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The Structure of Primary Modules

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This article characterizes the class of associative rings with identities to which the structural theory of p-primary abelian groups can be carried over. The paper deals with the study of the theory of basic submodules and Ulm-Zippin theory.

В статье оцисывается класс ассоциативных колец с единицей на которые переносится теория строения p-примарных абелевых групп. В работе изучается теория базисных подмодулей и теория Ульма-Цыпина.

V článku je charakterizována třída asociativních okruhů s jednotkovým prvkem, na něž lze přenést strukturní teorii p-primárních abelových grup. Práce se zabývá studiem teorie bazisních podmodulů a Ulm-Zippinovy teorie.

I. Introduction

The present paper continues my investigations [2]. For the rings satisfying the conditions (1I) and (2I) (see § 2 below) one can build almost all the theory known for the p-primary abelian groups. As in [2] we shall restrict ourselves to such structural properties of I-Loewy modules that are logically equivalent to the conditions (1I) and (2I). The Ulm-Zippin's theory for countably generated p-groups can also be carried over to countably generated I-Loewy modules. The only difficulty in proving Ulm's theorem for I-Loewy modules over rings with conditions (1I) and (2I) lies in the fact that the finitely generated I-Loewy modules are not finite, in general. Lemma 5.5 below substitutes this property of abelian p-groups sufficiently and the proof of Ulm's theorem for I-Loewy modules over the rings satisfying the conditions (1I) and (2I) runs then without change.

Although almost all the results on abelian p-groups can be proved for I-Loewy modules over the ring satisfying the conditions (1I) and (2I), we shall restrict ourselves to the fundamental results, only. It will be clear from the theory explained below (and in [2]) in what a way any other result must be formulated and proved.

At the end of the paper the fundamental results contained here and in [2] are summarized.

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2. Preliminaries

In what follows R stands for an associative ring with identity. Unless stated otherwise, by the word module we shall always mean a unitary right R-module. A module M is said to be simple if it is non-zero and has no proper submodules. The socle $S(M)$ of the module M is the submodule of M generated by all simple submodules of M. The Loewy series of M (often called socle sequence, in the literature) is defined for ordinals by $S_1(M) = S(M)$, $S_{\alpha+1}(M) / S_\alpha(M) = S(M/ S_\alpha(M))$ and $S_\alpha M = \bigcup_{\beta<\alpha} S_\beta(M)$, α limit. The smallest ordinal $\alpha = \alpha(M)$ for which $S_\alpha(M) = S_{\alpha+1}(M)$ is the Loewy length of M. M is a Loewy module if $S_\alpha(M) = M$ for some α. A Loewy module with finite Loewy length is said to be bounded. For an element m of a module M the annihilator $(0 : m) = \{r \in R, mr = 0\}$ of m is said to be the order of m.

Let I be a maximal right ideal of R. The I-socle $S(M, I)$ of a module M is the submodule of M generated by all simple submodules of M isomorphic to R/I. As before, we can define the I-Loewy series, I-Loewy length, I-Loewy module (sometimes called primary, or I-primary module).

A right ideal K of R is called Loewy right ideal if R/K is a Loewy module. The Loewy length of K is that of R/K. An I-Loewy right ideal is defined in the obvious way. These definitions are taken from Shores [10].

Let I be a maximal right ideal of R, M an I-Loewy module and $m \in M$. The I-height $h_I^M(m)$ of m in M is defined as the supremum of the set of all integers k (including 0) for which $m \in MI^k$. An I-Loewy module M is said to be divisible if every element of M is of infinite I-height. An I-Loewy module M is said to be I-quasicyclic if $S_{\alpha+1}(M)/ S_\alpha(M)$ is either 0 or isomorphic to R/I, for all ordinals α, and if M is not bounded. An I-Loewy module M is called reduced if it contains no I-quasicyclic submodules.

Note that in the case of abelian groups the Loewy module is the ordinary torsion group, I-Loewy module is the primary group, I-quasicyclic module is the group $C(p^\infty)$ and the I-height coincides with the p-height for the corresponding prime p. Recall, that a cyclic module mR is said to be ideal cyclic if the order of m is a two-sided ideal.

Now we are going to formulate the conditions (1I) and (2I) (see [2]).

(1I) Every right I-Loewy ideal of R is two-sided.

(2I) If I is a two-sided ideal of R which is maximal as a right ideal then I/I^2 is either trivial or simple as both right and left R-module.

For the convenience of the reader we shall formulate the following results of [2].

2.1. Lemma: Let I be an ideal of a ring R which is maximal as a right ideal and let R satisfy the condition (2I). If $a \in I \dot{-} I^2$ is an arbitrary element then $I^k = Ra^k + I^{n+k} = a^kR + I^{n+k}$ for all integers n, k.

Proof: See [2], Lemma 2.2.

2.2. Lemma: If K is an I-Loewy right ideal of a ring R satisfying the conditions (1I) and (2I) then $K = I^n$ for some integer n.

Proof: See [2], Lemma 2.3.

2.3. Lemma: Let I be a maximal right ideal of a ring R satisfying the conditions (1I) and (2I). If M is an I-Loewy module and $m \in M$, $a \in I \dot{-} I^2$ are arbitrary elements then $h_I^M(m) = \sup \{k, xa^k = m$ is solvable in $M\}$.

Proof: See [2], Proposition 2.4.

2.4. Lemma: Let I be a maximal right ideal of a ring R satisfying the conditions (1I) and (2I) and $I^n \supsetneqq I^{n+1}$ for every integer n. Then the following statements on an I-Loewy module are equivalent:

(i) M is a direct sum of I-quasicyclic modules,

(ii) M is divisible,

(iii) M is injective with respect to the full subcategory of I-Loewy modules.

Proof: See [2], Theorem 3.2.

2.5. Lemma: Let R be a ring satisfying the conditions (1I) and (2I). If an I-Loewy module M is a finite direct sum of ideal cyclic submodules, $M = \dot{\sum}_{j=1}^{n} C_j$ then every submodule N of M is a finite direct sum of ideal cyclic submodules, $N = \dot{\sum}_{k=1}^{m} D_k$ and $m \leq n$.

Proof: It follows immediately from Kulikov's criterion ([2], Theorem 4.1, see also Theorem 5.10 below) that N is a direct sum of ideal cyclic submodules. The inequality $m \leq$ n follows at once from the obvious inclusion $S(N) \leq S(M)$.

2.6. Lemma: Let R be a ring satisfying the conditions (1I) and (2I). Then every countably generated I-Loewy module M can be expressed as the union $M = \bigcup_{n=1}^{\infty} M_n$ where $M_n \subseteq M_{n+1}$ and M_n is a finite direct sum of ideal cyclic submodules.

Proof: Let M be generated by the elements $u_1, u_2, \ldots$ and let M_n be the submodule of M generated by $u_1, u_2, \ldots, u_n$. Then $M = \bigcup_{n=1}^{\infty} M_n$ and every M_n is a finite direct sum of ideal cyclic submodules by [2], Theorem 4.2 (see also Theorem 5.10 below).

2.7. Proposition: Let R be a ring satisfying the conditions (1I) and (2I). Then every submodule of a countably generated I-Loewy module M is countably generated.

Proof: Let N be a submodule of M. By 2.6 $M = \bigcup_{n=1}^{\infty} M_n$ where $M_n \subseteq M_{n+1}$ and M_n is a finite direct sum of ideal cyclic submodules. Consequently, $N = \bigcup_{n=1}^{\infty} (N \cap M_n)$ where every $N \cap M_n$ is finitely generated by 2.5. Thus N is countably generated.

2.8. Proposition: Let R be a ring satisfying the conditions (1I) and (2I) and let $I^n = I^{n+1}$ for some n. Then every I-Loewy module is a direct sum of ideal cyclic submodules of orders at most I^n.

Proof: Let M be an I-Loewy module. Clearly $mI^n = 0$ for every $m \in M$, so that $MI^n = 0$ and M is bounded. Now it suffices to use Theorem 4.2 of [2].

2.9. Notation: The above Proposition completely describes the I-Loewy modules over the rings with $I^n = I^{n+1}$ for some n. Thus in the rest of this paper I will always denote a maximal right ideal of R such that $I^n \supsetneqq I^{n+1}$ for every natural integer n.

3. Purity

3.1. Definition: A submodule N of an I-Loewy module M is said to be pure in M if $MI^n \cap N = NI^n$ for every natural integer n.

3.2. Lemma: Let R be a ring satisfying the conditions (1I) and (2I) and let M be an a-Loewy module. If $u = \sum_{i=1}^{k} u_i r_i$, $u_i \in M$, $r_i \in I^n$ then there exists $u' \in M$ such that $u = u'a^n$, where $a \in I \dot{-} I^2$.

Proof: We can clearly assume $u \neq 0$. If I^t annihilates all the u_i, $i = 1, 2, \ldots, k$ then by Lemma 2.1 $r_i = s_i a^n + s_i'$, $s_i \in R$, $s_i' \in I^t$ and $u = \sum_{i=1}^{k} u_i r_i = \sum_{i=1}^{k} u_i s_i a^n = (\sum_{i=1}^{k} u_i s_i)\, a^n$.

3.3. Proposition: Let R be a ring satisfying the conditions (1I) and (2I) and let $a \in I \dot{-} I^2$ be an arbitrary element. Then a submodule N of an I-Loewy module M is pure in M iff for every natural integer n and every $u \in N$ the solvability of the equation $xa^n = u$ in M implies its solvability in N.

Proof: Let N be pure in M and let the equation $xa^n = u$, $u \in N$ be solvable in M. Then $u \in MI^n \cap N = NI^n$, i.e. $u = \sum_{i=1}^{k} u_i r_i$, $u_i \in N$, $r_i \in I^n$. By 3.2 $u = u'a^n$ for some $u' \in N$.

Conversely, let $u \in MI^n \cap N$ be an arbitrary element. Then $u = \sum_{i=1}^{k} u_i r_i \in N$, $u_i \in M$, $r_i \in I^n$. By Lemma 3.2 $u = u'a^n$ for some $u' \in M$ and by the hypothesis there exists an element $v \in N$ with $u = va^n$ and hence $u \in NI^n$. Thus $MI^n \cap N \subseteq NI^n$ and consequently N is pure in M, the converse inclusion being obvious.

3.4. Lemma: Let R be a ring satisfying the conditions (1I) and (2I). If $B \subseteq N$ are submodules of an I-Loewy module M such that B is pure in M and N/B is pure in M/B then N is pure in M.

Proof: Owing to 3.3, the proof is the same as that for abelian groups (see [5], §23, M)) and it will therefore be omitted.

3.5. Lemma: Let R be a ring satisfying the conditions (1I) and (2I). If M is an I-Loewy module such that every element of the socle of M is of infinite I-height in M then M is divisible.

Proof: We shall use the induction on the orders of the elements. Let l be a natural integer, $u \in M$ be an element of order I^k and let every element of order less than I^k be of infinite I-height. It follows from Lemma 2.1 that the element ua^{k-1} is of the order I, so that $va^{k+l-1} = ua^{k-1}$ for some $v \in M$. The element $va^l - u$ is of the order I^{k-1} and the induction hypothesis yields $wa^l = va^l - u$ for some $w \in M$ and the assertion follows easily.

3.6. Proposition: Let R be a ring satisfying the conditions (1I) and (2I). If S is a submodule of an I-Loewy module M such that S is a direct sum of ideal cyclic submodules of the same order I^t then the following are equivalent:

(i) S is a direct summand of M,

(ii) S is pure in M,

(iii) $MI^t \cap S = 0$.

Proof: (i) implies (ii) trivially.

(ii) implies (iii). By Definition 3.1. $MI^t \cap S = SI^t = 0$.

(iii) implies (i). Let T be a submodule of M maximal with respect to $Ma^t \subseteq T$ and $T \cap S = 0$. Assume the existence of an element $u \in M \dot{-} (S \dot{+} T)$ with $ua \in S \dot{+} T$, $ua = v + w$, $v \in S$, $W \in T$. Then $ua^t = va^{t-1} + wa^{t-1}$ and consequently $va^{t-1} = 0$ since $ua^t \in Ma^t \subseteq T$. Thus $za = v$ for some $z \in S$ owing to the form of S and Lemma 2.1. Now $u - z \in S \dot{+} T$ and $(u - z) a = w \in T$. By the choice of T, $(T + (u - z) R) \cap S \neq 0$, so that there is $0 \neq s = x + (u - z) r$, $x \in T$, $r \in R$. Further, $r \in I$, since $(u - z) r = s - x \in S \dot{+} T$. Let I^l be the order of $u - z$. By Lemma 2.1, $r = ar' + r''$, $r' \in R$, $r'' \in I^l$ and $s = x + (u - z) ar' \in S \cap T = 0$ which contradicts the choice of s. Therefore $M = S \dot{+} T$.

3.7. Corollary: Let R be a ring satisfying the conditions (1I) and (2I). Then every non-divisible I-Loewy module M contains a (non-zero) ideal cyclic direct summand.

Proof: M contains an element u of finite I-height. By Lemma 3.5 we can suppose that u lies in the socle of M. Let $va^k = u$, where $k = h_I^M(u)$. Owing to Proposition 3.6 it remains only to show that vR is pure in M. Let $xa^l = vr \neq 0$, $x \in M$, $r \in R$. It follows from 2.1 that $r = sa^i + t$, $s \in R \dot{-} I$, $t \in I^{k+1}$, $i \leq k$ and $sa^k = a^k s' + t'$, $s' \in R \dot{-} I$, $t' \in I^{k+1}$, from which $xa^{l+k-i} = vra^{k-i} = vsa^k = va^k s' = us'$. Now, again by 2.1 $s's'' + z = 1$, $z \in I$, $s'' \in R \dot{-} I$ and $a^{l+k-i} s'' = s''' a^{l+k-i} + z'$, $s''' \in R - I$, $z' \in I^{l'}$ where $I^{l'}$ is the order of x. Consequently, $xs''' a^{l+k-i} = xa^{l+k-i} s'' = us's'' = u$ and $l + k - i \leq k$, $l \leq i$. Finally $vr = vsa^i = (vsa^{i-l})\, a^l$ and vR is pure in M by Proposition 3.3.

3.8. Proposition: Let R be a ring satisfying the conditions (1I) and (2I) and let M be an I-Loewy module. Then a submodule N of M is pure in M iff every coset $u + N$, $u \in M$ contains an element having the same order as $u + N$.

Proof: If N is pure in M and if $u + N$ is of order I^k then $ua^k = v$, $v \in N$ and consequently $u'a^k = v$ for some $u' \in N$. Now the order of $u - u'$ is at most I^k by 2.1 and hence $u - u'$ is of order I^k.

Conversely, for $ua^k = v$, $u \in M$, $v \in N$ let u' be the element of $u + N$ having the same order as $u + N$. So $u'a^k = 0$, $u - u' \in N$ and $(u - u')\, a^k = v$, as desired.

3.9. Proposition: Let R be a ring satisfying the conditions (1I) and (2I). If N is a pure submodule of an I-Loewy module M such that M/N is direct sum of ideal cyclic modules then N is a direct summand of M.

Proof: It follows easily from 3.8 (cf. Theorem 25.2 of [5]).

4. Basic submodules

4.1. Definition: A submodule B of an I-Loewy module M is said to be a basic submodule of M if

(i) B is a direct sum of ideal cyclic submodules,
(ii) B is pure in M,
(iii) M/B is a direct sum of I-quasicyclic modules.

4.2. Theorem: The following are equivalent for a ring R satisfying the condition (1I):

(i) Every I-Loewy module has a basic submodule,
(ii) R satisfies the condition (2I).

Proof: (i) implies (ii). Let M be a finitely generated I-Loewy module. No non-zero factor of M is a direct sum of I-quasicyclic modules, since such modules clearly have no finite set of generators. Thus M is its own basic submodule and R satisfies the conditions (1I) and (2I) by [2], Theorem 4.2 (see Theorem 5.10 below).

(ii) implies (i). The set of elements $\{a_\lambda,\ \lambda \in \Lambda\}$ of an I-Loewy module M will be called purely independent if the sum $\sum_{\lambda \in \Lambda} a_\lambda R$ is direct and pure in M. Using the Zorn's lemma one can easily see that in every I-Loewy module a maximal purely independent set exist. Let $L = \{a_\lambda,\ \lambda \in \Lambda\}$ be a maximal purely independent set of an I-Loewy module M. Then $a_\lambda R$ are ideal cyclic by (1I) and $B = \dot{\sum}_{\lambda \in \Lambda} a_\lambda R$ is pure in M. Suppose that M/B is not a direct sum of I-quasicyclic modules. Then M/B is not divisible by Lemma 2.4 and consequently it contains an ideal cyclic direct summand N/B by Corollary 3.7. Since N/B is clearly pure in M/B, N is pure in M by Lemma 3.4. Morevover, by Proposition 3.9 $N = B \dot{+} C$ where C is ideal cyclic, contradicting the maximality of M, and we are through.

5. Ulm-Zippin's theory

Throughout this section we shall assume that all I-Loewy modules considered are reduced.

5.1. Definition: Let τ be an ordinal. A well-ordered sequence $M_0, M_1, \ldots, M_\alpha, \ldots, \alpha < \tau$ of non-zero I-Loewy modules is called the Ulm sequence of the type τ if

(i) every M_α, $\alpha < \tau$ is a direct sum of ideal cyclic modules,

(ii) every M_α, $\alpha + 1 < \tau$ is unbounded.

We shall call the Ulm sequence countable if τ is a countable ordinal and every M_α, $\alpha < \tau$ is countably generated.

Two Ulm sequences $M_0, M_1, \ldots, M_\alpha, \ldots, \alpha < \tau$ and $M_0', M_1', \ldots, M_\alpha' \ldots$, $\alpha < \tau'$ are said to be isomorphic if $\tau = \tau'$ and $M_\alpha \cong M_\alpha'$ for every $\alpha < \tau$.

5.2. Definition: Let M be an I-Loewy module. Let us define the sequence of submodules M^α of M in the following way: $M^0 = M$, $M^{\alpha+1}$ consists of all the elements of M^α that are of infinite I-height in M^α and $M^\alpha = \bigcap_{\beta<\alpha} M^\beta$, α limit. Further, for every α we put $M_\alpha = M^\alpha / M^{\alpha+1}$. If the sequence just defined is the Ulm sequence then we shall say that M has the Ulm sequence.

5.3. Lemma: Let R be a ring satisfying the conditions (1I) and (2I) and let φ be an epimorphism of an I-Loewy module M onto N such that the kernel K of φ contains only the elements of infinite I-heights. Then $h_I^M(m) = h_I^N(\varphi(m))$ for every $m \in M$.

Proof: Let $a \in I \dot{-} I^2$ be an arbitrary element and let the equation $xa^k = \varphi(m)$ be solvable in N. Then for some $u \in K$ the equation $xa^k = m + u$ is solvable in M. Since $h_I^M(u) = \infty$ by the hypothesis, $h_I^M(m) \geq k$ by Lemma 2.3. Thus $h_I^M(m) — h_I^N(\varphi(m))$ and we are through, the converse inequality being obvious.

5.4. Definition: Let M be an I-Loewy module, $m \in M$. If $m \in M^\gamma \dot{-} M^{\gamma+1}$ and $h_I^{M^\gamma}(m) = n$ then the pair (γ, n) is called the generalized I-height of m in M and is denoted by $H_I^M(m)$.

The crucial step in the next is the following:

5.5. Lemma: Let R be a ring satisfying the conditions (1I) and (2I). If U is a finitely generated submodule of an I-Loewy module M then the set $\{H_I^M(m), m \in U\}$ is finite.

Proof: We shall use the induction on the number of generators of U. Let $U = x_1R$. By Lemma 2.2, $x_1R \cong R/I^t$ for some natural integer t. It follows from Lemma 2.1 that every element of x_1R can be written in the form x_1a^ir, where $a \in I \dot{-} I^2$ is an arbitrary element (fixed in the sequel) and $r \in R \dot{-} I$. Thus for suitable $s \in R$, $s' \in I^t$ we have $rs = 1 + s'(R/I^t$ being local) and $x_1a^irs = x_1a^i$. Now using Lemma 2.1 we get $H_I^M(x_1a^i) = H_I^M(x_1a^irs) \geqslant H_I^M(x_1a^ir) \geqslant H_I^M(x_1a^i)$ and consequently $\{H_I^M(x_1), H_I^M(x_1a), \ldots, H_I^M(x_1a^{t-1})\}$ equals to the set considered.

Let us suppose that every submodule of M with at most $l - 1$ generators has the desired property and let $U = x_1R + x_2R + \ldots + x_lR$ (and U cannot be generated by less than l elements). Let us suppose that for the elements $u_n = \sum_{i=1}^{l} x_ir_i^{(n)}$, $n = 1, 2, \ldots$, we have $H_I^M(u_n) < H_I^M(u_{n+1})$ for all $n = 1, 2, \ldots$. We can clearly assume that all the $x_lr_l^{(n)}$ are non-zero since the converse would contradict to the induction hypothesis. Further, we can suppose that every $r_l^{(n)}$ is equal to some a^k (it follows from the first part of the proof that if we multiply every u_n by a suitable element of R we obtain a new sequence having the desired property and the same generalized I-heights). Moreover, since $x_l R \cong R/I^t$ by Lemma 2.2, we can assume without loss of generality that all $r_l^{(n)}$ are equal to the same a^k, $k < t$. Now considering the differences $u_1 - u_2$, $u_2 - u_3$, ..., $u_n - u_{n+1}$, ... we obtain an infinite set of elements of $g_1 R + \dots + g_{l-1} R$ with pair-wise different generalized I-heights, which contradicts the induction hypothesis.

5.6. Theorem: The following conditions are equivalent for a ring satisfying the condition (1I):

(i) Every countably generated I-Loewy module has countable Ulm sequence,

(ii) every countably generated I-Loewy module has Ulm sequence,

(iii) R satisfies the condition (2I).

Proof: (i) implies (ii). Obvious.

(ii) implies (iii). Since every countably generated I-Loewy module without elements of infinite I-heights has the Ulm sequence of type 1, it is a direct sum of ideal cyclic submodules and it suffices to use Theorem 4.9 of [2] (see Theorem 5.10 below).

(iii) implies (i). Let M be a countably generated I-Loewy module. By Lemma 5.3 no M_α contains elements of infinite I-height. By Proposition 2.7, every M^α and hence every M_α is countably generated and thus M_α is a direct sum of ideal cyclic submodules by Theorem 4.2 of [2].

Now let M_α be bounded, $M_\alpha a^k = 0$. Then $M^\alpha a^k \subseteq M^{\alpha+1}$. For every $m \in M^{\alpha+1}$ and every natural integer l the equation $xa^{k+l} = m$ is solvable in M^α. However, $xa^k \in M^{\alpha+1}$ and $(xa^k)\, a^l = m$ for arbitrary $m \in M^{\alpha+1}$ and a natural integer l shows $M^{\alpha+1}$ is divisible and consequently $M^{\alpha+1} = 0$, M being reduced (see Lemma 2.4). Thus $\alpha + 1 = \tau$. It follows from Lemma 5.5 that for a countably generated module M the set $\{H_I^M(m),\ m \in M\}$ is countable and consequently τ is a countable ordinal.

5.7. Proposition: Let R be a ring satisfying the conditions (1I) and (2I). Then to every countable Ulm sequence M_α, $\alpha < \tau$ there exists a countably generated I-Loewy module M, the Ulm sequence of which is isomorphic to M_α, $\alpha < \tau$.

Proof: The proof of Zippin's theorem (see [5], Theorem 36.1) can easily be adapted to our case (instead of Theorem 16.1 of [5] Lemma 2.4 must be used).

5.8. Definition: We shall say that the Ulm-Zippin's theory holds for I-Loewy modules if the map assigning to each I-Loewy module M the sequence M_α from 5.2 induces a one-to-one correspondence between the isomorphism classes of reduced countably generated I-Loewy modules and the isomorphism classes of countable Ulm sequences.

5.9. Theorem: Let R be a ring satisfying the condition (1I). Then the following are equivalent:

(i) The Ulm-Zippin's theory holds for I-Loewy modules,

(ii) R satisfies the condition (2I).

Proof: (i) implies (ii). It follows immediately from Theorem 5.6 (ii) imples (i). By Theorem 5.6 every countably generated I-Loewy module has the countable

Ulm sequence and by Proposition 5.7 to every countable Ulm sequence M_α, $\alpha < \tau$ there exists a countably generated I-Loewy module M, the Ulm sequence of which is isomorphic to M_α, $\alpha < \tau$. Thus it remains to show that two countably generated I-Loewy modules with isomorphic Ulm sequences are isomorphic.

Let U be a submodule of an I-Loewy module M. As in [5] we shall call an element $m \in M$ proper with respect to U, if $H_I^M(m) \geqslant H_I^M(m + u)$ for all $u \in U$. If U is finitely generated, $m' \notin U$, $m'a \in U$, then the coset $m' + U$ consists of the elements of the finitely generated submodule $U + m'R$ of M and Lemma 5.5 shows the existence of an element $m'' \in m' + U$ which is proper with respect to U. Moreover, the same Lemma shows that among all such elements there is at least one element m with maximal $H_I^M(ma)$. Now one can easily adapt the proof of Ulm's Theorem ([5], Theorem 37.1) to finish the proof of our Theorem. The details will be omitted.

At the end of this paper we summarize some results obtained here and in [2].

5.10. Theorem: For a ring R satisfying the condition (1I) (especially for a subcommutative or commutative ring R) the following conditions are equivalent:

(i) The Kulikov's criterion holds for I-Loewy modules, i.e. an I-Loewy module M is a direct sum of ideal cyclic submodules iff M is the union of an ascending chain of submodules M_n such that the I-heights of elements of M_n in M remain under a finite bound k_n,

(ii) every bounded I-Loewy module is a direct sum of ideal cyclic submodules,

(iii) every finitely generated I-Loewy module is a direct sum of ideal cyclic submodules,

(iv) every countably generated I-Loewy module without elements of infinite I-heights is a direct sum of ideal cyclic submodules,

(v) every I-Loewy module has a basic submodule,

(vi) the Ulm-Zippin theory holds for I-Loewy modules,

(vii) R satisfies the condition (2I).

Proof: See [2], Theorem 4.2 and Theorem 5.9 above.

References

[1] H. Bass: Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc. 95 (1960), 466–488.

[2] L. Bican: Kulikov's criterion for modules (to appear).

[3] S. E. Dickson: A torsion theory for abelian categories, Trans. Amer. Math. Soc. 121 (1966), 223–235.

[4] S. E. Dickson: Direct decompositions of radicals, Proceedings of the Conference on Categorical Algebra, La Jolla 1965, 366–374.

[5] L. Fuchs: Abelian groups, Budapest, 1966.

[6] J. Lambek: Lectures on rings and modules, Waltham, Mass., 1966.

[7] A. P. Mišina, Skornjakov: Abelevy gruppy i moduli, Nauka, Moskva 1969.

[8] A. Rosenberg, D. Zelinski: Finiteness of the injective hulls, Math. Z. 70 (1959), 372–380.

[9] T. SHORES: Decompositions of finitely generated modules, Proc. Amer. Math. Soc. 30 (1972), 445–450.
[10] T. SHORES: The structure of Loewy modules, J. reine angew. Math. 254 (1972), 204–220.
[11] B. STENSTRÖM: Rings and modules of quotients, Lecture Notes in Mathematics 237, Springer Verlag, 1971.