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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 17 (1976), No. 1, 65--72

Persistent URL: http://dml.cz/dmlcz/142380

1976 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 17, NO. 1, Pag. 65–72

Fredholm Points of Compactly Perturbed Bounded Linear Operators

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Received – September 1974, revised 15 January 1975

Several sufficient conditions are shown which guarantee that the additional points of the spectrum of an operator $T = U + V$ generated as a sum of a selfadjoint bounded linear operator V and a compact perturbation U are located outside the circle $\{\lambda : |\lambda| \leqq r(V)\}$, where $r(V)$ is the spectral radius of V. In particular, a condition is presented under which the intersection of the spectrum $\sigma(T)$ and the set $\{\lambda : |\lambda| = r(T)\}$ contains exactly one point. A conterexample is constructed which shows that in general none of additional points of $\sigma(U+V)$ lies outside the circle $\{\lambda : |\lambda| \leqq r(V)\}$.

Фредгольмовы точки ограниченных линейных операторов с компактными возмущениями. В статье содержится несколько достаточных условий гарантирующих следующую структуру спектра $\sigma(T)$ оператора T, который является суммой ограниченного самосопряжённого оператора V и компактного линейного возмущения U: Все точки $\mu \in \sigma(T)$, и которые находятся вне круга $\{\lambda : |\lambda| \leqq r(V)\}$, где $r(V)$ спектральный радиус оператора V, являются полюсами резольвентного оператора $(\lambda I - T)^{-1}$. Более того, множество точек $\mu \in \sigma(T)$, $|\mu| > r(V)$, непусто и не более чем счетно. Есть указаны условия, выполнение которых влечёт за собой, что $\{\mu \in \sigma(T) : |\mu| = r(T) > r(V)\}$ содержит единственную точку $\mu = r(T)$. Построен пример — показывающий, что без некоторых дополнительных предположений, множество $\{\mu \in \sigma(T) : |\mu| > r(V)\}$, где $T = U + V$ и где U и V подчиняются приведённым выше условиям, пусто.

Fredholmovy body ohraničených lineárních operátorů s kompaktními poruchami. Obsahem článku je několik postačujících podmínek zaručujících, že některé body spektra operátoru T, který vznikl jakožto součet samoadjungovaného ohraničeného lineárního operátoru V a kompaktní lineární poruchy U, leží vně kruhu $\{\lambda : |\lambda| \leqq r(V)\}$, kde $r(V)$ značí spektrální poloměr operátoru V, $T = U + V$. Je též ukázána podmínka, při jejímž splnění průnik spektra $\sigma(T)$ a množiny $\{\lambda : |\lambda| = r(T)\}$ obsahuje právě jeden bod. Je též sestrojen příklad, který ukazuje, že obecně žádný bod spektra operátoru $T = U + V$ neleží vně kruhu $\{\lambda : |\lambda| \leqq r(V)\}$.

One of the celebrated Weyl's theorems says [10, p. 395] that a bounded linear self adjoint operator V and the operator $T = U + V$, where U is any compact symetric operator have the same limit spectra $L\sigma(T) = L\sigma(V)$. As usual, we say

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that a spectral point λ belongs to the limit spectrum $L\sigma(A)$ of a linear operator A in a Banach space X, if λ has at least one of the three properties:

a) λ belongs to the continuous spectrum $C\sigma(A)$;
b) λ is a limit point of the point spectrum $P\sigma(A)$;
c) λ is an eigenvalue having infinite multiplicity.

A description of the spectrum of $T = U + V$, where V is a self-adjoint operator and U a compact, generally nonsymetric, operator on a Hilbert space $\mathfrak{H}$, is given by Gochberg and Krein in [1], Theorem 5.1. Neither of the results by Weyl nor by Gochberg and Krein gives information concerning the existence of Fredholm points in the spectrum of T. In particular, information is needed concerning the structure of the peripheral part of the spectrum. In some problems of neutron transport theory [3] and reactor physics [9] it is very important to know that the peripheral spectrum is discrete and contains only isolated poles of the resolvent operator. Such behaviour of the peripheral spectrum also has quite interesting consequences concerning some spectral properties for certain classes of cone preserving operators [4], [8].

The aim of this note is to show several simple sufficient conditions upon operators V and U each of which guarantees that the peripheral spectrum of $U + V$ contains only isolated poles of the resolvent operator $R(\lambda, U + V) = (\lambda I - U - V)^{-1}$.

Let $\mathfrak{Y}$ be a real Banach space, $\mathfrak{Y}'$ its dual and $[\mathfrak{Y}]$ the space of linear endomorphisms of $\mathfrak{Y}$ into $\mathfrak{Y}$. If $y \in \mathfrak{Y}$ and $\|y\|_{\mathfrak{Y}}$ is the norm of y in $\mathfrak{Y}$, then $y' = \sup\{|y'(x)| : x \in \mathfrak{Y}, \|x\|_{\mathfrak{Y}} = 1, \|T\|_{[\mathfrak{Y}]} = \sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{Y}, \|x\|_{\mathfrak{Y}} \leqq 1\}$.

We assume that $\mathfrak{K}$ is a generating and normal cone in $\mathfrak{Y}$, i.e.

(α) $\mathfrak{K} + \mathfrak{K} \subset \mathfrak{K}$,
(β) $\alpha\mathfrak{K} \subset \mathfrak{K}$, $\alpha \geqq 0$, α real,
(γ) $\mathfrak{K} \cap (-\mathfrak{K}) = \{0\}$,
(δ) $\mathfrak{K} - \mathfrak{K} = \mathfrak{Y}$,
(ε) $\exists\delta > 0 : \forall\, x, y \in \mathfrak{K} \Rightarrow \|y + y\|_{\mathfrak{Y}} \geqq \delta\|x\|_{\mathfrak{Y}}$,
(η) $x_n \in \mathfrak{K}$, $\|x_n - x\|_{\mathfrak{Y}} \to 0 \Rightarrow x \in \mathfrak{K}$.

It follows from (α) — (η) that

$$\mathfrak{K}' = \{y' \in \mathfrak{Y}' : \langle y, y' \rangle \geqq 0 \quad \forall y \in \mathfrak{K}\}$$

is a cone in $\mathfrak{Y}'$ and $\mathfrak{K}'$ fulfils (α) — (η).

In the case that $\mathfrak{Y}$ is a Hilbert space we denote the inner product in $\mathfrak{Y}$ by (x, y), $x, y \in \mathfrak{Y}$ and we denote the space by the a symbol $\mathfrak{H}$. We also consider the dual $\mathfrak{H}'$ as identical with $\mathfrak{H}$ associating to every $y' \in \mathfrak{H}'$ the representative $y_{y'} \in \mathfrak{H}$ according to the Riesz theorem: $y'(x) = (x, y_{y'})$.

We call an operator $T \in [\mathfrak{Y}]$ positive [2] (more precisely $\mathfrak{K}$-positive) if $Tx \in \mathfrak{K}$ whenever $x \in \mathfrak{K}$; a positive operator T is called **indecomposable** (originally introduced in [11] and called there semi-non-support operator), if to every pair

$x \in \mathfrak{K} \setminus 0$, $x' \in \mathfrak{K}' \setminus 0$ there corresponds a positive integer $p = p(x, x')$ such that the value $x'(T^p x) = \langle T^p x, x' \rangle$ is positive.

An indecomposable operator T is called **$\mathfrak{K}$-primitive** if there is a positiv integer p_0 such that $\langle T^p x, x' \rangle > 0$ for $p \geqq p_0$, where $x \in \mathfrak{K} \setminus 0$ and $x' \in \mathfrak{K}' \setminus 0$ (originally such an operator was in [11] named non-support operator).

We say that y is a **quasiinterior element** of the cone $\mathfrak{K}$ if $\langle y, x' \rangle > 0$ for all $x' \in \mathfrak{K}' \setminus 0$. A linear form $x' \in \mathfrak{K}'$ is called **strictly positive** if $\langle x, x' \rangle > 0$ for all $x \in \mathfrak{K} \setminus 0$.

We denote by $\widetilde{\mathfrak{Y}}$ the **complex extension** of $\mathfrak{Y}$, i.e. $z \in \widetilde{\mathfrak{Y}}$ if and only if $z = x + \mathrm{i}y$, where $x, y \in \mathfrak{Y}$, and $\mathrm{i}^2 = -1$. The norm in $\widetilde{\mathfrak{Y}}$ is defined by

$$\|z\|_{\widetilde{\mathfrak{Y}}} = \sup \{\|x \cos \vartheta + y \sin \vartheta\|_{\mathfrak{Y}} : 0 \leqq \vartheta \leqq 2\pi\}.$$

If $\mathfrak{Y} = \mathfrak{H}$ is a Hilbert space then $\widetilde{\mathfrak{Y}}$ also may have Hilbert space structure

$$(z, w)_{\widetilde{\mathfrak{H}}} = [(x, u)_{\mathfrak{H}} + (y, v)_{\mathfrak{H}}] + \mathrm{i}[(y, u)_{\mathfrak{H}} - (x, v)_{\mathfrak{H}}], \qquad \text{where} \qquad z = x + \mathrm{i}y,$$

$w = u + \mathrm{i}v$, $x, y, u, v \in \mathfrak{H}$.

Let $T \in [\mathfrak{Y}]$. We let $\widetilde{T}z = Tx + \mathrm{i}Ty$ for $z = x + \mathrm{i}y$, $x, y \in \mathfrak{Y}$ and call $\widetilde{T}$ the **complex extension of** T.

Let $\sigma(\widetilde{T})$ be the spectrum of $\widetilde{T}$ and $r(\widetilde{T}) = \mathbf{sup}\, \{|\lambda| : \lambda \in \sigma(\widetilde{T})\}$ its spectral radius. By definition we set $\sigma(T) = \sigma(\widetilde{T})$ and $r(T) = r(\widetilde{T})$.

Let $T \in [\mathfrak{Y}]$. Then the set $\pi\sigma(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$ is never empty and we call it **the peripheral** (part of the) **spectrum of** T.

Our first criterion requires no restriction upon the space $\mathfrak{Y}$.

Theorem 1. *Let U and V, both in $[\mathfrak{Y}]$, be $\mathfrak{K}$-positive operators. Let U be compact and $\varrho > r(U+V)$. Let us assume that for every $\varepsilon > 0$ there exists a $v'_\varepsilon \in \mathfrak{K}'$ such that*

$$R(\varrho, V')\, v'_\varepsilon = \frac{1}{\varrho - r(V)}\, v'_\varepsilon + w'_\varepsilon, \quad w'_\varepsilon + \nu\varepsilon v'_\varepsilon \in \mathfrak{K}', \quad \tau\varepsilon v'_\varepsilon - w'_\varepsilon \in \mathfrak{K}', \ \textit{where}\ \nu$$

and τ are positive numbers independent of ε. Furthermore, let there exist a $\delta > 0$ independent of ε and x that for every sufficiently small $\varepsilon > 0$ the following relation

$$\langle UR(\varrho, U)\, x, \ v'_\varepsilon \rangle \geqq \delta \langle x, v' \rangle$$

holds for all $x \in \mathfrak{K}$. Then the peripheral spectrum $\pi\sigma(T)$, where $T = U + V$, contains only isolated poles of the resolvent operator $R(\lambda, T) = (\lambda I - T)^{-1}$, I being the identity operator.

Proof. According to [7] it is enough to show that

$$r(V) < r(T)\,. \tag{1}$$

It is known that $r(T) \in \pi\sigma(T) \subset \sigma(T)$ and also that $r(V) \in \pi\sigma(V)$ [12] and that $r(T) \geqq \max\,[r(U), r(V)]$ [5]. It follows that for $\varrho > r(T)$ we have that $r(R(\varrho, T)) = [\varrho - r(T)]^{-1}$ and $r(R(\varrho, V)) = [\varrho - r(V)]^{-1}$.

Let us consider the quantity

$$s_{v'_\varepsilon}(R(\varrho, T)) = \mathbf{sup}\, \{\nu\ \mathbf{real} : \langle R(\varrho, T)\, x, \ v'_\varepsilon \rangle \geqq \nu \langle x, v'_\varepsilon \rangle, \ \langle x, v'_\varepsilon \rangle \neq 0, \ x \in \mathfrak{K} \setminus 0\}. \tag{2}$$

Since

$$R(\varrho, T) = R(\varrho, V) + R(\varrho, V)\, UR(\varrho, T)\,,$$

we derive easily that

$$\langle R(\varrho, T)\, x,\ v_\varepsilon'\rangle \geqq \langle R(\varrho, V)\, x,\ v_\varepsilon'\rangle + \langle R(\varrho, V)\, UR(\varrho, T)\, x,\ v_\varepsilon'\rangle \geqq$$

$$\geqq \frac{1}{\varrho - r(V)} \langle x,\ v_\varepsilon'\rangle - \frac{\delta}{\varrho - r(V)} |x,\ v_\varepsilon'\rangle| +$$

$$+ \langle UR(\varrho, U)\, x,\ R(\varrho, V')\, v_\varepsilon'\rangle \geqq$$

$$\geqq \frac{1}{\varrho - r(V)} [\langle x,\ v_\varepsilon'\rangle - \varepsilon|\langle x,\ v_\varepsilon'\rangle|] +$$

$$+ \delta \frac{1}{\varrho - r(V)} [\langle x,\ v_\varepsilon'\rangle - \varepsilon|\langle x,\ v_\varepsilon'\rangle|] .$$

Since $\varepsilon > 0$ can be taken arbitrary small this implies that

$$s_{v_\varepsilon'}(R(\varrho, T)) > \frac{1}{\varrho - r(V)} = r(R(\varrho, V)) \text{ for such } \varepsilon.$$

According to Lemma 3.4 in [5] we deduce that

$$\frac{1}{\varrho - r(T)} = r(R(\varrho, T)) \geqq s_{v_\varepsilon'}(R(\varrho, T)) > \frac{1}{\varrho - r(V)} = r(R(\varrho, V))$$

and finally that (1) holds. This completes the proof.

Remark. It is an obvious consequence of the normality of the cone $\mathfrak{K}'$ that (1) holds if $r(U) > r(V)$. Thus, $\pi\sigma(T)$ contains only isolated poles of $R(\lambda, T)$ as well. We also see that the assumptions of Theorem 1 and the Theorems given below are essentially needed only if $r(V) \geqq r(U)$. If $\mathfrak{Y} = \mathfrak{H}$ is a Hilbert space then we have the following result.

Theorem 2. *Let U and V both in $[\mathfrak{H}]$ be $\mathfrak{K}$-positive and self-adjoint operators. In addition, let U be compact and let a quasiinterior element u_0 of $\mathfrak{K}$ correspond to its spectral radius $r(U) : Uu_0 = r(U)u_0$, $\|u_0\|_{\mathfrak{H}} = 1$. We assume that there is a system $\{v_\varepsilon\}$, $v_\varepsilon \in \mathfrak{K}\setminus 0$, $\|v_\varepsilon\|_{\mathfrak{H}} \neq 0$* such that $Vv_\varepsilon = r(V)v_\varepsilon + y_\varepsilon$ with $\|y_\varepsilon\|_{\mathfrak{H}} \leqq \varepsilon\|v_\varepsilon\|$ and there is a constant ν independet of ε such that

$$\left|\frac{(y_\varepsilon, u_0)}{(v_\varepsilon, u_0)}\right| \leqq \nu\varepsilon .$$

Then the relation (1) *holds, where $T = U + V$ and therefore, the peripheral spectrum $\pi\sigma(T)$ contains only isolated poles of the resolvent operator $R(\lambda, T)$.*

Proof. Let $\varrho > r(T)$ and let $\varepsilon > 0$ be given. Then, similarly as in the proof of Theorem 1,

$$\frac{(R(\varrho, T)\, v_\varepsilon, u_0)}{(v_\varepsilon, u_0)} \geqq \frac{1}{\varrho - r(V)} + \frac{(y_\varepsilon, u_0)}{(v_\varepsilon, u_0)} + \frac{1}{\varrho} \frac{(v_\varepsilon, R(\varrho, U)\, Uu_0)}{(v_\varepsilon, u_0)}$$

and, according to the assumptions,

$$\frac{(R(\varrho, T)\, v_\varepsilon, u_0)}{(v_\varepsilon, u_0)} \geqq \frac{1}{\varrho - r(V)} - \varepsilon\nu + \frac{1}{\varrho} \frac{r(U)}{\varrho - r(U)} .$$

Since $\varepsilon > 0$ can be chosen sufficiently small, we deduce that

$$\frac{(R(\varrho, T)\, v_\varepsilon, u_0)}{(v_\varepsilon, u_0)} > \frac{1}{\varrho - r(V)} = r(R(\varrho, V))\,. \tag{3}$$

On the other hand,

$$r(R(\varrho, T)) = \|R(\varrho, T)\| = \mathbf{sup}\left\{\frac{(R(\varrho, T)\, x, u_0)}{(x, u_0)} : x \in \mathfrak{H},\ (x, u_0) \neq 0\right\}$$

and hence, according to (3)

$$\frac{1}{\varrho - r(T)} = r(R(\varrho, T)) > r(R(\varrho, V)) = \frac{1}{\varrho - r(V)}$$

which implies the required relation (1) and actually completes the proof.

Remark. The condition concerning the system of approximate eigenvectors $\{v_\varepsilon\}_{\varepsilon>0}$ is obviously satisfied if $r(V)$ is an eigenvalue of V with an eigenvector $v_0 \in \mathfrak{K}$. In this case $v_\varepsilon = v_0$ and $y_\varepsilon = 0$ for all $\varepsilon > 0$.

According to Theorem 1 and 2 the peripheral spectrum of T is a finite set. It is some times needed that it contains exactly one point. We present a simple condition which guarantees that $\operatorname{card} \pi\sigma(T) = 1$.

Theorem 3. *Let U and V both in $[\mathfrak{Y}]$ be $\mathfrak{K}$-positive operators. Let V satisfy the conditions of either Theorem 1 or Theorem 2 and U let be compact and $\mathfrak{K}$-primitive, and, in addition, if V satisfies the assumption of Theorem 2, let U be self-adjoint. Then the peripheral spectrum of T contains exactly one point: $\pi\sigma(T) = \{r(T)\}$ and the corresponding eigenspace is one-dimensional.*

Proof. It is easy to see that the conclusions of either Theorem 1 or Theorem 2 are valid. Hence $\pi\sigma(T) = \{r(T) = \lambda_0, \lambda_1, \ldots, \lambda_q\}$, $q < +\infty$. The $\mathfrak{K}$-positivity of V implies that $T = U + V$ is $\mathfrak{K}$-primitive too. The required conclusions then follows using standard arguments [5]. The proof is complete.

Next we show a model situation which is typical for some neutron transport problems [6], [9].

Let μ be a nonnegative regular measure on a Euclidean space R^d, $d \geqq 1$. Let $\Omega \subset R^d$ be a given closed set with $\mu(\Omega) = 1$. As usual we denote by $\mathfrak{C}(\Omega)$ the Banach space of continuous functions on Ω with the supremum norm and by $\mathfrak{L}^p(\Omega, \mu)$, $p \geqq 1$, the Banach space of classes of μ-integrable functions with p-th power on with the norm

$$\|u\|^p_{L^p(\Omega,\mu)} = \int_\Omega |u(s)|^p d\mu\,.$$

In particular, $\mathfrak{L}^2(\Omega, \mu)$ has Hilbert space structure with the inner product

$$(u, v) = \int_\Omega u(s)\, v(s)\, d\mu$$

and $\mathfrak{L}^\infty(\Omega, \mu)$ is the space of classes of essentially bounded μ-measurable functions with the norm

$$\|u\|_{L^\infty(\Omega,\mu)} = \mathbf{sup\ ess}\ u = \mathbf{sup}\,\{|u(s)| : s \in \Omega \setminus E,\ \mu(E) = 0\}\,.$$

Let $f \in \mathfrak{L}^\infty(\Omega, \mu)$ be such that $f(s) \geqq 0$ μ-a.e. in Ω and let $U = U(s, t)$ be square $\mu \times \mu$-integrable on $\Omega \times \Omega$ and furthermore let $U(s, t) \geqq 0$ $\mu \times \mu$-a.e. in $\Omega \times \Omega$.

We define operators V and U by putting

$$Vx = y \Leftrightarrow y(s) = f(s)\, x(s), \quad s \in \Omega \tag{4}$$

$$Ux = y \Leftrightarrow y(s) = \int_\Omega U(s, t)\, x(t)\, \mathrm{d}\mu(t)\,. \tag{5}$$

Theorem 4. Let the kernel $U = U(s, t)$, $s, t \in \Omega$, satisfy the following conditions: (i) to every pair $0 \neq u$ and $0 \neq v$ in $\mathfrak{L}^2(\Omega, \mu)$, where $u(s) \geqq 0$, $v(s) \geqq 0$ μ-a.e. in Ω, there exists an index $p = \mathrm{p}(u, v)$ such that

$$0 < \int_\Omega \ldots \int_\Omega U(s, t_1) \ldots U(t_{p-1}, t_p)\, u(t_p)\, v(s)\, \mathrm{d}\mu(t_1) \ldots \mathrm{d}\mu(t_p), \mathrm{d}\mu(s) \tag{6}$$

$$U(s, t) = U(t, s), \quad s, t \in \Omega\,. \tag{ii}$$

If U and V defined by (4) and (5) are considered as operators on $\mathfrak{L}^2(\Omega, \mu)$ then the peripheral spectrum $\pi\sigma(T)$ of $T = U + V$ contains only isolated simple poles of the resolvent operator $R(\lambda, T)$.

Proof. Let $\mathfrak{Y} = \mathfrak{L}^2(\Omega, \mu)$, $\mathfrak{K} = \widetilde{\mathfrak{K}}$ be the set of $\mathfrak{L}^2(\Omega, \mu)$-elements whose representatives assume only nonnegative values. Obviously, $\widetilde{\mathfrak{K}}$ satisfies (α) — (η) and its dual is identical with $\widetilde{\mathfrak{K}}$ if we associate the elements in $\widetilde{\mathfrak{K}}'$ with the corresponding representatives via the Riesz representation theorem:

$$\langle x, y' \rangle = (x, y_{y'}), \quad y' \in \mathfrak{K}', \quad x, y\ \mathfrak{L}^2(\Omega, \mu)\,.$$

It is evident that U is $\widetilde{\mathfrak{K}}$-indecomposable, self-adjoint and compact. Thus, there is an eigenvector $u_0 \in \widetilde{\mathfrak{K}}$, $\|u_0\| = 1$, such that $u_0(s) > 0$ μ-a.e. in Ω.

If $f(s) = 0$ μ-a.e. in Ω there is nothing to prove because $r(V) = 0$. Thus, let us fix any representative for the class f and let us denote it also by f. Let

$$\Omega_\varepsilon = \{s \in \Omega : f(s) > r(V) - \varepsilon\}\,.$$

Since $f \neq 0$ $\mu(\Omega_\varepsilon) > 0$ for every $\varepsilon > 0$. Define

$$v_\varepsilon(s) = \begin{cases} 1 & \text{for } s \in \Omega_\varepsilon \\ 0 & \text{elsewhere.} \end{cases}$$

It follows that for $\varrho > r(T)$

$$[(\varrho I - V)^{-1} v_\varepsilon]\,(s) = \begin{cases} \dfrac{1}{\varrho - f(s)} & \text{for } s \in \Omega_\varepsilon \\ 0 & \text{elsewhere,} \end{cases}$$

and hence,

$$R(\varrho, V)\, v_\varepsilon = \frac{1}{\varrho - r(V)}\, v_\varepsilon - y_\varepsilon\,,$$

where

$$y_\varepsilon(s) = \begin{cases} \dfrac{1}{\varrho - r(V)} - \dfrac{1}{\varrho - f(s)} & \text{for } s \in \Omega_\varepsilon \\ 0 & \text{elsewhere,} \end{cases}$$

or else

$$y_\varepsilon(s) = \frac{r(V) - f(s)}{[\varrho - r(V)]\,[\varrho - f(s)]}\, v_\varepsilon(s), \quad s \in \Omega\,.$$

We have that $(v_\varepsilon, u_0) > 0$ and

$$|y_\varepsilon(s)| \leqq \varepsilon v_\varepsilon(s) \qquad \mu\text{-a.e.} \quad \text{in } \Omega\,,$$

which implies that

$$\left| \frac{(y_\varepsilon, u_0)}{(v_\varepsilon, u_0)} \right| \leqq \varepsilon\,.$$

To complete the proof it is enough to show that $r(T)$ is a simple pole of $R(\lambda, T)$. But this is a consequence of the fact that each element of $\pi\sigma(T)$ is a pole of $R(\lambda, T)$ and the $\mathfrak{K}$-indecomposability of T [11]. The proof of the Theorem 4 is complete.

Theorem 5. *Let f be a continuous and nonnegative function on Ω. Let $s_0 \in \Omega$ be such that*

$$f(s_0) = \mathbf{max}\,\{f(s) : s \in \Omega\}\,.$$

Let the kernel $U = U(s, t)$ be continuous and nonnegative on $\Omega \times \Omega$ and let $U(s_0, t) > 0$ for $t \in \Omega \setminus \Omega_0$, where $\operatorname{mes} \Omega_0 = 0$. *Then the peripheral spectrum $\pi\sigma(T)$, where $T = U + V$, and U and V considered as operators on $\mathfrak{C}(\Omega)$, contains only isolated poles of the resolvent operator $R(\lambda, T)$.*

Proof. Let $\hat{U}$, $\hat{V}$ denote the extension of U and V respectively on the space $\mathfrak{L}^2(\Omega, \mu)$ in which $\mathfrak{C}(\Omega)$ forms a dense subset. We set $\hat{T} = \hat{U} + \hat{V}$. Hence, $\hat{U}$ and $\hat{V}$ remain bounded and $\hat{U}$ compact.

Using the previous notation we easily see that $\hat{U}$ and $\hat{V}$ are $\tilde{\mathfrak{K}}$-positive. An argument similar to that used in the proof of Theorem 4 shows that all the assumptions of Theorem 1 are fulfilled if we put e.g.

$$v'_\varepsilon = v_\varepsilon\,, \quad w'_\varepsilon = y_\varepsilon\,.$$

It follows that $\pi\sigma(\hat{T})$ contains only isolated poles of the resolvent operator $R(\lambda, \hat{T})$. Since every eigenfunction of $\hat{T}$ is continuous, we conclude that $\sigma(T) = \sigma(\hat{T})$ and hence $\pi\sigma(T)$ has the required form. The proof is complete.

Theorem 6. *Let the operators U and V defined by* (4) *and* (5) *fulfil the assumptions of Theorem* 4 *and, in addition, let $U(s, t) > 0$ $\mu \times \mu$-a.e. in $\Omega \times \Omega$. Then $\pi\sigma(T) = \{r(T)\}$ if T considered as an operator on $\mathfrak{L}^2(\Omega, \mu)$ and $r(T)$ is an eigenvalue of T with a one-dimensional eigenspace.*

Proof. Let u and v be in $\tilde{\mathfrak{K}} / 0$. Then

$(Uu, v) > 0$

and thus, U is $\tilde{\mathfrak{K}}$-primitive and so is T as well. The assertion is then a consequence of Theorem 4 and 3 and this completes the proof of Theorem 6.

The following example shows that the conclusions of the previous theorems are not valid for quite arbitrary $\mathfrak{K}$-positive operators U and V, where V is bounded and U compact.

Let $\mathfrak{Y} = \mathfrak{L}^2(0, 1)$ be the Hilbert space of classes of Lebesgue square integrable functions on [0, 1]. Let $\tilde{\mathfrak{K}}$ be the cone of nonnegatively valued functions in

$\mathfrak{L}^2(0, 1)$. We define

$$f(s) = \begin{cases} 0 & s \in \left[0, \frac{1}{2}\right] \\ 2s - 1 & s \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and

$$U(s, t) = \begin{cases} \frac{1}{2} & \text{for} \quad s, t \in \left[0, \frac{1}{2}\right] \\ 0 & \text{for} \quad s \in \left(\frac{1}{2}, 1\right], \quad t \in [0, 1] \text{ and } s \in [0, 1], \quad t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Let U and V be operators in $\mathfrak{L}^2(0, 1)$ defined by (5) and (4). It is easy to see that

$$[Tx](s) = \begin{cases} [Ux](s) & \text{for} \quad s \in \left[0, \frac{1}{2}\right], \\ [Vx](s) & \text{for} \quad s \in \left(\frac{0}{2}, 1\right], \end{cases}$$

and thus, $r(T) = \|T\| = \max(\|U\|, \|V\|) = \|V\| = r(V) = 1$. It follows that $\pi\sigma(T) = \{r(T)\} = \{1\}$ however, $\lambda = 1$ is not an isolated point of the spectrum $\sigma(T)$ and belongs to the continuous spectrum $C\sigma(T)$.

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