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Λ, R-Transitive Groupoids

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1. Introduction

If M is a set then S_M will be the monoid of all mappings of M into M. Let G be a groupoid. Define

$$\mathscr{L}_G = \{(\lambda, \varrho) \mid \lambda, \varrho \in S_G, \lambda(xy) = \varrho(x) \, . \, y \;\forall x, y \in G\},$$
$$\mathscr{R}_G = \{(\lambda, \varrho) \mid \lambda, \varrho \in S_G, \lambda(xy) = x \, . \, \varrho(y) \;\forall x, y \in G\},$$
$$\mathscr{M}_G = \{(\lambda, \varrho) \mid \lambda, \varrho \in S_G, \lambda(x) \, . \, y = x \, . \, \varrho(y) \;\forall x, y \in G\},$$
$$\Lambda_G = \{\lambda \mid \lambda \in S_G, \exists \varrho \in S_G(\lambda, \varrho) \in \mathscr{L}_G\},$$
$$\Lambda_G^* = \{\lambda \mid \lambda \in S_G, \exists \varrho \in S_G(\varrho, \lambda) \in \mathscr{L}_G\},$$
$$R_G = \{\lambda \mid \lambda \in S_G, \exists \varrho \in S_G(\lambda, \varrho) \in \mathscr{R}_G\},$$
$$R_G^* = \{\lambda \mid \lambda \in S_G, \exists \varrho \in S_G(\varrho, \lambda) \in \mathscr{R}_G\},$$
$$\Phi_G = \{\lambda \mid \lambda \in S_G, \exists \varrho \in S_G(\lambda, \varrho) \in \mathscr{M}_G\},$$
$$\Phi_G^* = \{\lambda \mid \lambda \in S_G, \exists \varrho \in S_G(\varrho, \lambda) \in \mathscr{M}_G\},$$
$$\tilde{\Lambda}_G = \{\lambda \mid \lambda \in S_G, (\lambda, \lambda) \in \mathscr{L}_G\},$$
$$\tilde{R}_G = \{\lambda \mid \lambda \in S_G, (\lambda, \lambda) \in \mathscr{R}_G\},$$
$$\tilde{\Phi}_G = \{\lambda \mid \lambda \in S_G, (\lambda, \lambda) \in \mathscr{M}_G\}.$$

The mappings from $\Lambda_G(R_G, \Phi_G)$ are sometimes called the left (right, middle) regular mappings. Some properties of these regular mappings can be found e.g. in [1], [3] and [4].

Let G be a groupoid. We shall say that G is Λ-transitive if for all $x, y \in G$ there exists $\lambda \in \Lambda_G$ with $\lambda(x) = y$. Similarly we define the Λ^*-transitivity, etc.

A groupoid G is said to be a μ-homotope of a groupoid $G(\circ)$ (having the same underlying set), provided there exist two mappings α, β of G onto G such that $xy = \alpha(x) \circ \beta(y)$ for all $x, y \in G$.

If G is a groupoid and $x \in G$ then $L_x(R_x)$ will be the left (right) translation by x (i.e. $L_x(y) = xy, R_x(y) = yx$).

2. Main results. The following three lemmas are obvious.

2.1. Lemma. Let G be a groupoid and $\lambda, \varrho \in S_G$. Then:

(i) $(\lambda, \varrho) \in \mathscr{L}_G$ iff $\lambda R_x = R_x \varrho \;\forall x \in G$.

(ii) $(\lambda, \varrho) \in \mathscr{L}_G$ iff $\lambda L_x = L_{\varrho(x)} \;\forall x \in G$.

(iii) $(\lambda, \varrho) \in \mathscr{R}_G$ iff $\lambda L_x = L_x\varrho \; \forall \; x \in G$.
(iv) $(\lambda, \varrho) \in \mathscr{R}_G$ iff $\lambda R_x = R_{\varrho(x)} \; \forall x \in G$.
(v) $(\lambda, \varrho) \in \mathscr{M}_G$ iff $L_x\varrho = L_{\lambda(x)} \; \forall \; x \in G$.
(vi) $(\lambda, \varrho) \in \mathscr{M}_G$ iff $R_x\lambda = R_{\varrho(x)} \; \forall \; x \in G$.

2.2. Lemma. Let G be a groupoid. Then:
(i) The sets $\mathscr{L}_G$, $\mathscr{R}_G$ are submonoids in the monoid $S_G \times S_G$.
(ii) The set $\mathscr{M}_G$ is a submonoid in the monoid $S_G \times S_G^\circ$ (S_G° is the opposite monoid of S_G).
(iii) The sets Λ_G, Λ_G^*, R_G, R_G^*, Φ_G, Φ_G^*, $\widetilde{\Lambda}_G$, $\widetilde{R}_G$ are submonoids of the monoid S_G.
(iv) If $\lambda, \varrho \in \widetilde{\Phi}_G$ and $\lambda\varrho = \varrho\lambda$, the $\lambda\varrho \in \widetilde{\Phi}_G$.
(v) $\Lambda_G^* \cap R_G^* \subseteq \widetilde{\Phi}_G$.

2.3. Lemma. Let G be a commutative groupoid. Then:
(i) $\mathscr{L}_G = \mathscr{R}_G, \Lambda_G = R_G, \Lambda_G^* = R_G^*$.
(ii) If $(\lambda, \varrho) \in \mathscr{M}_G$ then $(\varrho, \lambda) \in \mathscr{M}_G$. In particular, $\Phi_G = \Phi_G^*$.
(iii) $\Lambda_G^* \subseteq \widetilde{\Phi}_G$.

2.4. Proposition. Any Λ-transitive (R-transitive) groupoid is a right (left) division groupoid.

Proof. Let G be a Λ-transitive groupoid and $x, y, z \in G$ be arbitrary. There is $(\lambda, \varrho) \in \mathscr{L}_G$ such that $\lambda(zx) = y$. Hence $y = \lambda(zx) = \varrho(z) \,.\, x$. Similarly for the second case.

2.5. Proposition. Let G be such a groupoid that $G \,.\, G = \{xy \mid x, y \in G\} = G$. Then:
(i) $\lambda\varrho = \varrho\lambda \; \forall \; \lambda \in \Lambda_G \; \forall \; \varrho \in R_G$.
(ii) If $(\lambda, \varrho), (\sigma, \tau) \in \mathscr{L}_G(\mathscr{R}_G)$ and if $\varrho\tau = \tau\varrho$, then $\lambda\sigma = \sigma\lambda$.
(iii) If G is Φ-transitive ($\Phi^\star$-transitive) then G is a left (right) division groupoid.

Proof. (i) Let $y \in G$ be arbitrary, $y = ab$ for some $a, b \in G$. We have $\lambda\varrho(y) = \lambda\varrho(ab) = \lambda(a \,.\, \beta(b)) = \alpha(a) \,.\, \beta(b) = \varrho(\alpha(a) \,.\, b) = \varrho\lambda(ab) = \varrho\lambda(y)$, where $(\lambda, \alpha) \in \mathscr{L}_G$ and $(\varrho, \beta) \in \mathscr{R}_G$.
(ii) $\lambda\sigma(xy) = \varrho\tau(x) \,.\, y = \tau\varrho(x) \,.\, y = \sigma\lambda(xy) \; \forall \; x, y \in G$.
(iii) Let G be Φ-transitive and $x, y \in G$. There are $a, b \in G$ and $(\varphi, \psi) \in \mathscr{M}_G$ such that $ab = y$ and $\varphi(x) = a$. Hence $y = ab = \varphi(x) \,.\, b = x \,.\, \psi(b)$. Similarly, if G is $\Phi^\star$-transitive.

2.6. Proposition. Let in a groupoid G there be at least one element x such that the mapping R_x is one-to-one. Then:
(i) $\varphi\lambda = \lambda\varphi \; \forall \; \varphi \in \Phi_G \; \forall \; \lambda \in \Lambda_G^*$.
(ii) If $(\lambda, \varrho), (\sigma, \tau) \in \mathscr{L}_G$ and $\lambda\sigma = \sigma\lambda$, then $\varrho\tau = \tau\varrho$.
(iii) If G is Λ-transitive then G is $\Lambda^\bullet$-transitive.
(iv) If G is $R^\bullet$-transitive then G is a right cancellation groupoid.
(v) If $(\varphi, \psi), (\alpha, \beta) \in \mathscr{M}_G$ and $\psi\beta = \beta\psi$, then $\varphi\alpha = \alpha\varphi$.

Proof. (i) Let $(\varrho, \lambda) \in \mathscr{L}_G, (\varphi, \psi) \in \mathscr{M}_G$ and $y \in G$ be arbitrary. We may write $R_x \, \lambda\varphi(y) = \lambda\varphi(y) \,.\, x = \varrho(\varphi(y) \,.\, x) = \varrho(y \,.\, \psi(x)) = \lambda(y) \,.\, \psi(x) = \varphi\lambda(y) \,.\, x = R_x\varphi\lambda(y)$. Hence $\varphi\lambda(y) = \lambda\varphi(y)$.

(ii) $\varrho\tau(y) \, . \, x = \lambda\sigma(yx) = \sigma\lambda(yx) = \tau\varrho(y) \, . \, x$.

(iii) Let $a, b \in G$ be arbitrary. There is $(\lambda, \varrho) \in \mathscr{L}_G$ such that $\lambda(ax) = bx$. However $R_x(b) = bx = \lambda(ax) = \varrho(a) \, . \, x = R_x\varrho(a)$, and therefore $\varrho(a) = b$.

(iv) If $ay = by$ for some $a, b, y \in G$, then $ax = a \, . \, \varrho(y) = \lambda(ay) = \lambda(by) = bx$, where $(\lambda, \varrho) \in \mathscr{R}_G$ is such that $\varrho(y) = x$. Hence $ax = bx$, and consequently $a = b$.

(v) Obvious.

2.7. Proposition. Let in a groupoid G there be at least one element x such that the mapping L_x is one-to-one. Then:

(i) $\varphi\lambda = \lambda\varphi \; \forall \, \varphi \in \Phi_G^* \; \forall \, \lambda \in R_G^*$.

(ii) If $(\lambda, \varrho), (\sigma, \tau) \in \mathscr{R}_G$ and $\lambda\sigma = \sigma\lambda$, then $\varrho\tau = \tau\varrho$.

(iii) If G is R-transitive then G is R^*-transitive.

(iv) If G is Λ^*-transitive then G is a left cancellation groupoid.

(v) If $(\varphi, \psi), (\alpha, \beta) \in \mathscr{M}_G$ and $\varphi\alpha = \alpha\varphi$, then $\psi\beta = \beta\psi$.

Proof. The proof is dual to that of 2.6.

2.8. Proposition. Let G be a Λ^*-transitive (R^*-transitive) groupoid and let there be at least one element $x \in G$ such that the mapping $R_x(L_x)$ is onto G. Then G is Λ-transitive (R-transitive).

Proof: For the first case only. If $a, b \in G$ are arbitrary, then there are $y, z \in G$ and $(\lambda, \varrho) \in \mathscr{L}_G$ such that $yx = b, zx = a$ and $\varrho(z) = y$. Hence $\lambda(a) = \lambda(zx) = \varrho(z) \, . \, x = yx = b$.

2.9. Lemma. Let G be an R-transitive (Λ-transitive) groupoid. Then:

(i) Any mapping from $\Lambda_G(R_G)$ is a mapping onto G.

(ii) If $\lambda, \sigma \in \Lambda_G(R_G)$ and $\lambda(a) = \sigma(a)$ for some $a \in G$, then $\lambda = \sigma$.

Proof. For the first case only.

(i) G is a left division groupoid (by 2.4) and 2.1 (ii) yields the result.

(ii) Since G is a left division groupoid, $G \, . \, G = G$. Let $x \in G$ be an element. By the hypothesis there is $\alpha \in R_G$ with $\alpha(a) = x$. Applying 2.5 (i) we get $\lambda\alpha = \alpha\lambda$ and $\sigma\alpha = \alpha\sigma$. Hence $\lambda(x) = \lambda\alpha(a) = \alpha\lambda(a) = \alpha\sigma(a) = \sigma\alpha(a) = \sigma(x)$.

2.10. Theorem. Let G be a Λ and R-transitive groupoid. Then:

(i) G is a division groupoid.

(ii) Λ_G and R_G are mutually isomorphic groups and $\mathit{card}\ \Lambda_G = \mathit{card}\ R_G = \mathit{card}\ G$.

(iii) Any mapping from Λ_G and R_G is a permutation.

Proof. (i) See 2.4.

(ii) First we show that Λ_G is a group. To this purpose it is enough to prove that it is a right division groupoid (since Λ_G is a monoid). For let $\lambda, \varrho \in \Lambda_G$ and $x \in G$ be arbitrary. There is $\tau \in \Lambda_G$ such that $\tau\lambda(x) = \varrho(x)$. However $\tau\lambda \in \Lambda_G$ and 2.9 yields now $\tau\lambda = \varrho$. Similarly we can prove that R_G is a group. Further, for any $\lambda \in \Lambda_G$ there is a uniquely determined $\varrho \in R_G$ with $\lambda(x) = \varrho(x)$ (by the hypothesis and by 2.9). Setting $\varrho = A(\lambda)$ we get, for all $\alpha, \beta \in \Lambda_G$, $A(\alpha\beta)(x) = \alpha\beta(x) = \alpha \, A(\beta)(x)$. But $\alpha A(\beta) = A(\beta)\alpha$ due to (i) and 2.5 (i). Hence $A(\alpha\beta)(x) = \alpha A(\beta)(x) = A(\beta)\,\alpha(x) = A(\beta)A(\alpha)(x)$ and so $A : \Lambda_G \to R_G$ in an antihomomorphism. Using 2.9, we may check easily that A is

a biunique mapping. Thus the groups Λ_G and R_G are antiisomorphic and therefore isomorphic. The equality $card\ G = card\ \Lambda_G = card\ R_G$ is obvious from 2.9.
(iii) Since Λ_G and R_G are groups having the identity mapping 1_G as the unit element, it is evident that every mapping from Λ_G or R_G is a permutation.

2.11. Theorem. Let G be a Λ^* and Φ-transitive groupoid and let there be at least one element $x \in G$ such that the mapping R_x is one-to-one. Then:
(i) Λ^*_G and Φ_G are mutually isomorphic groups and $card\ G = card\ \Phi_G = card\ \Lambda^*_G$.
(ii) Any mapping from Λ_G and Φ_G is a permutation.
(iii) If $G . G = G$ then G is a left division groupoid.

Proof. The proof is similar to that of 2.10.

For ease of reference we give the following proposition; it is proved in [4, Theorem 9].

2.12. Proposition. Let G be a groupoid. Then the following are equivalent:
(i) G is a μ-homotope of a groupoid possessing a unit.
(ii) There are two elements $x, y \in G$ satisfying

$$(\alpha)\ L_x, R_y \text{ are onto,}$$
$$(\beta)\ \forall\, u, v, z \in G, uy = vy \text{ implies } uz = vz\,,$$
$$(\gamma)\ \forall\, u, v, z \in G, xu = xv \text{ implies } zu = zv\,.$$

2.13. Theorem. Let G be a groupoid. Then the following conditions are equivalent:
(i) G is $\Lambda, \Lambda^*, R, R^*, \Phi, \Phi^*$-transitive.
(ii) G is $\Lambda, \Lambda^*, R, R^*$-transitive.
(iii) G is a Λ^*, R^*-transitive division groupoid.
(iv) G is Λ^*, R^*-transitive and there exist $x, y \in G$ such that the mappings L_x, R_y are onto.
(v) G is a μ-homotope of a group.

Proof. (i) implies (ii) and (iii) implies (iv) trivially.
(ii) implies (iii) by 2.4.
(iv) implies (v). We show that the elements x, y satisfy $(\alpha), (\beta), (\gamma)$ from 2.12. For let $uy = vy$ and $z \in G$. There is $(\lambda, \varrho) \in \mathscr{R}_G$ with $\varrho(y) = z$. So $uz = u . \varrho(y) = \lambda(uy) = \lambda(vy) = v . \varrho(y) = vz$ and we have proved (β). Similarly (γ). Thus G is a μ-homotope of a groupoid $G(\circ)$, which has a unit. By [4, Theorem 7], $G(\circ)$ must be a group.
(v) implies (i). By the hypothesis there are two mappings α, β of G onto G and a group $G(\circ)$ such that $ab = \alpha(a) \circ \beta(b)$ for all $a, b \in G$. For $a, u \in G$ let $\gamma_u(a) = u \circ a$, $\delta_u(a) = a \circ u$. Then obviously $(\gamma_u, \sigma_u) \in \mathscr{L}_G, (\delta_u, \tau_u) \in \mathscr{R}_G$ and $(\lambda_u, \varrho_u) \in \mathscr{M}_G$ where $\sigma_u, \tau_u, \lambda_u, \varrho_u \in S_G$ are arbitrary mappings satisfying:
$\gamma_u\alpha = \alpha\sigma_u, \delta_u\beta = \beta\tau_u, \beta\varrho_u = \gamma_u\beta, \alpha\lambda_u = \delta_u\alpha\,.$
From this we can easily deduce that G is a $\Lambda, \Lambda^*, R, R^*, \Phi, \Phi^*$-transitive groupoid.

2.14. Corollary. Let G be a groupoid and let there be $x, y \in G$ such that the mappings L_y, R_x are one-to-one. Then the following are equivalent:
(i) G is Λ, Φ-transitive.
(ii) G is R, Φ^*-transitive.
(iii) G is Λ, R-transitive.

(iv) G is $\Lambda, \Lambda^*, R, R^*, \Phi, \Phi^*$-transitive.
(v) G is a quasigroup isotopic to a group.

Proof. (i) implies (v). G is a division groupoid (by 2.4 and 2.5) and hence from the hypothesis and from 2.12 we see that G is a μ-homotope of a groupoid $G(\circ)$ having a unit. However (see [4, Theorem 7]) $G(\circ)$ is a group and so G is Λ^* and R^*-transitive according to 2.13. Further, by 2.6 and 2.7 G is a cancellation groupoid and consequently a quasigroup.
(ii) implies (v). Similarly.
(iii) implies (v) by 2.6, 2.7 and 2.13.
(v) implies (iv), (iii), (ii) and (i). See 2.13.

2.15. Remark. The author does not know, whether there exists a Λ, R-transitive groupoid not being a μ-homotope of a group.

3. $\tilde{\Lambda}$-transitive groupoids. If G is a groupoid then let $A_G(B_G)$ be the submonoid in S_G generated by all the mappings $R_x(L_x)$, $x \in G$.

3.1. Lemma. Let G be a $\tilde{\Lambda}$-transitive ($\tilde{R}$-transitive) groupoid and let there be $a, b \in G$ such that $ab = a(ba = a)$. Then b is a right (left) unit in G.

Proof. Given $x \in G$ there is $\lambda \in \tilde{\Lambda}_G$ with $\lambda(a) = x$, and hence $xb = \lambda(a) \cdot b = \lambda(ab) = \lambda(a) = x$.

3.2. Theorem. Let G be a groupoid. Then the following are equivalent:
(i) G is a $\tilde{\Lambda}$-transitive division groupoid and a right quasigroup.
(ii) G is $\tilde{\Lambda}$-transitive and there is $a \in G$ such that the mapping L_a is onto.
(iii) G is a μ-homotope of a group and G possesses a right unit.
(iv) There are a group $G(\circ)$ and a mapping δ of G onto G such that $xy = x \circ \delta(y)$ for all $x, y \in G$.

Proof. (i) implies (ii) trivially.
(ii) implies (iii). Since L_a is onto, there is $j \in G$ with $L_a(j) = a$, and consequently j is a right unit in G (by 3.1). Further, the pair a, j satisfies the conditions $(\alpha), (\beta), (\gamma)$ from 2.12. Indeed, (α) and (β) are obvious since j is a right unit and L_a is onto. For (γ) we use the $\tilde{\Lambda}$-transitivity. If $au = av$ for some $u, v \in G$ and $z \in G$ is an element, then $z = \lambda(a)$ where $\lambda \in \tilde{\Lambda}_G$ is suitable. Hence $zu = \lambda(a)u = \lambda(au) = \lambda(av) = zv$. Thus G is a μ-homotope of a groupoid with a unit and an application of [4, Theorem 7] yields (iii).
(iii) implies (iv). We have, for all $x, y \in G$, $xy = \alpha(x) \circ \beta(y)$; $G(\circ)$ is a group and α, β are mappings of G onto G. Since G has a right unit j, $xj = x = \alpha(x) \circ \beta(j)$ for all $x \in G$. Hence $\alpha(x) = x \circ (\beta(j))^{-1}$ and $xy = \alpha(x) \circ \beta(y) = x \circ (\beta(j))^{-1} \circ \beta(y) = x \circ \delta(y)$.
(iv) implies (i). Obvious.

If G is a quasigroup then $C_G(D_G)$ will be the right (left) multiplication group corresponding to G.

3.3. Theorem. Let G be a groupoid. Then the following are equivalent:
(i) G is a quasigroup and $C_G = \{Rx \mid x \in G\}$ (i.e. for all $a, b \in G$ there are $c, d \in G$ with $R_a R_b = R_c$ and $R_a^{-1} = R_d$).
(ii) G is a division groupoid, $A_G = \{R_x \mid x \in G\} \cup \{1_G\}$ (i.e. for all $a, b \in G$ there is $c \in G$ with $R_c = R_a R_b$), and there exists $x \in G$ such that the mapping L_x is one-toone.

(iii) There are a group $G(\circ)$ and a permutation δ of the set G such that $ab = a \circ \delta(b)$ for all $a, b \in G$.
(iv) G is a quasigroup possessing a right unit and G is isotopic to a group.
(v) G is a $\widetilde{\Lambda}$-transitive groupoid and there exist $x, y \in G$ such that L_x is onto and L_y is one-to-one.

Proof. (i) implies (ii). It is obvious, since $\{R_x \mid x \in G\} \cup \{1_G\} \subseteq A_G \subseteq C_G$.
(ii) implies (iv). By the hypothesis there exists a binary operation $\circ$ on the set G with the property $c \,.\, (a \circ b) = (ca) \,.\, b$ for all $a, b, c \in G$. We can write, for all $u, v, z \in G$, $x(u \circ (v \circ z)) = (xu)(v \circ z) = (xu \,.\, v)\, z = (x(u \circ v))\, z = x((u \circ v) \circ z)$. However the mapping L_x is one-to-one, and so $u \circ (v \circ z) = (u \circ v) \circ z$, i.e. $G(\circ)$ is a semigroup. On the other hand, $G(\circ)$ is a division groupoid, as it is easy to see, and consequently $G(\circ)$ is a group. Further, $ab = (x \,.\, L_x^{-1}(a))\, b = L_x(L_x^{-1}(a) \circ b)$ for all $a, b \in G$. From this it is obvious that G is a quasigroup and that the unit of $G(\circ)$ is a right unit in G.
(iv) implies (v). By 3.2.
(v) implies (iii). According to 3.2, there are a group $G(\circ)$ and a mapping δ of G onto G such that $ab = a \circ \delta(b)$ for all a, b. Hence $L_y = \gamma_y \delta$ where $\gamma_y(a) = y \circ a$ for all $a \in G$, and consequently δ is a one-to-one mapping (since L_y is so).
(iii) implies (i). Given $a, b \in G$ we have $R_a R_b(z) = zb \,.\, a = z \circ \delta(b) \circ \delta(a) =$
$= z \circ \delta\delta^{-1}(\delta(b) \circ \delta(a)) = z \circ \delta(c) = R_c(z)$ and $R_a^{-1}(z) = z \circ (\delta(a))^{-1} =$
$= z \circ \delta\delta^{-1}(\delta(a))^{-1} = z \circ \delta(d) = R_d(z)$ for all $z \in G$.

3.4. Corollary. Let G be a groupoid. Then the following are equivalent:
(i) G is $\widetilde{\Lambda}$ and $\widetilde{R}$-transitive.
(ii) G is a division groupoid, $A_G = \{R_x \mid x \in G\} \cup \{1_G\}$ and $B_G = \{L_x \mid x \in G\} \cup \{1_G\}$.
(iii) G is a group.

Proof. (i) implies (iii). Since G is $\widetilde{\Lambda}$, $\widetilde{R}$-transitive, G is a division groupoid, and consequently G has a unit (by 3.1). So G is a group (see [4, Theorem 3]).
(ii) implies (iii). By the hypothesis there are two mappings $\alpha, \beta : G \times G \to G$ such that $ab \,.\, c = a \,.\, \alpha(b, c)$ and $b \,.\, ca = \beta(b, c) \,.\, a$ for all $a, b, c \in G$. Hence $R_c \in R_G$ and $L_b \in \Lambda_G$; all $b, c \in G$. Since G is a division groupoid, G is Λ-transitive and R-transitive. By 2.10, any mapping from R_G and Λ_G is a permutation and therefore G is a quasigroup. Applying 3.3 (and the dual theorem) we see that G possesses a unit and so it is a group ([4, Theorem 3]).
(iii) implies (i) and (ii) trivially.

4. Applications. If G is a groupoid and $x_1, \ldots, x_n \in G$, then we set

$$(x_1, \ldots, x_n) = x_1(x_2(x_3(\ldots x_{n-2}(x_{n-1} \,.\, x_n))))$$
$$[x_1, \ldots, x_n] = ((((x_1 x_2)\, x_3) \ldots x_{n-2})\, x_{n-1})\, x_n \,.$$

4.1. Proposition. Let G be a groupoid. Then the following statements are equivalent:
(i) G is a division groupoid and there exists $n \geq 3$ such that $(x_1, \ldots, x_n) =$
$= (x_1, \ldots, x_{n-1}) \,.\, x_n$ for all $x_1, \ldots, x_n \in G$.
(ii) There are a group $G(\circ)$ and an automorphism δ of $G(\circ)$ such that $\delta^{n-2} = 1_G$ and

$ab = a \circ \delta(b)$ for all $a, b \in G$. In this case G is a quasigroup and δ is an automorphism of G.

Proof. (i) implies (ii). We have, for all $x_1, \ldots, x_{n-2}, a, b \in G$,
$L_{x_1}L_{x_2} \ldots L_{x_{n-2}} (ab) = (x_1, \ldots, x_{n-2}, a, b) = (x_1, \ldots, x_{n-2}, a) \,.\, b =$
$= L_{x_1}L_{x_2} \ldots L_{x_{n-2}} (a) \,.\, b$. Hence $L_{x_1}L_{x_2} \ldots L_{x_{n-2}} \in \widetilde{\Lambda}_G$ and since G is a division groupoid, G is $\widetilde{\Lambda}$-transitive. According to 3.2, there exist a group $G(\circ)$ and a mapping δ of G onto G such that $ab = a \circ \delta(b)$ for all $a, b \in G$. Further, G has a right unit j and with respect to [4, Theorem 11] and [4, Lemma 15] we may assume (without loss of generality) that j is also the unit element in $G(\circ)$ and $\delta(j) = j$. Now let us write $\delta^{n-1}(a) =$
$= j \circ \delta(j \circ \delta(j \circ \delta(\ldots \delta(j \circ \delta(a))))) = (j, \ldots, j, a) = (j, \ldots, j) \,.\, a = ja = \delta(a)$.
So $\delta^{n-1} = \delta$. However δ is a mapping onto G, and hence $\delta^{n-2} = 1_G$. In particular, δ is a quasigroup. Finally $\delta(a \circ b) = \delta(a \circ \delta^{n-2}(b)) =$
$= j \circ \delta(a \circ \delta(j \circ \delta(\ldots \delta(j \circ \delta(b))))) = (j, a, j, \ldots, j, b) = (j, a, j, \ldots, j) \,.\, b = ja \,.\, b =$
$= \delta(a) \circ \delta(b)$. Thus δ is an automorphism of $G(\circ)$ and consequently of G, too.
(ii) implies (i). If $x_1, \ldots, x_n \in G$, then by the hypothesis
$(x_1, \ldots, x_n) = x_1 \circ \delta(x_2 \circ \delta(\ldots \delta(x_{n-1} \circ \delta(x_n)))) =$
$= x_1 \circ \delta(x_2) \circ \delta^2(x_3) \circ \ldots \circ \delta^{n-2}(x_{n-1}) \circ \delta^{n-1}(x_n) =$
$= x_1 \circ \delta(x_2 \circ \delta(x_3 \circ \delta(\ldots \delta(x_{n-2} \circ \delta(x_{n-1}))))) \circ \delta(x_n) = (x_1, \ldots, x_{n-1}) \,.\, x_n$,
and we are through.

4.2. Proposition. Let G be a division groupoid satisfying the identity $(x_1, \ldots, x_n) = [x_1, \ldots, x_n]$ for some $n \geq 3$. Then G is a group.

Proof. We see immediately that $L_{x_1}L_{x_2} \ldots L_{x_{n-2}} \in \Lambda_G$ and $R_{x_3}R_{x_4} \ldots R_{x_n} \in R_G$ for all $x_1, \ldots, x_n \in G$. Since G is a division groupoid, G is Λ and R-transitive. Hence, by 2.10, any mapping from Λ_G and R_G is a permutation, and therefore G is a quasigroup. Now, according to [2, Theorem 4], there are a group $G(\circ), \varphi, \psi \in \mathrm{Aut}\, G(\circ)$ and $c \in G$ such that $ab = \varphi(a) \circ c \circ \psi(b)$ for all $a, b \in G$.
Hence
$\varphi(x_1) \circ c \circ \psi\varphi(x_2) \circ \psi(c) \circ \ldots \circ \psi^{n-2}\varphi(x_{n-1}) \circ \psi^{n-2}(c) \circ \psi^{n-1}(x_n) =$
$= \varphi^{n-1}(x_1) \circ \varphi^{n-2}(c) \circ \varphi^{n-2}\psi(x_2) \circ \ldots \circ \varphi(c) \circ \varphi\psi(x_{n-1}) \circ c \circ \psi(x_n)$ for all $x_1, \ldots x_n \in G$. In particular, $\varphi(x_1) = \varphi^{n-1}(x_1)$ for each $x_1 \in G$, and so $\varphi^{n-2} = \ {}_G$.
Further, $\varphi(x_1) \circ c \circ \psi\varphi(x_2) = \varphi^{n-1}(x_1) \circ \varphi^{n-2}(c) \circ \varphi^{n-2}\psi(x_2) = \varphi(x_1) \circ c \circ \psi(x_2)$,
i.e. $\psi\varphi(x_2) = \psi(x_2)$. From this, $\varphi = 1_G$. Similarly $\psi = 1_G$, and consequently G is a group.

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