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Limits in Generalized Algebraic Categories – Contravariant Case

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Dualization of various types of limits by contravariant set functors is investigated. The results obtained are used to the study of the limits in generalized algebraic categories.

В статье сначала рассмотривается, в каком случае контравариянтный множественный функктор дуализирюет предели. Полученые ресультаты применены при изучению обобщенных алгебраических категорий.

Článek má dvě části. Nejprve se vyšetřuje, kdy kontravariantní množinový funktor převádí limity diagramů na kolimity. Získané výsledky jsou pak aplikovány při studiu limit v zobecněných algebraických kategoriích.

The paper has two parts. In the first one we prove that, roughly speaking, given any „non-trivial" diagram scheme $\mathscr{D}$, no non-constant contravariant set functor dualizes limits over $\mathscr{D}$. The next part is devoted to generalized algebraic categories: Given two contravariant set functors F, G, we form a category $A(F, G)$. Objects of $A(F, G)$, algebras, are pairs (X, ω) where X is a set and $\omega : F(X) \to G(X)$ is a mapping, and morphisms are $f : (X, \omega) \to (X', \omega')$ where $f : X \to X'$ is a mapping satisfying $G(f)\,\omega' = \omega\,F(f)$. We show, roughly speaking, that $A(F, G)$ has never products and that it has equalizers iff G dualizes unions (i.e. carries unions of subobjects into co-unions of factor-objects). More in detail, given functors F, G, we characterize those schemes $\mathscr{D}$ such that $A(F, G)$ has limits over $\mathscr{D}$.

This paper continues the investigation started by V. Trnková and P. Goralčík (see [8]) – they proved that $A(F, G)$ has not products as soon as F and G are faithful. Related results were obtained by J. Adámek (see mainly [3]) whose methods we adopt sometimes.

We were introduced to this topic on a seminar lead by V. Trnková. We are extremely grateful to her also for the attention paid to our work.

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Preliminaries. We shall denote by *Set* the category of sets and mappings. Given a set X, the symbols $p_X : X \to 1, \vartheta_X : \Phi \to X$ and $1_X : X \to X$ denote the constant mapping onto the standard one-point set 1, the void mapping into the set X, Φ is the void set, the identity mapping, respectively. If $Y \subset X$ denote by $i_X^Y : Y \to X$ the corresponding inclusion mapping.

As usual, any cardinal is a set.

We shall deal with contravariant set functors only. For a mapping $p : M \to N$ define the constant functor $C_{N,p,M}$ as follows:

$$C_{N,p,M}(\Phi) = N,\ C_{N,p,M}(X) = M \quad \text{for} \quad X \neq \Phi,$$
$$C_{N,p,M}(1_\Phi) = 1_N,\ C_{N,p,M}(\vartheta_X) = p \quad \text{for} \quad X \neq \Phi,$$
$$C_{N,p,M}(f) = 1_M \quad \text{for} \quad f : X \to Y,\ X \neq \Phi\,.$$

We shall write simpler C_M instead of $C_{M,1_M,M}$ and $C_{M,\Phi}$ instead of $C_{M,\vartheta_M,\Phi}$.

I. Dualization of limits

Convention: Given a cardinal α and a functor F, put $P_\alpha^F(X) = \bigcup F(f)\,F(X) - \bigcup F(g)\,F(Y)$ where the first union is taken over all mappings $f : X \to \alpha$ and the second union is taken over all $g : X \to Y$ where $card\, Y < \alpha$. Denote by $\mathscr{A}_F$ the class of all cardinals $\alpha, \alpha > 1$ such that $P_\alpha^F(\alpha) \neq \Phi$.

Note that $\mathscr{A}_F = \Phi$ iff $F = C_{N,p,M}$ for some mapping $p : M \to N$ (see [4]).

Proposition 1.1: If $\alpha \in \mathscr{A}_F$ and $card\, X \geq \alpha$ then $card\, P_\alpha^F(X) = max\,(card\, 2^X,\ card\, P_\alpha^F(\alpha))$. If $\alpha \notin \mathscr{A}_F$ then $P_\alpha^F(X) = \Phi$ for every set X. If $f : X \to Y$ is an epimorphism then, for every cardinal α, $F(f)\,P_\alpha^F(Y) \subset P_\alpha^F(X)$. Further, $P_\alpha^F(X) = \bigcup F(g)\,P_\alpha^F(\alpha)$ where the union is taken over all epimorphisms $g : X \to \alpha$.

Proof see [4].

Definition: A couple of epimorphisms $f, g : X \to Y$ is called a diverse couple if there exists a set $Z, Z \subset X$ such that either $g(Z) = Y$ and $card f(Z) < card\, Y$ or $f(Z) = Y$ and $card\, g(Z) < card\, Y$.

Proposition 2.1: If $\alpha \in \mathscr{A}_F$ and a couple of epimorphisms $f, g : X \to \alpha$ is diverse then $F(f)\,P_\alpha^F(\alpha) \cap F(g)\,P_\alpha^F(\alpha) = \Phi$.

Proof: see [4].

Definition: Let $f : X \to Y,\ g : X \to Z$ be mappings. We say that g is coarser than f if there exists a mapping $h : Y \to Z$ such that $hf = g$.

Proposition 3.1: If $f : X \to Y$ is a mapping then $F(f)\,P_\alpha^F(X) = \bigcup F(g)\,P_\alpha^F(\alpha)$ where the union is taken over all epimorphisms $g : X \to \alpha$ which are coarser than f.

Proof: Evidently, if $g : X \to \alpha$ is coarser than $f : X \to Y$ then $F(f)\,F(Y) \supset F(g)\,F(\alpha)$. Conversely, there exists an epimorphism $k : X \to Z$ and a monomorphism $h : Z \to Y$ with $f = h\,k$. Of course, $F(f)\,F(Y) \subset F(k)\,F(Z)$. If $x \in P^F_\alpha(Z)$ then there is an epimorphism $\bar{k} : Z \to \alpha$ such that $x \in F(\bar{k})\,P^F_\alpha(\alpha)$. Thus, for, $g = \bar{k}\,k$, $F(k)\,(x) \in F(g)\,P^F_\alpha(\alpha)$ and so $F(f)\,P^F_\alpha(X) = \bigcup F(g)\,P^F_\alpha(\alpha)$.

Proposition 4.1: Let $f, g : X \to Y$ be mappings and let F be a functor. Then $F(p_X)\,F(1) \subset F(f)\,F(Y) \cap F(g)\,F(Y)$ and for every $x \in F(p_Y)\,F(1)$, $F(f)\,(x) = F(g)\,(x)$.

Proof is evident.

Proposition 5.1: Let $f : X \to Y$ be a mapping. Then $F(f)\,F(Y) = \bigcup F(g)\,F(\alpha)$ where the union is taken over all epimorphisms $g : X \to \alpha$, $\alpha \in \mathscr{A}_F$ which are coarser than f.

Proof: It follows immediately from Propositions 3.1 and 4.1.

Proposition 6.1: Let F be a functor, $\alpha \in \mathscr{A}_F$. Assume that for mappings $f : X \to Y$, $h : X \to Z$ we have: a couple of epimorphisms $g, k : X \to \alpha$ is diverse whenever g is coarser than f and k is coarser than h. Then $F(f)\,F(Y) \cap F(h)\,F(Y) \cap P^F_\alpha(X) = \Phi$.

Proof: It follows from Proposition 3.1.

Proposition 7.1: Let $\alpha \in \mathscr{A}_F$ for a given functor F. Let a mapping $f : X \to Z$ and a subset $Y \subset X$ have the following property: for every epimorphism $g : X \to \alpha$ such that $g(X - Y)$ is a one-point set the set $F(g)\,F(\alpha)$ is a subset of $F(f)\,F(Z)$. Then there exists a set $U, U \subset Y, card\, U < \alpha$ such that if $x \in Y - U$ then $f(x) \neq f(x')$ for all $x' \in X$.

Proof: Suppose the contrary. Then there is a set V, $V \subset Y$, $card\, V = \alpha$ such that for every $x \in V$ there exists a point $y \in X$, $y \neq x$ with $f(y) = f(x)$. If $card\, f(V) < \alpha$ then we can choose a mapping $g : X \to \alpha$ such that g/V is a bijection from V onto α. Thus, according to Proposition 1.1, $F(g)\,P^F_\alpha(\alpha) \cap F(f)\,F(Z) = \Phi$ and this is impossible.

Hence $card\, f(V) = \alpha$. Thus there is a subset W of V with $card$ W $= \alpha$ and $f/$W is one-to-one. Let $g : X \to \alpha$ be a mapping such that $g\,(X -$ W$)$ is a one-point set and $g/$W is a bijection from W onto the set α. Let $k : X \to \alpha$ be an epimorphism coarser than f. Since $k/X -$ W is onto so g, k is a diverse couple. Applying Proposition 6.1 we derive a contradiction. The proof is finished.

A scheme $\mathscr{D}$, i.e. a small category, is called indecomposable if it is not a sum of two non-void categories. It is called decomposable in the contrary case. A maximal indecomposable subscheme of $\mathscr{D}$ is a component of $\mathscr{D}$. We say that $\mathscr{D}$ has a weakinitial object if there is an object $a \in \mathscr{D}^o$ such that, for every $b \in \mathscr{D}^o$, there exists a morphism $l : a \to b$.

We shall characterize those functors which turn limits into colimits. We say that a functor F dualizes limits over a scheme $\mathscr{D}$ if for every diagram $D : \mathscr{D} \to Set$ with a limit $(A, \pi_a : A \to D(a) \mid a \in \mathscr{D})$, $(F(A), F(\pi_a) : F\,D(a) \to F(A) \mid a \in \mathscr{D})$ is a colimit of FD. In particular, we say that F dualizes unions if the following holds: For every set X and subsets $X_i \subset X$, $i \in I$, the equalities $F(i_X^{X_i})(x) = F(i_X^{X_i})(y)$ for all $i \in I$ imply $F(i_X^U)(x) = F(i_X^U)(y)$ where $U = \bigcup_{i \in I} X_i$.

Proposition 8.1: The functor F dualizes unions iff it dualizes unions of pairwise disjoint subsets.

Proof is easy.

Proposition 9.1: If F does not dualize the union of $\{X_i, X_i \subset X \mid i \in I\}$ then for every set Y, F does not dualize the union of $\{X_i \vee Y, X_i \vee Y \subset X \vee Y \mid i \in I\}$.

Proof is easy.

Definition: Let $(A, \pi_a : A \to D(a) \mid a \in \mathscr{D})$ be the limit of a diagram $D : \mathscr{D} \to Set$. For a given functor F denote by $(B, \sigma_a : F\,D(a) \to B \mid a \in \mathscr{D})$ the colimit of FD. Then there exists exactly one mapping $\varphi : B \to F(A)$ such that $\varphi\,\sigma_a = F(\pi_a)$ for every $a \in .\mathscr{D}$ We say that

1. F spreads the colimit of D if φ is not an epimorphism (see also [3]). In this case we put

$$R_F^D = F(A) - \varphi(B) .$$

2. F shrinks the colimit of D if φ is an epimorphism but it is not an isomorphism.

We say that F spreads (shrinks) colimits over a scheme $\mathscr{D}$ if there exists a diagram $D : \mathscr{D} \to Set$ such that F spreads (shrinks) the colimit of D.

Theorem 1.1: Let $\mathscr{D}$ be a decomposable scheme. F dualizes limits over $\mathscr{D}$ iff $F = C_\Phi$. Moreover, if $\mathscr{A}_F \neq \Phi$ then for every cardinal α there exists a diagram $D : \mathscr{D} \to Set$ such that F spreads the limit $(A, \pi_a : A \to D(a) \mid a \in \mathscr{D})$ of D and $card\, A \geq \alpha$, $card\, R_F^D \geq card\, 2^A$.

Proof: If $F \neq C_\Phi$ then $F(\Phi) \neq \Phi$. Let $D : \mathscr{D} \to Set$ be the constant diagram to Φ. Then $(\Phi, 1_\Phi : \Phi \to D(a) \mid a \in \mathscr{D})$ is the limit of D but $F(\Phi), F(1_\Phi) : F\,D(a) \to F(\Phi) \mid a \in \mathscr{D})$ is not colimit of FD. Conversely, C_Φ dualizes all limits.

Now, assume $\mathscr{A}_F \neq \Phi$. Denote by I the set of all components od $\mathscr{D}$ ane put $\beta = max\,(\alpha, min\, \mathscr{A}_F, card\, 2^I)$. Define the diagram $D : \mathscr{D} \to Set$ as the constant diagram to β, i.e. $D(a) = \beta$, $D(l) = 1_\beta$ for every $a \in \mathscr{D}^o$, $l \in \mathscr{D}^m$. Let $(\beta^I, \pi_i : \beta^I \to \beta \mid i \in I)$ be the I-th power of β. Then $(\beta^I, \pi_a : \beta^I \to D(a) \mid a \in \mathscr{D}^o)$ is the limit of D where $\pi_a = \pi_i$ whenever the i-th component of $\mathscr{D}$ contains a. We are to prove $card\, R_F^D \geq card\, 2^\beta = card\, 2^{\beta^I}$. Take a well-ordering of I and choose a mapping $f : \beta^I \to \beta$ such that $f(\{\gamma_i \mid i \in I\}) = \min_{i \in I} \gamma_i$. If an epimorphism $g : X \to min\, \mathscr{A}_F$ is coarser than π_a for some

$a \in \mathscr{D}^o$ and an epimorphism $h : X \to min\, \mathscr{A}_F$ is coarser than f then g, h is a diverse couple and the inequality follows from Proposition 1.1 and 6.1.

Theorem 2.1: Let $\mathscr{D}$ be a indecomposable scheme without the weak initial object. Then F dualizes limits over $\mathscr{D}$ iff $F = C_M$. Moreover, if $\mathscr{A}_F \neq \Phi$ then for every cardinal α there exists a diagram $D : \mathscr{D} \to Set$ such that F spreads the limit $(A, \pi_a : A \to D(a) \mid a \in \mathscr{D}^o)$ of D and $card\, A \leq \alpha$, $card\, R_F^D \geq card\, 2^A$.

Proof: Clearly, if $F = C_M$ then F dualizes the limits in question.
a) Suppose there are objects $a, b \in \mathscr{D}^o$ such that for no object $c \in \mathscr{D}^o$, there are two morphisms $l : c \to a$, $l' : c \to b$. Define a diagram $D : \mathscr{D} \to Set$ as follows (see also [3]; the sets A, B, C and the mappings $p : A \to C$, $r : B \to C$ will ve determined later):
If $d \in \mathscr{D}^o$ and there is a morphism $l : d \to a$ then, $D(d) = A$,
if $d \in \mathscr{D}^o$ and there is a morphism $l : d \to b$ then $D(d) = B$, $D(d) = C$ otherwise. Let $l : d \to d'$ be a morphism of $\mathscr{D}^m$. If $D(d') = A$ then $D(l) = 1_A$, if $D(d') = B$ then $D(l) = 1_B$, if $D(d) = C$ then $D(l) = 1_C$. If $D(d) = A$ and $D(d') = C$ then $D(l) = p$, if $D(d) = B$ and $D(d') = C$ then $D(l) = r$.

Now, if $\mathscr{A}_F \neq \Phi$ put $\beta = max\,(\alpha, min\, \mathscr{A}_F)$, $C = 1$, $A = B = \beta$ and $p = r = p_\beta$. Then $(A \times B, \pi_a : A \times B \to D(a) \mid a \in \mathscr{D}^o)$ is the limit of D and it follows in the same way as in Theorem 1.1 that F spreads this limit and $R_F^D \geq card\, 2^{A \times B}$.

There remains only the case $F = C_{N,p,M}$. Then take for A, B arbitrary non-void sets and put $C = A \vee B$, $p = i_C^A$, $r = i_C^B$. Thus, $(\Phi, \vartheta_{D(a)} : \Phi \to D(a) \mid a \in \mathscr{D}^o)$ is the limit of D and F does not dualize this limit.
b) If the case a) does not take place then for every object $a \in \mathscr{D}^o$ there is an object $b \in \mathscr{D}^o$ such that there is no morphism $l : a \to b$ but there is a morphism $l' : b \to a$. We can easily construct a chain $\{a_i \mid i \in \gamma\}$ of objects of $\mathscr{D}$ having the following properties:
i) γ is a regular cardinal;
ii) Given $i, j \in \gamma$, there exists a morhism $l : a_i \to a_j$ iff $j \leq i$;
iii) If $c \in \mathscr{D}^o$ then for some a_i there is no morphism $g : c \to a_i$.

Define a diagram $D : \mathscr{D} \to Set$ as follows (the sets $A, A_i, i \in \gamma$ and the mappings $\delta_i : A_i \to A$, $\delta_{ij} : A_i \to A_j$ $i, j \in \gamma$, $i \geq j$ will be determined later):
If $a \in \mathscr{D}^o$ and the set $M^a = \{j \in \gamma \mid$ there is a morphism $l : a \to a_j\}$ is non-void then put $D(a) = A_i$ where $i = sup\, M^a$, if $M^a = \Phi$ then put $D(a) = A$.
If $l : a \to b$ is a morphism of $\mathscr{D}$ then
i) $D(l) = \delta_{ij}$ if $D(a) = A_i$, $D(b) = A_j$
ii) $D(l) = 1_A$ if $D(a) = D(b) = A$
iii) $D(l) = \delta_i$ if $D(a) = A_i$, $D(b) = A$.

If $\mathscr{A}_F \neq \Phi$ put $\beta = max\,(\alpha, min\, \mathscr{A}_F)$, $A = \beta$, $A_0 = F(A)$, $A_i = F(\underset{j<i}{sup}$

card A_j) and δ_i, δ_{ij} arbitrary epimorphisms for $i > j$, $\delta_{ij}\delta_{jh} = \delta_{ik}$, $\delta_{ij}\delta_j = \delta_i$, $\delta_{ii} = 1_{Ai}$. Let $(B, \pi_a : B \to D(a) \mid a \in \mathscr{D}^o)$ be the limit of D. Then, $card\, B \geq card\, A_i$ for all $i \in \gamma$ and, according to Proposition 1.1, $card\, F(B) \geq card\, 2^B$. Let $(C, \varphi_a : F\, D(a) \to C \mid a \in \mathscr{D}^o)$ be a colimit of FD. Then $card\, C \leq \sup_{i \in \gamma} card\, F(A_i) = \sup_{i \in \gamma} card\, A_{i+1} \leq card\, B$ and therefore F spreads the limit of D and $R_F^D \geq card\, 2^B$. If $F = C_{N,p,M}$ then it suffices to take $A = A_0$, $A_j \subset A_i$ iff $i \geq j$, $A_i \neq \Phi$ for all $i \in \gamma$ and $\bigcap_{i \in \gamma} A_i = \Phi$; $\delta_i = i_A^{Ai}$, $\delta_{ij} = i_{Aj}^{Ai}$. Then $(\Phi, \vartheta_{D(a)} : \Phi \to D(a) \mid a \in \mathscr{D}^o)$ is the limit of D and F does not dualize this limit. This completes the proof.

Theorem 3.1: Let $\mathscr{D}$ be a scheme with a weak initial object but without the initial object. Then F dualizes limits over $\mathscr{D}$ iff $F = C_M$. Moreover, if $\mathscr{A}_F \neq \Phi$ then for every cardinal α there exists a diagram $D : \mathscr{D} \to Set$ such that $card\, D(a) \geq \alpha$ for all $a \in \mathscr{D}^o$ and F shrinks the limit of D.

Proof: It follows from an easy observation that for any cardinal α, there exists a diagram $D : \mathscr{D} \to Set$ such that $card\, D(a) > \alpha$ for all objects of $\mathscr{D}$, $D(l)$ is an epimorphism for all morphisms of $\mathscr{D}$ and $(\Phi, \vartheta_{D(a)} : \Phi \to D(a) \mid a \in \mathscr{D}^o)$ is the limit of D.

Proposition 10.1: Let $\mathscr{D}$ be a scheme. Then $\mathscr{D}$ has the initial object iff every functor dualizes limits over $\mathscr{D}$.

Proof is evident.

2. Limits in generalized algebraic categories

Let us begin with the limit over the void scheme i.e. with the terminal object in $A(F, G)$.

Theorem 1.2: The category $A(F, G)$ has the terminal object iff $F = C_{M,\Phi}$ or $G = C_1$ or $G = C_{1,\Phi}$ or $F = C_{M,p,N}$ and G is a contravariant homfunctors.

Proof: If $A(F, G)$ has a terminal object and $F1 \neq \Phi$, $G \neq C_1$, $G1 \neq \Phi$ then analogously as in [2] we can prove that $F = C_{M,p,N}$ and G is a contravariant homfunctor. The rest is evident.

Let us note before getting into further Lemmas that if the scheme $\mathscr{D}$ has the initial object then $A(F, G)$ has limits over $\mathscr{D}$ for any choice of F, G.

Lemma 1.2: Let $\mathscr{D}$ be a scheme without a weak initial object, let F be a functor such that $\mathscr{A}_F \neq \Phi$. If either $\mathscr{A}_G \neq \Phi$ or $card\, G(1) > 1$ then $A(F, G)$ has not limits over $\mathscr{D}$.

Proof: By Theorem 1.1 and 1.2 there exists a diagram $D : \mathscr{D} \to Set$ such that F spreads the limit of D and $card\, A > \alpha$ where $(A, \tau_a : A \to D(a) \mid a \in \mathscr{D}^o)$ is the limit (the cardinal α will be determined later). Let $b \in G(1)$. We define $\omega_{D(a)} : FD(a) \to$

$\to GD(a)$ as the constant mapping to $G(p_{D(a)})(b.)$ Further, define $\omega_A : F(A) \to$ $\to G(A)$ as the constant mapping to $G(p_A)$ and finally, define $D' : \mathscr{D} \to A(F, G)$ as follows: $D'(a) = (D(a), \omega_{D(a)})$, $D'(l) = D(l)$ for every $a \in \mathscr{D}^o$, $l \in \mathscr{D}^m$. Then $((A, \omega_A), \tau_a : (A, \omega_A) \to D'(a) \mid a \in \mathscr{D}^o)$ is a bound of D'.

Assume that $((S, \omega_S), \pi_a : (S, \omega_S) \to D'(a) \mid a \in \mathscr{D}^o)$ is a limit of D'.

Then there exist $k : S \to A$, $k' : A \to S$ such that $\tau_a k = \pi_a$, $\pi_a k' = \tau_a$ for every $a \in \mathscr{D}^o$. Hence $\tau_a k k' = \tau_a$ and so $k k' = 1_A$. Therefore k and $F(k')$ are epimorphisms and then $card\, S \geq card\, A$, $F(S) \neq \bigcup_{a \in \mathscr{D}^o} F(\pi_a)\, D(a)$.

a) Assume that $card\, G(1) > 1$ and $\mathscr{A}_G = \Phi$. Put $\alpha = 0$. Define two mappings $\omega_S^1, \omega_S^2 : F(S) \to G(S)$ as follows:
ω_S^1 is the constant mapping to $G(p_S)\,(b)$, $\omega_S^2(c) = G(p_S)\,(b)$ whenever $c \in \bigcup_{a \in \mathscr{D}^o} F(\pi_a)\, F\, D'(a)$, $\omega_S^2(c) = G(k)\,(d)$ otherwise. Here $d \in G(S)$, $d \neq G(p_S)\,(b)$. Of course, $((S, \omega_S^1), \pi_a : (S, \omega_S^1) \to D'(a) \mid a \in \mathscr{D}^o)$ and $((S, \omega_S^2), \pi_a : (S, \omega_S^2) \to D'(a) \mid a \in \mathscr{D}^o)$ are bounds of D' and so there are some mappings $h_1, h_2 : S \to S$ such that $h_i : (S, \omega_S^i) \to (S, \omega_S)$ and such that $\pi_a = \pi_a h_i$ for $i = 1, 2$, $a \in \mathscr{D}^o$. Thus we have $k\, h_i = k, i = 1, 2$. Choose $z \in F(A) - \bigcup_{a \in \mathscr{D}^o} F(\pi_a) D(a)$. Now, using Proposition 4.1 and the fact that $\mathscr{A}_G = \Phi$, we obtain $G(p_S)\,(b) = \omega_S^1\, F(k)\,(z) = \omega_S^1\, F(h_1)\, F(k)\,(z) = G(h_1)\, \omega_S\, F(k)\,(z) =$ $= G(h_2)\, \omega_S\, F(k)\,(z) = \omega_S^2\, F(h_2)\, F(k)\,(z) = \omega_S^2\, F(k)\,(z) = d$ where $z \in FA - \bigcup_{a \in \mathscr{D}^o} F(\tau_a)(D(a))$, a contradiction because $G(p_S)\,(b) \neq d$.

b) Now assume $\mathscr{A}_G \neq \Phi$. Put $\alpha = max\,(card\, P_\beta^F(\beta), card\, P_\gamma^G(\gamma))$ where $\beta = min\, \mathscr{A}_F$, $\gamma = min\, \mathscr{A}_G$. Let V be an infinite set, $card\, V > card\, S$. Put $X = A \vee V$. Choose a point $c \in A$ and define $\mu_a : X \to D'(a)$ such that $\mu_a | A = \tau_a$, $\mu_a(v) = \tau_a(c)$ for every $v \in V$.

Let V_{ij}, $(i, j) \in I \times \mathcal{J}$ be a decomposition of V such that $card\, V_{ij} = card\, V$ and $card\, I = card\, \mathcal{J} = card\, X$. Choose a well ordering $\leqslant$ of I and, for any limit $i_0 \in I$, put $M_{i_0} = \{Z;\, Z \subset \bigcup_{\substack{i \leqslant i_0 \\ j \in \mathcal{J}}} V_{ij},\, card(Z \cap V_{ij}) \leq 1,\, card\, Z = card\, V,$ $(i, j) \in I \times \mathcal{J}\}$. Further, define $M_Z^F = \bigcup F(g)\, F(\beta)$ where the union is taken over all epimorphisms $g : X \to \beta$ such that $g(x) = g(y)$ whenever $x, y \in X - Z$.

Similarly, define $M_Z^G = \bigcup G(g)\, G(\gamma)$ where the union is taken over all epimorphisms $g : X \to \gamma$ such that $g(x) = g(y)$ whenever $x, y \in X - Z$. Finally, put $V_i = \bigcup_{j \in \mathcal{J}} V_{ij}$. It is easily seen that we can choose a mapping $\omega_X : F(X) \to$ $\to G(X)$ as follows:

$\omega_X(c) = G(p_X)\,(b)$	whenever	$c \in \bigcup_{a \in \mathscr{D}^o} F(\mu_a)\, F(D'(a))$.
ω_X / R_F^D	is a mapping onto	$M_{V_0}^G$,
$\omega_X / M_{V_{ij}}^F$	is a mapping onto	$M_{V_{i+1}}^G$.
ω_X / M_Z^F	is a mapping onto	$M_{V_{i_0}}^G$, $Z \in M_{i_0}$, i_0 is limit.

It suffices to use that for every $(i,j) \in I \times \mathcal{J}$ and $Z \in M_{i_0}, i_0$ limit we have $card\, R_F^D = card\, M_{V_0}^G = card\, M_{V_{ij}}^F = card\, M_{V_i}^G = card\, M_Z^F$. (Theorems 1.1, 2.1 and Proposition 1.1).

Evidently, $((X, \omega_X), \mu_a : (X, \omega_X) \to D'(a) \mid a \in \mathscr{D}^o)$ is a bound of D' and so there is a morphism $h : (X, \omega_X) \to (S, \omega_S)$ with $\tau_a h = \mu_a, a \in \mathscr{D}^o$. Thus, if $x \in A, x \neq c$ then the equality $h(x) = h(y)$ implies $x = y$ (k' is a monomorphism). We will prove by induction that for every $i \in I$ there is a set T, $T \subset V_i, card\, T < card\, A$ such that for any $x \in V_i - T$ the equality $h(x) = h(y)$ implies $x = y$. As $card\, X > card\, S \geq card\, A$ we will obtain $card\, h(X) = card\, I = card\, X$ which is imposible. Of course, $M_{V_0}^G \subset G(h)\, G(X)$ and via Proposition 7.1 the assertion holds for $i = 0$. Assume that the assertion holds for all $i \lll i'$. If i' is non-limit then these is some $i'' \in I$ such that $i'' + 1 = i'$ and so there is a $j \in \mathcal{J}$ such that $M_{V_{i''j}}^F \subset F(h)\, F(X)$. Then $M_{V_{i'}}^G \subset G(h)\, G(X)$ and the assertion for i' follows from Proposition 7.1. If i' is limit then for every $i \lll i'$ there is a set $\mathcal{J}_i \subset \mathcal{J}$ such that, for every $x \in V_{ij}, j \in \mathcal{J}_i$, the equality $h(x) = h(y)$ implies $x = y$. Moreover, $card\, \mathcal{J}_i = card\, V$ (Proposition 7.1). Hence there is a set $Z, Z \in M_i$, with the following property: If $x \in Z, h(x) = h(y)$ then $x = y$. So, $M_Z^F \subset F(h)\, F(x)$ and therefore $M_{V_{i'}}^G \subset G(h)\, G(X)$. Again, Proposition 7.1 yields the assertion for i' and the proof is complete.

Convention: Denote by $\square$ the natural forgetful functor from $A(F, G)$ into *Set*.

Lemma 2.2: Suppose that F shrinks the limits over $\mathscr{D}$. If G dualizes unions and if for every diagram $D : \mathscr{D} \to A(F, G)$ there exists a mapping $\omega_\Phi : F(\Phi) \to G(\Phi)$ such that $((\Phi, \omega_\Phi), \vartheta_{D(a)} : (\Phi, \omega_\Phi) \to D(a) \,/\, a \in \mathscr{D}^o)$ is a bound of D then $A(F, G)$ has limits over $\mathscr{D}$.

Proof: Let $D : \mathscr{D} \to A(F, G)$ be a diagram. Let $(A, \pi_a : A \to \square\, D(a) \,/\, a \in \mathscr{D}^o)$ be the limit of $\square\, D$. Denote by $\mathscr{B}$ the set of all bounds $((X, \omega_X), \tau_a : (X, \omega_X) \to D(a) \in \mathscr{D}^o)$ such that for every distinct points x, y of X there exists an $a \in \mathscr{D}^o$ with $\tau_a(x) \neq \tau_a(y)$. For every bound $((X, \omega_X), \tau_a : (X, \omega_X) \to D(a) \,/\, a \in \mathscr{D}^o)$ there exists $h : X \to A$ such that $\tau_a = \pi_a h$. Denote by U the union of all $h(X)$ over all bounds belonging to $\mathscr{B}$. Thus $G(i_A^U)$ is the co-union of all $G(i_A^{h(X)})$. This, together with the fact that F shrinks limits over $\mathscr{D}$, makes it possible to find a unique mapping $\omega_U : F(U) \to G(U)$ such that $((U, \omega_U), \pi_a|U : (U, \omega_U) \to D(a) \mid a \in \mathscr{D}^o)$ is a bound od D. One can easily prove that this is the limit of D.

Lemma 3.2: Let $\mathscr{D}$ be a decomposable scheme. If $card\, G(\Phi) = 1$ then for every $D : \mathscr{D} \to A(F, G)$ there exists a mapping $\omega_\Phi : F(\Phi) \to G(\Phi)$ such that $((\Phi, \omega_\Phi), \vartheta_{D(a)} : (\Phi, \omega_\Phi) \to D(a) \,/\, a \in \mathscr{D}^o)$ is a bound of D. If $card\, G(\Phi) > 1$ and $F(\Phi) \neq \Phi$ then $A(F, G)$ has not limits over $\mathscr{D}$.

Proof is easy.

Lemma 4.2: Let $\mathscr{D}$ be an indecomposable scheme. If $card\ G(\vartheta_1)\, G(1) = 1$

then for every $D : \mathscr{D} \to A(F, G)$ there exists a mapping $\omega_\Phi : F(\Phi) \to G(\Phi)$ such that $((\Phi, \omega_\Phi), \vartheta_{D(a)} : (\Phi, \omega_\Phi) \to D(a) \,/\, a \in \mathscr{D}^o)$ is a bound of D. If F shrinks colimits over the scheme $\mathscr{D}$ and $card\, G(\vartheta 1)\, G(1) > 1$ then $A(F, G)$ has not limits over $\mathscr{D}$.

Proof: If $\Box\, D(a) = \Phi$ for every $a \in \mathscr{D}^o$ then D is the constant diagram and evidently, there exists the required mapping. If $\Box\, D(a) \neq \Phi$ for some $a \in \mathscr{D}$ then take for ω_Φ the constant mapping to $x, x \in G(\vartheta_1)\, G(1)$. Clearly $((\Phi, \omega_\Phi), \vartheta_{D(a)} : (\Phi, \omega_\Phi) \to D(a) \mid a \in \mathscr{D}^o)$ is a bound of D. Now, assume that F shrinks colimits over the scheme $\mathscr{D}$ and $card\, G(\vartheta_1)\, G(1) > 1$. Let $D : \mathscr{D} \to Set$ be a diagram which F shrinks. Denote by $(A, \pi_a : A \to D(a) \,/\, a \in \mathscr{D}^o)$ the limit of D and by $(B, \sigma_a : F\, D(a) \to B \mid a \in \mathscr{D}^o)$ the colimit of FD. So, we have a mapping $\varphi : B \to F(A)$ such that $F(\pi_a) = \varphi\, \sigma_a$ for every $a \in \mathscr{D}^o$. There exist disctinct points $u, v \in B$ with $\varphi(u) = \varphi(v)$ and distinct points $x, y \in G(\vartheta_1)\, G(1)$. Define a diagram $D' : \mathscr{D} \to A(F, G)$, $D'(a) = (D(a), \omega_{D(a)})$, $D'(l) = D(l)$ for every $a \in \mathscr{D}^o$, $l \in \mathscr{D}^m$. The mapping $\omega_{D(a)} : FD(a) \to GD(a)$ is defined as follows: $\omega_{D(a)}(z) = G(p_{D(a)})\,(x_1)$ if $\sigma_a(z) \neq v$, $\omega_{D(a)}(z) = G(p_{D(a)})\,(y_1)$ if $\sigma_a(z) = v$ where $x_1, y_1 \in G(1)$ such that $G(\vartheta_1)\,(x_1) = x$, $G(\vartheta_1)\,(y_1) = y$. We will prove that there is no bound of D'. Assume $((Z, \omega_Z), \tau_a : (Z, \omega_Z) \to D'(a) \mid$ $\mid a \in \mathscr{D}^o)$ is such a bound. Then there exist $a_1, a_2 \in \mathscr{D}^o$ and $u_1 \in D(a_1)$, $v_1 \in D(a_2)$ with $\sigma_{a_1}(u_1) = u$, $\sigma_{a_2}(v_1) = v$. Further, there is a $\psi : Z \to A$ such that $\tau_a = \pi_a \psi$ for all $a \in \mathscr{D}^o$ and therefore we obtain successively $F(\tau_{a_1})\,(u_1) =$ $F(\psi)\, F(\pi_a)\,(u_1) = F(\psi)\, \varphi\, \sigma_a(u_1) = F(\psi)\, \varphi(u) = F(\psi)\, \varphi(v) = F(\psi)\, \varphi\, \sigma_{a_2}(v_1) =$ $= F(\psi)\, F(\pi_{a_2})\,(v_1) = F(\tau_{a_2})\,(v_1)$.

On the other hand $\omega_Z F(\tau_{a_1})\,(u_1) = G(\tau_{a_1})\, \omega_{D(a)}(u_1) = G(\tau_{a_1})\, G\, p_{D(a_1)}(x_1) =$ $= G(p_Z)\,(x_1)$, $\omega_Z F(\tau_{a_2})\,(v_1) = G(\tau_{a_2})\, \omega_{D(a_2)}(v_1) = G(\tau_{a_2})\, G(p_{D(a_2)})\,(y_1) =$ $= G(p_Z)\,(y_1)$ – a contradiction.

Lemma 5.2: Let $\mathscr{D}$ be a scheme with a weak initial object a_0. If F shrinks the non-void limits over $\mathscr{D}$ and G does not dualize unions then $A\,(F, G)$ has not limits over $\mathscr{D}$.

Proof: Let $D : \mathscr{D} \to Set$ be a diagram such that F shrinks the colimit od D. Denote by $(A, \pi_a : A \to D(a) \mid a \in \mathscr{D}^o)$ the limit of D and by $(C, \sigma_a : FD(a) \to$ $\to C \mid a \in \mathscr{D}^o)$ the colimit of FD. Let $\{B_i, B_i \subset B \mid \alpha \in I\}$ be a system of subsets of B such that G does not dualize its union. Define a diagram $D' : \mathscr{D} \to$ $\to Set$, $D'(a) = D(a) \vee B$, $D'(l) = D(l) \vee 1_B$. Then $(A \vee B, \pi_a \vee 1_B : A \to$ $\to D'(a) \mid a \in \mathscr{D}^o)$ is a limit of D'. Denote by $(C', \sigma'_a : F\, D'(a) \to C' \mid a \in \mathscr{D}^o)$ the colimit of FD'. First we will prove that F shrinks the colimit of D'. There exist $x, y \in F\, D(a_0)$ such that $F(\pi_{a_0})\,(x) = F(\pi_{a_0})\,(y)$ and $\sigma_{a_0}(x) \neq \sigma_{a_0}(y)$. Choose $\varphi_0 : D'(a_0) \to D(a_0)$, $\varphi_1 : A \vee B \to A$ such that $\varphi_0 i^{D(a_0)}_{D'(a_0)} = 1_{D(a_0)}$, $\varphi_1\, i^A_{A \vee B} = 1_A$, $\pi_{a_0}\varphi_0 = \varphi_1(\pi_{a_0} \vee 1_B)$. Then $F(\varphi_0)\,(x) \neq F(\varphi_0)\,(y)$ and $\sigma'_{a_0}(x) \neq \sigma'_{a_0}(y)$ but $F(\pi_{a_0} \vee 1_B)\, F(\varphi_0)\,(x) = F(\pi_{a_0} \vee 1_B)\, F(\varphi_0)\,(y)$. Hence F

shrinks the colimit of D'. By Proposition 9.1, there are $u, v \in G(A \vee B)$ such that $G(i^{B_i}_{A \vee B})(u) = G(i^{B_i}_{A \vee B})(v)$ and $G(i^{U}_{A \vee B})(u) \neq G(i^{U}_{A \vee B})(v)$ where $U = \bigcup_{i \in I} B_i$. Further, there are $u_a, v_a \in G\, D'(a)$ such that $G(\pi_a)(u_a) = u$, $G(\pi_a)(v_a) = v$. Hence we can define a diagram $D'' : \mathscr{D} \to A(F, G)$ such that $D''(a) = (D'(a), \omega_{D'(a)})$ where $\omega_{D'(a)}(z) = u_a$ if $\sigma'_a(z) \neq i^A_{A \vee B} \sigma_{a_0}(x)$, $\omega_{D'(a)}(z) = v_a$ otherwise, $D''(l) = D'(l)$. Suppose $((P, \omega_P), \tau_a : (P, \omega_P) \to D''(a) \mid a \in \mathscr{D}^o)$ is the limit of D''. Then, for every $i \in I$, $((B_i, \omega_{B_i}), (\pi_a \vee 1_B)\, i^{B_i}_{A \vee B} : (B_i, \omega_{B_i}) \to D''(a) \mid a \in \mathscr{D}^o)$ with ω_{B_i} the constant mapping to $G(i^{B_i}_{A \vee B})(u)$ is a bound of D''. As a consequence, $U \subset P$, but this is not possible.

Lemma 6.2: Let $\mathscr{D}$ be a decomposable scheme. If $F = C_{N,p,M}$ with $M \neq \Phi$ and if G does not dualize unions then $A(F, G)$ has not limits over $\mathscr{D}$.

Proof: Take a system $\{A_i, A_i \subset A \mid i \in I\}$ of subsets of A such that G does not dualize its union. Let $D : \mathscr{D} \to Set$ be the constant diagram to A. Denote by $(B, \pi_a : B \to D(a) \mid a \in \mathscr{D}^o)$ the limit of D. Clearly, $B = A^J$ where J is the set of all components of $\mathscr{D}$. Let $\triangle : A \to B$ be the inclusion mapping to the diagonal. There exist $u, v \in G(A)$ such that $G(i^{A_i}_A)(u) = G(i^{A_i}_A)(v)$ but $G(i^U_A)(u) \neq G(i^U_A)(v)$ where $U = \bigcup_{i \in I} A_i$. Clearly G does not dualize the union of the system $\{B_i, B_i \subset B \mid i \in I\}$ where $B_i = \triangle(A_i)$. It holds $G(i^{B_i}_B) G(\pi_{a_1})(u) = G(i^{B_i}_B)\, G(\pi_{a_1})(v) = G(i^{B_i}_B)\, G(\pi_{a_2})(v) = G(i^{B_i}_B)\, G(\pi_{a_2})(u)$ for every $a_1, a_2 \in \mathscr{D}^o$ and $i \in I$. On the other hand, if $U' = \bigcup_{i \in I} B_i$ then $G(i^{U'}_B)\, G(\pi_{a_1})(u) \neq G(i^{U'}_B)\, G(\pi_{a_1})(v)$. Choose a component K of $\mathscr{D}$ and define $\omega_{D(a)} : FD(a) \to GD(a)$ such that $\omega_{D(a)}$ is the constant mapping to u if $a \notin K$, $\omega_{D(a)}$ is the constant mapping to v otherwise. Then, putting $D'(a) = (D(a), \omega_{D(a)})$, $D'(l) = D(l)$, we obtain a diagram $D' : \mathscr{D} \to A(F, G)$. It is easily seen then all $((B_i, \omega_{B_i}), \pi_a\, i^{B_i}_B : (B_i, \omega_{B_i}) \to D'(a)\, /\, a \in \mathscr{D}^o)$ are bounds of D'. If $((P, \omega_P), \varepsilon_a : (P, \omega_P) \to D'(a) \mid a \in \mathscr{D}^o)$ is the limit of D' then $U' \subset P$ and this is a contradiction.

Theorem 2.2: Suppose $F \neq C_{N,p,M}$, $G \neq C_{M,\Phi}$, $G \neq C_1$ and suppose $\mathscr{D}$ is a non-void scheme.

1) If $\mathscr{D}$ has not a weak initial object then $A(F, G)$ has not limits over $\mathscr{D}$.

2) Let $\mathscr{D}$ be a scheme with a weak initial object but without the initial object. If $F(\vartheta_1)$ is an epimorphism then $A(F, G)$ has limits over $\mathscr{D}$ iff G dualizes unions and $card\, G(\vartheta_1)(1) = 1$. If $F(\vartheta_1)$ is not an epimorphism then $A(F, G)$ has limits over $\mathscr{D}$ iff G dualizes unions and $G(\Phi) = 1$.

The proof follows from the previous Lemmas.

Examples: If $F \neq C_{N,p,M}$, $G \neq C_{M,\Phi}$, $G \neq C_1$ then $A(F, G)$ has not products nor pullbacks. If G is a contravariant hom-functor then $A(F, G)$ has equalizers.

Theorem 3.2: Suppose $F = C_{N,p,M}$, $N \neq \Phi$.

1) If $\mathscr{D}$ is a non-void indecomposable scheme without the initial object then

$A(F, G)$ has limits over $\mathscr{D}$ exactly in one of the following cases:
a) The mapping $p : M \to N$ is a bijection
b) The mapping $p : M \to N$ is an epimorphism and $card\ G(\vartheta_1) = 1$
c) $G(\Phi) = 1$
2) If $\mathscr{D}$ is a decomposable scheme then $A(F, G)$ has limits over $\mathscr{D}$ iff $G(\Phi) = 1$ and G dualizes unions.

The proof follows from the previous Lemmas.

Theorem 4.2: The category $A(F, G)$ is complete (has all limits) iff one of the following cases takes place:
1) $F = C_\Phi$.
2) $G = C_1$.
3) $F = C_{M,\Phi}$, $M \neq \Phi$ and $G(\Phi) = 1$.
4) $G = C_{1,\Phi}$.

Proof is easy.

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