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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 16 (1975), No. 2, 79--87

Persistent URL: `http://dml.cz/dmlcz/142371`

Extensive Varieties

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Received 4 March 1974

Consider universal algebras A of a given type Δ. We recall that Δ is a set of some operation symbols, each having assigned a non-negative integer (arity) and to any $F \in \Delta$ of arity n there corresponds in A an n-ary operation, denoted by F_A. An element $a \in A$ is said to be idempotent if the one-element subset $\{a\}$ is a subalgebra in A, or equivalently if $F_A(a, a, \dots, a) = a$ for any $F \in \Delta$ (if F is nullary, this means $F_A = a$). Let K be a variety (i.e. an equationally definable class) of algebras of type Δ. We shall say that K is an extensive variety if any algebra of K can be imbedded into an algebra of K having at least one idempotent.

Proposition 1. Let K be a variety of Δ-algebras. The following are equivalent:
(i) K is an extensive variety.
(ii) For any $A, B \in K$ there is an algebra $C \in K$, containing a subalgebra isomorphic to A and a subalgebra isomorphic to B.
(iii) For any subset $M \subseteq K$ there is an algebra $C \in K$ such that every $A \in M$ is isomorphic to a subalgebra of C.
(iv) The free K-product of any pair of K-algebras is formed by monomorphisms.
(v) The free K-product of any family of K-algebras is formed by monomorphisms.

Proof. The equivalence of (ii), (iii), (iv) and (v) is easy and belongs to the mathematical folklore.
(i) implies (ii). For $A, B \in K$ there are $C, D \in K$ with idempotents such that A is a subalgebra in C and B is a subalgebra in D. The cartesian product $C \times D$ is an element of K and contains subalgebras isomorphic to A and B.
(ii) implies (i). Let $A \in K$. There exists an algebra $B \in K$ such that A and the one-element algebra are subalgebras in B.

Let $x, y, z, \dots$ be a set of variables. Expressions containing variables and operational symbols from Δ are called Δ-terms. If t is a Δ-term then var (t) will be the set of all variables occurring in t. Further, by $W(\Delta)$ we shall denote the Δ-algebra of all the Δ-terms. Let $u, v \in W(\Delta)$ and $A \in K$. We shall say that the algebra A satisfies the equation $u = v$ if $f(u) = f(v)$ for all homomorphisms f of $W(\Delta)$ into A. Finally, if E is a set of equations, then Mod(E) denotes the variety of all Δ-algebras satisfying all equations from E.

Proposition 2. Let Δ be a type containing no nullary symbols and E be a set of equations such that var(u) = var (v) for every equation $u = v$ belonging to E. Then Mod(E) is an extensive variety.

Proof. Let $A \in$ Mod (E). Choose an element e not belonging to A and define a Δ-algebra B in the following way:

(i) $B = A \cup \{e\}$.

(ii) If $F \in \Delta$ is of arity n and $a_1, a_2, \ldots, a_n \in A$ then $F_B(a_1, \ldots, a_n) = F_A(a_1, \ldots, a_n)$.

(iii) If $F \in \Delta$ is of arity n, $a_1, \ldots, a_n \in B$ and $\{a_1, \ldots, a_n\} \nsubseteq A$ then $F_B(a_1, \ldots, a_n) = e$.

Since e is idempotent and A is a subalgebra in B, it is enough to prove $B \in$ Mod (E). For, let $u = v$ be an equation from E and f be a hommorphism of $W(\Delta)$ into B. If $f(x) \in A$ for any $x \in$ var (u), then $f(u) = f(v)$ follows from the validity of $u = v$ in A. In the opposite case we have $f(u) = f(v) = e$.

Proposition 3. Let Δ be a type containing no nullary operations and K be a variety of Δ-algebras. Let there exist two Δ-terms u, v such that var (u) and var (v) are disjoint sets and the equation $u = v$ is satisfied in K. The following conditions are equivalent:

(i) K is an extensive variety.

(ii) The equation $u = F(u, u, \ldots, u)$ is satisfied in K for all $F \in \Delta$.

Proof. (i) implies (ii). Let $A \in K$ be an arbitrary algebra. Then A is a subalgebra in an algebra $B \in K$ which possesses an idempotent e. If $f: W(\Delta) \to A$ is a homomorphism then we define a homomorphism g of $W(\Delta)$ into B in this way: $g(x) = e$ for all variables $x \in$ var (v) and $g(x) = f(x)$ for all variables $x \notin$ var (v). We have $f(u) = g(u) = g(v) = e$, so that $f(u) = f(F(u, u, \ldots, u)) = e$.

(ii) implies (i) trivially.

If Δ contains some nullary operations, then a variety K of Δ-algebras is extensive iff every algebra from K contains at least one idempotent, i.e. iff $F(c, c, \ldots, c) = d$ is valid in K for any $F \in \Delta$ and any two constants $c, d \in \Delta$.

In the following we restrict ourselves to the case of groupoids and quasigroups.

Proposition 4. Any variety of semigroups is extensive.

Proof. Let K be a variety of semigroups. If var (u) = var (v) for every equation $u = v$ valid in K then the assertion follows from Proposition 2. In the opposite case it is evident that there exist two different natural numbers n, m such that $x^n = x^m$ holds in every semigroup from K. Hence any cyclic semigroup form K is finite, and consequently it contains an idempotent.

Proposition 5 .Let t, u, v be three groupoid terms such that var (t) = var (u) = $= \{x\}$ and var $(v) = \{y\}$. Then Mod $(t = uv)$ and Mod $(t = vu)$ are extensive.

Proof. Let $A \in$ Mod $(t = uv)$. If $a \in A$, then we denote by f_a the homomorphism of W into A such that $f_a(z) = a$ for every variable z. Let $a, b \in A$ and $f_a(u) = f_b(u)$. We show that $f_a(t) = f_b(t)$. Indeed, if $g: W \to A$ is such a homomorphism that $g(x) = a$ and $g(y) = b$, then we can write $f_a(t) = g(t) = g(uv) = g(u)g(v) = f_a(u)f_b(v) =$ $= f_a(u)f_b(v) = f_b(u)f_b(v) = f_b(uv) = f_b(t)$.

Choose an element e not belonging to A and define a groupoid B in such a way:

(i) $B = A \cup \{e\}$.

(ii) A is a subgroupoid in B.

(iii) $ea = e$ for any $a \in B$.
(iv) If $a \in A$ and $a = f_b(u)$ for some $b \in A$ then $ae = f_b(t)$.
(v) If $a \in A$ and $a \neq f_b(u)$ for all $b \in A$ then $ae = e$.
It is easy to show that $B \in \mathrm{Mod}\,(t = uv)$. For $\mathrm{Mod}\,(t = vu)$ the proof is similar.

Proposition 6. Let t and u be two groupoid terms such that $\mathrm{var}\,(t) = \{x\}$, $\mathrm{var}\,(u) = \{x, y\}$, and let u contain no subterm having the form yv for some term v. Then $\mathrm{Mod}\,(t = u)$ is extensive.

Proof. Let $A \in \mathrm{Mod}\,(t = u)$. Take an element $c \in A$, an element e not belonging to A and set $B = A \cup \{e\}$. We extend the groupoid structure of A to B setting $ae = ac$, $eb = e$ for all $a \in A$ and $b \in B$. In order to prove the proposition it is sufficient to show $B \in \mathrm{Mod}\,(t = u)$. For, let f be a homomorphism of W into B. The following cases can arise:
(i) $f(x) \in A$ and $f(y) \in A$. Then $f(t) = f(u)$ follows from the validity of $t = u$ in A.
(ii) $f(x) = f(y) = e$. In this case, $f(t) = e = f(u)$.
(iii) $f(x) = e$ and $f(y) \in A$. If v is a term whose first variable is x then $f(v) = e$ as it is easy to prove by the induction on the length of v. However, by the hypothesis, x is the first variable in u, and therefore $f(t) = e = f(u)$.
(iv) $f(x) \in A$ and $f(y) = e$. Let g be a homomorphism of W into A such that $g(x) = f(x)$ and $g(y) = c$. Let us prove the following assertion using the induction:

If w is a term such that $w = y$, $\mathrm{var}\,(w) \subseteqq \{x, y\}$ and no subterm of w has the form yv, then $f(w) = g(w)$.

The assertion is trivial if w is a variable. Assume $w = rs$. We have $r \neq y$ and from the induction hypothesis it follows $f(r) = g(r)$. If $s \neq y$ then $f(s) = g(s)$, and hence $f(w) = g(w)$. If $s = y$ then we can write $f(w) = f(r)f(s) = g(r)e = g(r)c = g(r)g(y) = g(w)$.

The assertion is proved and may be applied to our case. We get $f(t) = g(t) = g(u) = f(u)$.

Proposition 7. Let t and u be two groupoid terms such that $\mathrm{var}(t) = \{x\}$, $\mathrm{var}\,(u) = \{x, y\}$ and let u contain no subterm having the form vy. Then $\mathrm{Mod}\,(t = u)$ is extensive.

Proof. The proof is similar to that of the proceding proposition.

Proposition 8. Let t be a groupoid term such that $\mathrm{var}\,(t) \subseteq \{x, y\}$. Then $\mathrm{Mod}\,(x = x \,.\, yt)$ and $\mathrm{Mod}\,(x = ty \,.\, x)$ are extensive.

Proof. Let $A \in \mathrm{Mod}\,(x = x \,.\, yt)$. Choose an element e not belonging to A and define a groupoid B as follows:
(i) $B = A \cup \{e\}$ and A is a subgroupoid in B.
(ii) $ea = e$ for all $a \in B$ and $ae = a$ for all $a \in A$.
The rest is obvious.

Proposition 9. The groupoid variety $K = \mathrm{Mod}\,(x = yx \,.\, y)$ has the following properties:
(i) $K = \mathrm{Mod}\,(x = y \,.\, xy)$.
(ii) Every groupoid from K is a quasigroup.
(iii) K is extensive.

Proof. The equation $x = yx \,.\, y$ implies (after substitution xy for y) $x = xy \,.\, x \,.\, xy = y \,.\, xy$. Similarly, $x = y \,.\, xy$ implies $x = yx \,.\, y$.
(ii) is an easy consequence of (i).
Let $A \in \text{Mod}\,(x = yx \,.\, y)$. Choose an element e not belonging to A and denote by Z the absolutely free groupoid generated by the set $A \cup \{e\}$. To avoid confusion, we denote the multiplication in Z by $\circ$. For $u, v \in Z$ we shall write usv if $v = u \circ w$ or $v = w \circ u$ for some $w \in Z$. The smallest reflexive and transitive relation on Z containing s will be denoted by t. Further, let B be the set of all the elements $z \in Z$ such that

$$\begin{gathered}\text{non } u \circ (v \circ u)\, tz \\ \text{non } (u \circ v) \circ utz \\ \text{non } e \circ etz \\ \text{non } a \circ btz\end{gathered}$$

for all $a, b \in A$ and all $u, v \in Z$. The set B, as one may check easily, possesses the following properties:
1. $A \cup \{e\} \subseteq B$.
2. If $u \circ v \in B$ then $u, v \in B$.
We shall define a binary operation $*$ on B in the following way:
(i) $a*b = ab$ for all $a, b \in A$ and $e*e = e$.
(ii) $u*v = u \circ v$ *if* $u, v \in B$ and $u \circ v \in B$.
(iii) Let $u, v \in B$ and let $u*v$ be not defined. Then there is uniquely determined $z \in Z$ such that $v = z \circ u$ or $u = v \circ z$ (here we use the fact that Z is an absolutely free groupoid). The property (2) of B yields $z \in B$ and we set $u*v = z$.
Now it remains to show $B \in \text{Mod}\,(x = yx \,.\, y)$. For, let $u, v \in B$.
If $u = v = e$ then $(u*v)*u = e = v$.
If $u, v \in A$ then $(u*v)*u = uv \,.\, u = v$.
If $u = v \circ z$ for some z then $(u*v)*u = z*u = z*(v \circ z) = v$.
If $v = z \circ u$ for some z then $(u*v)*u = z*u = z \circ u = v$.
In all other cases $(u*v)*u = (u \circ v)*u = v$.

Proposition 10. If the length of a groupoid term t is at most there, then the groupoid variety $\text{Mod}\,(x = t)$ is extensive.

Proof. If $\text{var}\,(t)$ contains at most two variables then either $\text{Mod}\,(x = t)$ is trivial or one of Propositions 2,5–9 applies. If $x \notin \text{var}\,(t)$ then $\text{Mod}\,(x = t)$ is trivial. The remaining case is $\text{var}\,(t) = \{x, y, z\}$. The variety $\text{Mod}\,(x = x \,.\, yz)$ is equal to $\text{Mod}\,(x = xy)$, $\text{Mod}\,(x = yz \,.\, x)$ is equal to $\text{Mod}\,(x = yx)$ and the varieties $\text{Mod}\,(x = yx \,.\, z)$, $\text{Mod}\,(x = y \,.\, xz)$ are trivial. Further, let $A \in \text{Mod}\,(x = xy \,.\, z)$. Put $B = A \cup \{e\}$ where e is an element not belonging to A. Take an element $c \in A$ and define $ea = e$ for all $a \in B$, $ae = ac$ for all $a \in A$. Then $B \in \text{Mod}\,(x = xy \,.\, z)$ is a groupoid with idempotent and A is a subgroupoid of B. Similarly we can show that $\text{Mod}\,(x = y \,.\, zx)$ is extensive.

Proposition 11. The groupoid variety $\text{Mod}\,(x = xy \,.\, yx)$ is extensive.

Proof. Let $A \in \text{Mod}\,(x = xy \,.\, yx)$. Choose an element e such that $e \notin A$ and set

$B = A \cup \{e\}$. We shall define the groupoid structure on B as follows: A will be a subgroupoid in B and $ae = a$, $ea = e$ for all $a \in B$. Obviously $B \in \text{Mod}\,(x = xy\,.\,yx)$.

Proposition 12. The groupoid variety $\text{Mod}\,(x = yx\,.\,xy)$ is extensive.

Proof. Let $A \in \text{Mod}\,(x = yx\,.\,xy)$ and let A have no idempotent elements. The mapping f of A into itself defined by $f(a) = aa$ possesses the following properties:

$$f(f(a)) = a \text{ for all } a \in A,$$
$$f \text{ is a permutation of } A,$$
$$f(a) \neq a \text{ for all } a \in A.$$

From this it follows that there are two disjoint sets $C \subseteq A$ and $D \subseteq A$ such that $A = C \cup D$ and f is a one-to-one mapping of C onto D and of D onto C. Choose an element e not belonging to A and define a groupoid B in this way:

(i) $B = A \cup \{e\}$ and A is a subgroupoid in B.

(ii) $ee = e$ and $ae = e$, $ea = aa$ for all $a \in C$.

(iii) $ae = aa$ and $ea = e$ for all $a \in D$.

It remains to show $B \in \text{Mod}\,(x = yx\,.\,xy)$. But $ae\,.\,ea = e\,.\,aa = e$, $ea\,.\,ae = aa\,.\,e = aa\,.\,aa = a$, $be\,.\,eb = bb\,.\,e = e$, $eb\,.\,be = e\,.\,bb = bb\,.\,bb = b$ for all $a \in C$ and $b \in D$.

Proposition 13. The groupoid variety $K = \text{Mod}\,(x = yy\,.\,xy)$ has the following properties:

(i) $K = \text{Mod}\,(x = yx\,.\,yy)$.

(ii) Any groupoid from K is a quasigroup.

(iii) K is extensive.

Proof. The equation $x = yy\,.\,xy$ implies:

$$(1)\ x = y(x\,.\,yy) \text{ (the substitution } yy \text{ for } y)$$
$$(2)\ xx = (yx\,.\,yx)\,y \text{ (since } xx = (yx\,.\,yx)\,(xx\,.\,yx) = (yx\,.\,yx)\,y)$$
$$(3)\ x = ((y\,.\,xx)\,(y\,.\,xx))\,y \text{ (as follows from (2) using the substitution } xx \text{ for } x)$$
$$(4)\ yx\,.\,yx = yy\,.\,xx \text{ (as, by (2), } yx\,.\,yx = yy\,.\,((yx\,.\,yx)\,y) = yy\,.\,xx).$$

Now we can write, using 1 and 4,

$yx\,.\,yy = yx\,.\,((xx\,.\,yx)\,(xx\,.\,yx)) = yx\,.\,((xx\,.\,xx)\,(yx\,.\,yx)) = yx\,.\,(x\,.\,(yx\,.\,yx)) = x\,.$ Similarly we can show that $x = yx\,.\,yy$ implies $x = yy\,.\,xy$. Let $A \in K$. The equations (1) and (3) show that A is a division groupoid. Let $a, b, c \in A$ and $ab = ac$. Then $b = ab\,.\,aa = ac\,.\,aa = c$. Similarly, if $ba = ca$ then $b = aa\,.\,ba = aa\,.\,ca = c$. Thus we have proved that A is a cancellation groupoid, and consequently A is a quasigroup Let $A \in K$ and e be an element not belonging to A. We denote by Z the absolutely free groupoid freely generated by the set $A \cup \{e\}$ and by $\circ$ the multiplication in Z. Define s and t in the same way as in the proof of Proposition 9. Further we shall define z' for all $z \in Z$ in this way: $e' = e$, $a' = aa$ for all $a \in A$ and $(u \circ v)' = u' \circ v'$ for all $u, v \in Z$. Obviously $z'' = z$ and $u = v'$ iff $u' = v$. Let B be the set of all $z \in Z$ such that

$$\text{non } u \circ utz$$
$$\text{non } a \circ btz$$
$$\text{non } u' \circ (v \circ u)\,tz$$
$$\text{non } (u \circ v) \circ u'\,tz\,.$$

Obviously, $A \subseteq B, z \in B$ iff $z' \in B$ and if $u \circ v \in B$ then $u, v \in B$.
We shall define a binary operation $*$ on the set B in the following way:
a) $a*b = ab$ for all $a, b \in A$ and $e*e = e$.
b) $z*z = z'$ for all $z \in B$.
c) If $z, w \in B$ and $z = u', w = v \circ u$ for some u, v then $z*w = v$.
d) If $z, w \in B$ and $z = u \circ v, w = u'$ for some u, v then $z*w = v$.
e) If $z, w \in B$ and $z*w$ is not yet defined then, as it is easy to see, $z \circ w \in B$ and we put $z*w = z \circ w$.
Now it remains to prove $B \in \mathrm{Mod}\,(x = yy\,.\,xy)$. For, let $u, v \in B$.
If $u = v$ then $(u*u)*(u*u) = u'*u' = u'' = u$.
If $u \neq v$ and $u, v \in A$ then $(u*u) * (v*u) = uu*vu = uu\,.\,vu = v$.
If $u \neq v, u = z \circ w$ and $v = w'$ for some z, w then $(u*u) * (v*u) = u'*z =$
$= (z' \circ w') \circ z = w' = v$.
If $u \neq v, u = z'$ and $v = z \circ w$ for some z, w then $(u*u) * (v*u) = u'*w = z*w =$
$= z \circ w = v$.
In all other cases, $(u*u) * (v*u) = u'*(v \circ u) = v$.

Proposition 14. Let u and v be two groupoid terms, each of them having length two, such that var $(u) \subseteq \{x, y\}$ and var $(v) \subseteq \{x, y\}$. Then the groupoid variety $\mathrm{Mod}\,(x = uv)$ is extensive.

Proof. We have sixteen possibilities. For each of them one of Propositions 2, 5, 6, 7, 11, 12, 13 gives the result with the exception of $\mathrm{Mod}\,(x = xy\,.\,yy)$ and $\mathrm{Mod}\,(x = yy\,.\,yx)$. If $A \in \mathrm{Mod}\,(x = xy\,.\,yy)$ then it is sufficient to take an element e and define B by the following way:
$B = A \cup \{e\}$, A is a subgroupoid in B, $ee = e$, $ae = a$ and $ea = e$ for all $a \in B$.
Similarly we can show that $\mathrm{Mod}\,(x = yy\,.\,yx)$ is extensive.

Remark. If t is an arbitrary groupoid term of length four then the problem whether the groupoid variety $\mathrm{Mod}\,(x = t)$ is extensive remains open. For example we do not know the answer for the variety $K = \mathrm{Mod}(x = y(x\,.\,xy))$. The groupoid $A = \{0, 1, 2, 3, 4, 5, 6\}$ with multiplication $ab = 2b - a + 1 \pmod 7$ belongs to K and has no idempotent.

Example 1. Let $\mathcal{J} = \mathrm{Mod}\,(x = xx\,.\,(yy\,.\,y))$,
$L = \mathrm{Mod}\,(yy\,.\,y = xx\,.\,((yy\,.\,y)(yy\,.\,y)))$, $K = \mathcal{J} \cap L$. Then:
(i) $\mathcal{J}, L$ are extensive and K is non-trivial.
(ii) If a groupoid A from K has an idempotent then it is the one-element groupoid.
(iii) Any non-trivial groupoid variety contained in K is not extensive. In particular, K is not extensive.
The fact that L and $\mathcal{J}$ are extensive follows from Proposition 5. Further, the groupoid $A = \{1, 2\}$ with multiplication $1.1 = 2, 1.2 = 1, 2.1 = 1$ and $2.2 = 1$ belongs to K, as one may check easily. Let $B \in K$ and $e \in B$ be an idempotent. Then, for all $x \in B$, we can write $x = xx\,.\,(ee\,.\,e) = xx\,.\,e = xx\,.\,((ee\,.\,e)\,(ee\,.\,e)) = ee\,.\,e = e$.
The assertion (iii) follows easily from (ii).

Example 2. Let $\mathcal{J} = \mathrm{Mod}\,(xx\,.\,yy = yy\,.\,xx)$, $L = \mathrm{Mod}\,(xx = (yy\,.\,y)\,x)$ and $K = \mathcal{J} \cap L$. Then:

(i) $\mathcal{J}$ and L are extensive.

(ii) Let $A \in K$ and $x, y \in A$ be such that $xx = yy \,.\, y$. Then either $xx \,.\, xx$ is idempotent or A has no idempotents.

(iii) Any groupoid from K has at most one idempotent.

(iv) Let $A \in K$ be a groupoid without idempotent elements. Then A can be imbedded into a groupoid from K having an idempotent iff $xx \neq yy \,.\, y$ for all $x, y \in A$.

(v) K is not extensive.

(vi) If $F \in K$ is free then F can be imbedded into a groupoid from K having an idempotent.

$\mathcal{J}$ and L are extensive by Propositions 2, 5. Further, let $A \in K, x, y \in A$ and $xx = yy \,.\, y$. Assume that A possesses an idempotent element e. We have $e = ee = (yy \,.\, y)\, e = xx \,.\, e = xx \,.\, ee = ee \,.\, xx = e \,.\, xx = (ee \,.\, e) \,.\, xx = xx \,.\, xx$.

(iii) follows immediately from (ii). Let $A \in K$ and let $xx \neq yy \,.\, y$ for all $x, y \in A$. Take an element e not belonging to A and define a groupoid B in such a way: $B = A \cup \{e\}$, A is a subgroupoid in B, $ex = xx$ for all $x \in B$, $xx \,.\, e = xx \,.\, xx$ for all $x \in A$ and $ze = e$ provided $z \in A$ and $z \neq xx$ for all $x \in A$. It is an easy exercise to show $B \in K$. In order to show that K is not extensive, let us consider the following groupoid:

$A = \{1, 2, 3\}$, $1.1 = 2$, $1.2 = 1$, $1.3 = 1$, $2.1 = 1$, $2.2 = 1$, $2.3 = 3$, $3.1 = 2$, $3.2 = 3$ and $3.3 = 1$.

For all $x \in A$ we have $xx \,.\, x = 1$ and $xx = 1, 2$. From this we can easily deduce that $A \in K$. However $(2.2) \,.\, 2 = 1 = 3.3$, and hence A cannot be imbedded into a groupoid from K having an idempotent. Finally, suppose that there exists a free groupoid $F \in K$ such that $xx = yy \,.\, y$ for some elements $x, y \in F$. Then, for all $A \in K$, there are $a, b \in A$ with $aa = bb \,.\, b$. However, the last assertion is not true since the groupoid B defined by

$$B = \{1, 2, 3\}, 1.1 = 2, 1.2 = 3, 1.3 = 3, 2.1 = 3, 2.2 = 1, 2.3 = 3, 3.1 = 2, 3.2 = 1, 3.3 = 1$$

belongs to K and has the property $xx \neq yy \,.\, y$ for all x, y.

An identity $u = v$, u and v being two groupoid terms, is said to be balanced if $\operatorname{var} u = \operatorname{var} v$ and if every variable has at most one occurence in u and v. It is easy to see that the groupoid variety $\operatorname{Mod}(u = v)$ is extensive. On the other hand, the problem is not so trivial if we consider $\operatorname{Mod}(u = v)$ as a quasigroup variety.

Proposition 15. The variety of all commutative quasigroups is extensive.

Proof. Let G be a commutative cancellation halfgroupoid (i.e., G is a set with a partial binary operation, $ab = ba$ provided ab or ba is defined and $ab \neq ac$ if ab, ac are defined and $b \neq c$). Choose (pair-wise different and not belonging to G) symbols $x(a, b), y(a, b)$ and put $H = G \cup \{x(a, b), y(a, b) \mid a, b \in G\}$. On the set H we shall define a partial binary operation $*$ as follows:

(i) G is a subhalfgroupoid in H.

(ii) If $a, b \in G$ and ab is not defined then $a*b = b*a = x(a, b)$.

(iii) If $a, b \in G$ and the equality $ac = b$ holds for no $c \in G$ then $a*y(a, b) = y(a, b) * a = b$.
As it is easy to see, H is a commutative cancellation halfgroupoid and the rest of the proof is obvious.

Proposition 16. Let $u = v$ be a balanced groupoid identity of length three. Then the quasigroup variety Mod $(u = v)$ is extensive.

Proof. It is obvious that Mod $(u = v)$ is equal to at least one of the following varieties.
$K_1 = \text{Mod}\,(x \,.\, yz = x \,.\, yz)$, $K_2 = \text{Mod}\,(x \,.\, yz = x \,.\, zy)$, $K_3 = \text{Mod}\,(x \,.\, yz = y \,.\, xz)$, $K_4 = \text{Mod}\,(x \,.\, yz = z \,.\, xy)$, $K_5 = \text{Mod}\,(x \,.\, yz = y \,.\, zx)$, $K_6 = \text{Mod}\,(xy \,.\, z = yx \,.\, z)$, $K_7 = \text{Mod}\,(xy \,.\, z = xz \,.\, y)$, $K_8 = \text{Mod}\,(xy \,.\, z = zx \,.\, y)$, $K_9 = \text{Mod}\,(xy \,.\, z = yz \,.\, x)$, $K_{10} = \text{Mod}\,(x \,.\, yz = xy \,.\, z)$, $K_{11} = \text{Mod}\,(x \,.\, yz = yx \,.\, z)$, $K_{12} = \text{Mod}\,(x \,.\, yz = yz \,.\, x)$, $K_{13} = \text{Mod}\,(x \,.\, yz = zy \,.\, x)$, $K_{14} = \text{Mod}\,(x \,.\, yz = zx \,.\, y)$, $K_{15} = \text{Mod}\,(x \,.\, yz = xz \,.\, y)$, $K_{16} = \text{Mod}\,(x \,.\, yz = z \,.\, yx)$, $K_{17} = \text{Mod}\,(xy \,.\, z = zy \,.\, x)$.
K_2, K_6 and K_{12}. Since every quasigroup is a cancellation and division groupoid, $K_2 = K_6 = K_{12}$ is the variety of all commutative quasigroups and we may apply Proposition 15.
K_1. The variety of all quasigroups is extensive, as it is easy to see.
K_3 and K_7. Let $A \in K_3$ and $a, b \in A$. There is $e \in A$ such that $eb = b$.
Then $e \,.\, ab = a \,.\, eb = ab$, and consequently e is a left unit in A. In particular, $ee = e$. Similarly we can prove that any quasigroup from K_7 has a right unit.
K_4 and K_5. Let $A \in K_5$ and $a, b \in A$. There is $e \in A$ such that $be = b$. We can write $e \,.\, ab = a \,.\, be = ab$. Hence e is a left unit in A, and therefore A contains at least one idempotent element. Further, $K_4 \subseteq K_5$, as one may check easily, and so K_4 is extensive.
K_8 and K_9. Since $K_8 \subseteq K_9$, it is enough to prove that any quasigroup from K_9 has a right unit. For, let $A \in K_9$ and $a, b \in A$. There exists $e \in A$ with $ea = a$. Then $ab = ea \,.\, b = ab \,.\, e$. K_{10}. As it is well known, K_{10} is the variety of all groups, hence being extensive.
K_{11}. If $A \in K_{11}$ and $a, b \in A$ then $ab = a \,.\, eb = ea \,.\, b$ where $e \in A$ is such that $eb = b$. However A is a cancellation groupoid, and therefore e is a left unit in A.
K_{13}. Let $A \in K_{13}$ and $a, b \in A$. There are $c, d \in A$ such that $ac = da = a$. Then $aa = a \,.\, ac = ca \,.\, a$, and hence $c = d$. Now we have $ba = b \,.\, ac = ca \,.\, b = ab$. K_{14} and K_{15}. Similarly as for K_{11}.
Let us note here that $K_4 = K_5 = K_8 = K_9 = K_{11} = K_{14} = K_{15}$ is the variety of all abelian groups.
K_{16}. Let $Q \in K_{16}$. By Theorem 17 and Theorem 18 [4] there are an abelian group $Q(+)$, its automorphism f and $x \in Q$ such that $ab = f^2(a) + f(b) + x$ for all $a, b \in Q$. Denote by M the set consisting of all ordered pairs (n, q) where n is an integer and $q \in Q$. Let $F(+)$ be the free abelian group freely generated by the set M and g be the automorphism of $F(+)$ which is determined by $g((n, q)) = (n + 1, q)$. For all $a, b \in F(+)$ we put $a*b = g^2(a) + g(b) + (0, x)$. Proceeding similarly as in the paragraph 9 of [2] we may prove that $F(*)$ is a free quasigroup in the variety K_{16}. The set $N = \{(0, q) \mid q \in Q,$

$q \neq x\} \cup \{0\}$ is a set of free generators of $F(*)$. Hence the mapping $h: N \to Q$, $h((0, q)) = q$ and $h(0) = x$, can be extended to a homomorphism h of $F(*)$ onto Q. In view of Lemma 27 [2], the mapping $k: F(+) \to Q(+)$ defined by $k(a) = h(a) - h(0)$ is a group homomorphism of $F(+)$ onto $Q(+)$ and $kg = fk$, $kg^{-1} = f^{-1}k$. Let R be the subring generated by g and g^{-1} in the ring End $F(+)$ of all endomorphisms of the abelian group $F(+)$. We can define an R-module structure on $Q(+)$ in the following way:
If $q \in Q$ and $r \in R$ then $r \circ q = k(r(a))$ where $a \in F(+)$ is such that $k(a) = q$. It is obvious that $g \circ q = f(q)$ and $g^{-1} \circ q = f^{-1}(q)$ for all $q \in Q$. Let $P(+, \circ)$ be an injective hull of the R-module $Q(+, \circ)$. For all $a, b \in P$ we define the product ab as follows:

$$ab = g^2 \circ a + g \circ b + x .$$

It is an easy exercise to show that P is a quasigroup (under this operation) and that $P \in K_{16}$. Further, R is a ring without zero divisors and $(P+, \circ)$ is injective. Hence $P(+, \circ)$ is a divisible module. In particular, since $g^2 + g - 1 \neq 0$ in R, there exists $p \in P$ such that $p = g^2 \circ p + g \circ p + x = pp$. Finally, Q is a subquasigroup in the quasigroup P and we are through.
K_{17}. Similarly as for the preceding case.

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