomodular lattice satisfying one of the conditions* (**i**), (**ii**), (**iii**) *of Theorem* 2.

*Then* $\mathcal{G}' \cong$ **Ref** $\mathcal{G}$ *and* $\mathcal{G} = \mathcal{G}' \times \mathcal{H}$ *where* $\mathcal{H} \cong$ **Coref** $\mathcal{G}$.

**Proof.** First, $\mathcal{G} \cong \mathcal{G}/\beta \times \mathcal{G}/\gamma$ by Theorem 2. By Proposition 7 we have $\mathcal{G}' \cong$ $\cong (\mathcal{G}/\beta)' \times (\mathcal{G}/\gamma)'$. Using Lemma 6 (**i**) we see that $\mathcal{G}/\gamma$ is a generalized orthomodular lattice. Hence, by [1, Proposition 2.7], $(\mathcal{G}/\gamma)' \cong 1$ and therefore $\mathcal{G}' \cong (\mathcal{G}/\beta)'$. Now, if $g \in G$, then from the proof of Theorem 2 we conclude that there is a finite chain

$$0 = a_0 \leqq a_1 \leqq \dots \leqq a_n = g$$

with the property $a_i \equiv a_{i+1}\,(\gamma \cup \beta)$ for every $i = 0, 1, \dots, n - 1$. But for the element $[g]$ of $\mathcal{G}/\beta$ this yields $[0] \equiv [g]\,(\gamma(\mathcal{G}/\beta))$. Hence $(\mathcal{G}/\beta)' = \mathcal{G}/\beta$ by [1, Proposition 3.1] and so $\mathcal{G}' \cong \mathcal{G}/\beta =$ **Ref** $\mathcal{G}$. Now,

$$\mathcal{G} \cong \mathcal{G}/\beta \times \mathcal{G}/\gamma$$

and

$$(\mathcal{G}/\beta \times \mathcal{G}/\gamma)' = (\mathcal{G}/\beta)' \times (\mathcal{G}/\gamma)' = \mathcal{G}_i\beta \times \langle 0 \rangle\,.$$

Let $f$ be an isomorphism of $\mathscr{G}/\beta \times \mathscr{G}/\gamma$ on $\mathscr{G}$. By Theorem 5

$$f((\mathscr{G}/\beta \times \mathscr{G}/\gamma)') = \mathscr{G}',$$

and we see that

$$f(\mathscr{G}/\beta \times \langle 0 \rangle) = \mathscr{G}'.$$

On the other hand,

$$\mathscr{G} = f(\mathscr{G}/\beta \times \langle 0 \rangle) \times f(\langle 0 \rangle \times \mathscr{G}/\gamma).$$

Therefore $\mathscr{G} = \mathscr{G}' \times \mathscr{H}$ where $\mathscr{H} = f(\langle 0 \rangle \times \mathscr{G}/\gamma) \cong \langle 0 \rangle \times \mathscr{G}/\gamma \cong \mathscr{G}/\gamma = \mathbf{Coref}\,\mathscr{G}$.

**Theorem 9.** *Let $\mathscr{L}$ be on orthomodular lattice of finite length. Then*

$$\mathscr{L}' = \mathscr{S}_1 \times \mathscr{S}_2 \times \ldots \times \mathscr{S}_k, k \geqq 0,$$

*where the lattices $\mathscr{S}_i$ of the direct product are simple orthomodular lattices which are not distributive. (Here, of course, if $k = 0$, $\mathscr{L}' = \mathbf{1}$). Under the same hypotheses, $\mathscr{L} = \mathscr{L}' \times \mathbf{2}^m$ where $\mathbf{2}^m$ $(m \geqq 1)$ denotes the direct product of $m$ copies of the two-element lattice $\mathbf{2}$, and $\mathbf{2}^0 = \mathbf{1}$.*

**Proof.** By **Dilworth** Theorem we have

$$\mathscr{L} = \mathscr{S}_1 \times \mathscr{S}_2 \times \ldots \times \mathscr{S}_k \times \mathscr{D}_1 \times \ldots \times \mathscr{D}_m$$

where $\mathscr{D}_i$ are simple distributive lattices of finite length. Hence $\mathscr{D}_i = \mathbf{2}$ and, by Proposition 7,

$$\mathscr{L}' = \mathscr{S}_1' \times \mathscr{S}_2' \times \ldots \times \mathscr{S}_k'.$$

Using Proposition 4, we get

$$\mathscr{L}' = \mathscr{S}_1 \times \mathscr{S}_2 \times \ldots \times \mathscr{S}_k.$$

## References


[1] L. Beran: On solvability of generalized orthomodular lattices (to appear in Pacific J. Math.).

[2] Iqbalunnisa: On types of lattices, Fund. Math. 59 (1966), 97–102.

[3] E. L. Marsden, Jr.: The commutator and solvability in a generalized orthomodular lattice, Pacific J. Math. 33 (1970), 357–361.