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Free Commutative Idempotent Abelian Groupoids and Quasigroups

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A geometrical construction of free objects in the variety of commutative idempotent abelian groupoids is given. It follows from the construction that the free objects are cancellation groupoids.

Свободные коммутативные идемпотентные абелевы группоиды и квазигруппы. — Дано геометрическое описание свободных объектов в многообразии всех коммутативных идемпотентных абелевых группоидов. Показывается, что эти свободные объекты — группоиды с сокращениями.

Volné komutativní idempotentní abelovy grupoidy a kvazigrupy. — Je dána geometrická konstrukce volných objektů ve varietě všech komutativních idempotentních abelových grupoidů. Z konstrukce plyne, že tyto volné objekty jsou grupoidy s krácením.

A groupoid G is called
— *commutative* if $ab = ba$ for all $a, b \in G$,
— *idempotent* if $aa = a$ for every $a \in G$, and
— *abelian* if $ab \, . \, cd = ac \, . \, bd$ for all $a, b, c, d \in G$.
The purpose of this paper is to give a description (and in fact a construction) of free groupoids (resp. quasigroups) in the variety $\mathscr{G}$ (resp. $\mathscr{H}$) of all commutative idempotent abelian groupoids (resp. quasigroups). For the sake of brevity, such groupoids (quasigroups) will be called CIA-groupoids (CIA-quasigroups). Some properties of CIA-groupoids and CIA-quasigroups were studied e.g. in [1], [2], [4], [5] and [6].

I. Free CIA-quasigroups

The aim of this section is to describe free CIA-quasigroups. We start with a universal-algebraic background.

Let a (finite or infinite) sequence $\Delta = \langle n_1, n_2, \ldots \rangle$ of non-negative integers be given. *Δ-algebras* are formations $A = \langle X, f_1, f_2, \ldots \rangle$ such that X is a non-empty set and f_i is an n_i-ary operation in X. (If $n_i = 0$, this means $f_i \in X$).

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For any variety K of Δ-algebras we define a variety K^* of Δ^*-algebras in this way: $\Delta^* = \langle 0, n_1, n_2, \ldots \rangle$; a Δ^*-algebra $\langle X, a, f_1, f_2, \ldots \rangle$ belongs to K^* iff $\langle X, f_1, f_2, \ldots \rangle$ belongs to K (and the element $a \in X$ is quite arbitrary).

1.1. Lemma. *Let K be a variety of Δ-algebras and let $\langle X, a, f_1, f_2, \ldots \rangle$ be a free K^*-algebra, freely generated by a set $Y \subseteq X$. Then the Δ-algebra $\langle X, f_1, f_2, \ldots \rangle$ is free in K; it is freely generated by the set $Y \cup \{a\}$.*

Proof is easy.

Let a variety K_1 of Δ_1-algebras and a variety K_2 of Δ_2-algebras be given. The varieties K_1 and K_2 are called *equivalent* if there exists a one-to-one mapping φ of K_1 onto K_2 such that the following two conditions are fulfilled:

(1) if $A \in K_1$, then the algebras A and $\varphi(A)$ have the same underlying sets;

(2) if $A, B \in K_1$ and if f is a mapping of the underlying set of A into the underlying set of B, then f is a homomorphism of A into B iff it is a homomorphism of $\varphi(A)$ into $\varphi(B)$.

1.2. Lemma. *If the varieties K_1 and K_2 are equivalent then an algebra $A \in K_1$ is free in K_1 iff $\varphi(A)$ is free in K_2; if Y is a set of free generators in A, then it is a set of free generators in $\varphi(B)$, as well.*

Proof is easy.

Now we shall construct two special varieties and show that they are equivalent.

The class $\mathscr{H}$ is a variety if CIA-quasigroups are considered as universal algebras with two binary operations (multiplication $\circ$ and division $:$). Algebras of the variety $\mathscr{H}^*$ have, moreover, one nullary operation.

The class $\mathscr{U}$ of all *uniquely 2-divisible abelian groups* (i.e. abelian groups such that the mapping $x \mapsto 2x$ is a permutation) is a variety if these groups are considered as algebras with two binary operations (addition $+$ and subtraction $-$) and one unary operation (denoted by $\frac{1}{2}x$).

1.3. Lemma. *The varieties $\mathscr{H}^*$ and $\mathscr{U}$ are quivalent. The mapping φ and its inverse φ^{-1} are defined in this way: if $A = \langle X, a, \circ, : \rangle \in \mathscr{H}^*$, then $\varphi(A) = \langle X, +, -, \frac{1}{2} \rangle$ where $x + y = (x \circ y) : a$ and $\frac{1}{2}x = a \circ x$; if $B = \langle X, +, -, \frac{1}{2} \rangle \in \mathscr{U}$, then $\varphi^{-1}(B) = \langle X, a, \circ, : \rangle$, where a is the zero element of B and $x \circ y = \frac{1}{2}(x + y)$.*

Proof. The assertion $\varphi(A) \in \mathscr{U}$ follows immediately from the more general Toyoda's theorem [8]; some similar constructions can be found in [3] and [7]. However, the direct proof of 1.3 is easy.

It follows from 1.1, 1.2 and 1.3 that for the description of free algebras in $\mathscr{H}$ it is sufficient to find a description of free algebras in $\mathscr{U}$.

Denote by R the set of all rational numbers which can be expressed as $2^{-m}c$ for some integer c and some natural number m. For any natural number n define n significant elements $e_1^n, \ldots, e_n^n$ of the cartesian power R^n:

$$e_1^n = \langle 1, 0, \ldots, 0 \rangle ,$$
$$e_2^n = \langle 0, 1, 0, \ldots, 0 \rangle ,$$
$$\ldots$$
$$e_n^n = \langle 0, \ldots, 0, 1 \rangle .$$

Put, moreover,

$$e_0^n = \langle 0, 0, \dots, 0 \rangle .$$

1.4. Lemma. *The set* R *is a uniquely* 2-*divisible abelian group with respect to the ordinary addition of rational numbers. The group* R^n, *with operations defined componentwise, is free in the variety* $\mathcal{U}$; *the elements* $e_1^n, \dots, e_n^n$ *are its free generators.*

Proof is well-known and easy.

1.5. Theorem. *The set* R^n *is a* CIA-*quasigroup with respect to the operation* $\circ$ *defined by*

$$\langle x_1, \dots, x_n \rangle \circ \langle y_1, \dots, y_n \rangle = \langle \tfrac{1}{2}(x_1 + y_1), \dots, \tfrac{1}{2}(x_n + y_n) \rangle .$$

This quasigroup R^n *is free in the variety* $\mathcal{H}$; *the elements* $e_0^n, e_1^n, \dots, e_n^n$ *are its free generators.*

Proof is a trivial combination of the previous lemmas.

The construction of free CIA-quasigroups of infinite ranks α (and the proof, as well) is quite analogous; the underlying set is the set of all those mappings f of α into R for which the set $\{j \in \alpha; f(j) \neq 0\}$ is finite.

2. Free CIA-groupoids

Denote by P the set of all rational numbers which can be expressed as $2^{-m}c$ for some integers m and c such that $m \geq 0$ and $0 \leq c \leq 2^m$. Given an integer $n \geq 1$, we denote by F_n the set of all $\langle a_1, \dots, a_n \rangle \in P^n$ such that $a_1 + \dots + a_n \leq 1$. Especially: $F_1 = P$. The set F_n is a groupoid with respect to the operation $\circ$ defined by

$$\langle x_1, \dots, x_n \rangle \circ \langle y_1, \dots, y_n \rangle = \langle \tfrac{1}{2}(x_1 + y_1), \dots, \tfrac{1}{2}(x_n + y_n) \rangle .$$

Define $n + 1$ significant elements $e_0^n, e_1^n, \dots, e_n^n$ of F_n in the same way as in Section 1. We shall prove that $\langle F_n, \circ \rangle$ is a free CIA-groupoid, freely generated by $\{e_0^n, e_1^n, \dots, e_n^n\}$. The construction is thus similar as in the case of CIA-quasigroups; in fact, the free CIA-groupoid of rank $n + 1$ is just the subgroupoid generated by free generators in the free CIA-quasigroup of rank $n + 1$. However, the proof is more complicated. The difficulty is that we do not know a priori that the free CIA-groupoid is cancellative.

2.1. Lemma. *The groupoid* $\langle F_n, \circ \rangle$ *is cancellative and belongs to* $\mathcal{G}$.

Proof is evident.

2.2. Lemma. *The groupoid* $\langle F_n, \circ \rangle$ *is generated by* $\{e_0^n, e_1^n, \dots, e_n^n\}$.

Proof. Denote by H the subgroupoid generated by $\{e_0^n, e_1^n, \dots, e_n^n\}$. We shall prove by the induction on m that whenever $c_1, \dots, c_n$ are non-negative integers such that $c_1 + \dots + c_n \leq 2^m$, then $\langle 2^{-m}c_1, \dots, 2^{-m}c_n \rangle \in H$. If $m = 0$, this follows from $\{e_0^n, \dots, e_n^n\} \subseteq H$. Let now $m \geq 1$ be fixed; we shall proceed by the induction on the number of those i for which c_i is odd. If $c_1, \dots, c_n$ are all even, then

$$\langle 2^{-m}c_1, \dots, 2^{-m}c_n \rangle = \langle 2^{-(m-1)}\, \tfrac{1}{2}c_1, \dots, 2^{-(m-1)}\, \tfrac{1}{2}c_n \rangle \in H$$

by the induction assumption on m. If c_i is odd for exactly one i then $c_1 + + \dots + c_n < 2^m$ and we may write

$$\langle 2^{-m}c_1, \dots, 2^{-m}c_n\rangle = \langle 2^{-m}c_1, \dots, 2^{-m}(c_i - 1), \dots, 2^{-m}c_n\rangle \circ \langle 2^{-m}c_1, \dots, 2^{-m}(c_i + 1), \dots, 2^{-m}c_n\rangle .$$

If there are two i, j $(1 \leq i < j \leq n)$ such that c_i and c_j are odd then

$$\langle 2^{-m}c_1, \dots, 2^{-m}c_n\rangle = \langle 2^{-m}c_1, \dots, 2^{-m}(c_i - 1), \dots, 2^{-m}(c_j + 1), \dots, 2^{-m}c_n\rangle \circ \langle 2^{-m}c_1, \dots, 2^{-m}(c_i + 1), \dots, 2^{-m}(c_j - 1), \dots, 2^{-m}c_n\rangle .$$

Lemma is thus proved.

Denote by T the set of all finite sequences $\langle x_1, \dots, x_p\rangle$ such that $p \geq 1$, every x_i is either 0 or 1 and whenever $p \geq 2$, then $x_{p-1} = 0$ and $x_p = 1$. For any two elements $u = \langle x_1, \dots, x_p\rangle$ and $v = \langle y_1, \dots, y_q\rangle$ of T define an element $u * v \in T$ by the induction on $p + q$:

(1) $0 * 0 = 0$; $1 * 1 = 1$; $0 * 1 = 1 * 0 = \langle 0,1\rangle$;

(2) if $p = 1$ and $q \geq 2$, put $u * v = \langle x_1, y_1, \dots, y_q\rangle$;

(3) if $p \geq 2$ and $q = 1$, put $u * v = \langle y_1, x_1, \dots, x_p\rangle$;

(4) in the case $p \geq 2$ and $q \geq 2$ we count $\langle x_2, \dots, x_p\rangle * \langle y_2, \dots, y_q\rangle = \langle z_1, \dots, z_r\rangle$ and put

$$u * v = \langle x_1, z_1, \dots, z_r\rangle \text{ if } x_1 = y_1;$$
$$u * v = \langle 0,1\rangle \text{ if } r = 2 \text{ and } x_1 \neq y_1;$$
$$u * v = \langle x_1, y_1, z_2, \dots, z_r\rangle \quad \text{if } r \geq 3, \quad x_1 \neq y_1 \text{ and } x_1 = z_1;$$
$$u * v = \langle y_1, x_1, z_2, \dots, z_r\rangle \quad \text{if } r \geq 3, \quad x_1 \neq y_1 \text{ and } y_1 = z_1.$$

2.3. Lemma. *Let a CIA-groupoid $\langle G, \circ\rangle$ and two elements $a, b \in G$ be given. The mapping $h_{a,b}$ of T into G, defined by*

$$h_{a,b}(0) = a,$$
$$h_{a,b}(1) = b \quad \textit{and}$$
$$h_{a,b}(x_1, \dots, x_p) = h_{a,b}(x_1) \circ h_{a,b}(x_2, \dots, x_p) \quad \textit{if} \quad p \geq 2,$$

is a homomorphism of $\langle T, *\rangle$ into $\langle G, \circ\rangle$.

Proof. The operation $*$ was defined so that this might be true.

2.4. Lemma. *If we put $\langle G, \circ\rangle = \langle F_1, \circ\rangle$, $a = 0$ and $b = 1$ in Lemma 2.3 then the mapping $h_{0,1}$ is an isomorphism of $\langle T, *\rangle$ onto $\langle F_1, \circ\rangle$.*

Proof. By 2.2 and 2.3 it is sufficient to show that the mapping $h = h_{0,1}$ is injective. We shall prove by the induction on $p + q$ that $h(x_1, \dots, x_p) = h(y_1, \dots, y_q)$ implies $\langle x_1, \dots, x_p\rangle = \langle y_1, \dots, y_q\rangle$. This is evident if either $p = 1$ or $q = 1$, as $r \geq 2$ implies $h(z_1, \dots, z_r) \notin \{0,1\}$. Let $p \geq 2$ and $q \geq 2$. Suppose $x_1 = 0$ and $y_1 = 1$. We have $0 \circ h(x_2, \dots, x_p) = 1 \circ h(y_2, \dots, y_q)$, which is possible only if $h(x_2, \dots, x_p) = 1$ and $h(y_2, \dots, y_q) = 0$; the last equality gives $q = 2$ and $y_2 = 0$, a contradiction with $\langle 1,0\rangle \notin T$. A similar contradiction can be derived from $x_1 = 1$, $y_1 = 0$. We get $x_1 = y_1$, so that

$$h(x_1) \circ h(x_2, \dots, x_p) = h(x_1) \circ h(y_2, \dots, y_q);$$

as $\langle F_1, \circ\rangle$ is cancellative, $h(x_2, \ldots, x_p) = h(y_2, \ldots, y_q)$ and the induction assumption can be applied.

2.5. Lemma. *The groupoid* $\langle F_1, \circ\rangle$ *is free in* $\mathcal{G}$; *it is freely generated by the set* $\{0,1\} = \{e_0^1, e_1^1\}$.

Proof follows from 2.2, 2.3 and 2.4.

Now we shall prove by the induction on n that $\langle F_n, \circ\rangle$ is freely generated by $\{e_0^n, e_1^n, \ldots, e_n^n\}$ in $\mathcal{G}$. The case $n = 1$ is settled by 2.5. Let $n \geq 2$.

We do not know yet that $\langle F_n, \circ\rangle$ is free. However, there exists a free groupoid in $\mathcal{G}$, freely generated by $\{e_0^n, e_1^n, \ldots, e_n^n\}$; one such groupoid denote by $\langle G_n, *\rangle$.

For every $i = 0,1, \ldots, n$ define a binary relation Θ_i in G_n by $\langle x, y\rangle \in \Theta_i$ iff $x * e_i^n = y * e_i^n$.

2.6. Lemma. *Every* Θ_i *is a congruence relation of* $\langle G_n, *\rangle$.

Proof is easy.

2.7. Lemma. *The congruence* $\Theta_0 \cap \Theta_1 \cap \ldots \cap \Theta_n$ *is trivial.*

Proof. Suppose $\langle x, y\rangle \in \Theta_0 \cap \Theta_1 \cap \ldots \cap \Theta_n$. Denote by D the set of all $a \in G_n$ such that $x * a = y * a$. By the definition of Θ_i we have $\{e_0^n, e_1^n, \ldots, e_n^n\} \subseteq D$. It is easy to see that D is a subgroupoid. Hence $D = G_n$, so that $x, y \in D$ and consequently

$$x = x * x = y * x = x * y = y * y = y\,.$$

Lemma is thus proved.

As $\langle F_{n-1}, \circ\rangle$ is free (by the induction assumption), there exists a homomorphism α of $\langle F_{n-1}, \circ\rangle$ into $\langle G_n, *\rangle$ such that $\alpha(e_0^{n-1}) = e_0^n$, $\alpha(e_1^{n-1}) = e_1^n, \ldots,$ $\alpha(e_{n-1}^{n-1}) = e_{n-1}^n$. By 2.5 there exists a homomorphism β of $\langle F_1, \circ\rangle$ into $\langle G_n, *\rangle$ such that $\beta(0) = e_0^n$ and $\beta(1) = e_n^n$. Define a mapping γ of F_n into G_n by

$$\gamma(x_1, \ldots, x_n) = \alpha(x_1, \ldots, x_{n-1}) * \beta(x_n)\,.$$

2.8. Lemma. γ *is a homomorphism of* $\langle F_n, \circ\rangle$ *into* $\langle G_n, *\rangle$.

Proof.
$$\begin{aligned}
&\gamma(\langle x_1, \ldots, x_n\rangle \circ \langle y_1, \ldots, y_n\rangle) = \\
&\gamma(\tfrac{1}{2}(x_1 + y_1), \ldots, \tfrac{1}{2}(x_n + y_n)) = \\
&\alpha(\tfrac{1}{2}(x_1 + y_1), \ldots, \tfrac{1}{2}(x_{n-1} + y_{n-1})) * \beta(\tfrac{1}{2}(x_n + y_n)) = \\
&\alpha(\langle x_1, \ldots, x_{n-1}\rangle \circ \langle y_1, \ldots, y_{n-1}\rangle) * \beta(x_n \circ y_n) = \\
&(\alpha(x_1, \ldots, x_{n-1}) * \alpha(y_1, \ldots, y_{n-1})) * (\beta(x_n) * \beta(y_n)) = \\
&(\alpha(x_1, \ldots, x_{n-1}) * \beta(x_n)) * (\alpha(y_1, \ldots, y_{n-1}) * \beta(y_n)) = \\
&\gamma(x_1, \ldots, x_n) * \gamma(y_1, \ldots, y_n)\,.
\end{aligned}$$

Denote by F_n' the set of all $\langle x_1, \ldots, x_n\rangle \in F_n$ such that $x_1 + \ldots + x_n \leq \frac{1}{2}$. Evidently, F_n' is a subgroupoid. Define a mapping φ of F_n into F_n' by $\varphi(x_1, \ldots, x_n) =$ $= \langle \frac{1}{2}x_1, \ldots, \frac{1}{2}x_n\rangle$.

2.9. Lemma. φ *is an isomorphism of* $\langle F_n, \circ\rangle$ *onto* $\langle F_n', \circ\rangle$.

Proof is easy.

Define an endomorphism ψ of $\langle G_n, *\rangle$ by $\psi(x) = x * e_0^n$.

2.10. Lemma. *ψ is a homomorphism of $\langle G_n, *\rangle$ onto the subgroupoid G'_n of $\langle G_n, *\rangle$ generated by $\{e_0^n, e_1^n * e_0^n, \ldots, e_n^n * e_0^n\}$.*

Proof is easy.

As $\langle G_n, *\rangle$ is free, there exists a homomorphism δ of $\langle G_n, *\rangle$ into $\langle F_n, \circ\rangle$ such that δ is identical on $\{e_0^n, e_1^n, \ldots, e_n^n\}$. Denote by δ_0 the restriction of δ to G'_n, so that δ_0 is a homomorphism of $\langle G'_n, *\rangle$ into (and, evidently, onto) $\langle F'_n, \circ\rangle$.

2.11. Lemma. *δ_0 is injective.*

Proof. The homomorphism $\gamma\varphi^{-1}\delta_0$ of $\langle G'_n, *\rangle$ into $\langle G_n, *\rangle$ is identical on the generating set $\{e_0^n, e_1^n * e_0^n, \ldots, e_n^n * e_0^n\}$ of $\langle G'_n, *\rangle$ and consequently identical on G'_n.

2.12. Lemma. *For every $i = 0, 1, \ldots, n$ the factor-groupoid G_n/Θ_i is cancellative.*

Proof. As the role of free generators is symmetrical, it is sufficient to prove that G_n/Θ_0 is cancellative. As Θ_0 is just the kernel of ψ, the homomorphism theorem gives an isomorphism of G_n/Θ_0 onto G'_n; by 2.11, G'_n can be embedded into the cancellative groupoid $\langle F_n, \circ\rangle$.

2.13. Lemma. *The groupoid $\langle G_n, *\rangle$ is cancellative.*

Proof. By 2.7 and 2.12, G_n is isomorphic to a subdirect product of cancellative groupoids.

Now we are able to finish the induction. By 2.13 and the definition of ψ, ψ is evidently an isomorphism of $\langle G_n, *\rangle$ onto $\langle G'_n, *\rangle$. By 2.11, δ_0 is an isomorphism of $\langle G'_n, *\rangle$ onto $\langle F'_n, \circ\rangle$. The mapping $\varphi^{-1}\delta_0\psi$ is thus an isomorphism of $\langle G_n, *\rangle$ onto $\langle F_n, *\rangle$; it maps $\{e_0^n, e_1^n, \ldots, e_n^n\}$ identically onto itself.

We have proved:

2.14. Theorem. *The groupoid $\langle F_n, \circ\rangle$ is free in $\mathcal{G}$ for any $n \geq 1$; it is freely generated by $\{e_0^n, e_1^n, \ldots, e_n^n\}$.*

$\mathcal{G}$-free groupoids of finite ranks are thus described. The case of infinite ranks is now easy:

2.15. Theorem. *Let a cardinal number α be given. Denote by F the set of all mappings f of α into P such that $f(j) \neq 0$ holds only for a finite number of elements $j \in \alpha$. For any $f, g \in F$ put $f \circ g = h$ where $h(j) = \frac{1}{2}(f(j) + g(j))$. The groupoid $\langle F, \circ\rangle$ is free in $\mathcal{G}$; it is freely generated by the set E of all $f \in F$ such that $f(j) \in \{0,1\}$ for all $j \in \alpha$ and $f(j) = 1$ for at most one j.*

Proof. F is the union of all subgroupoids generated by finite subsets of E.

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