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A Note on Simple Quasigroups

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Every countable quasigroup with at least three elements is isotopic to a quasigroup without proper subquasigroups.

Всякая счетная квазигруппа имеющая по крайней мере три элемента, изотопна квазигруппе, которая не имеет никаких собственных подквазигрупп.

Každá kvazigrupa o aspoň třech prvcích je izotopní kvazigrupě, která nemá žádné vlastní podkvazigrupy.

Let Q be a quasigroup. We shall say that Q is a 1-simple quasigroup if Q has no non-trivial normal congruences. Further we shall say that Q is a 2-simple quasigroup if Q has no proper subquasigroup. Finally, we shall say that Q is a 3-simple quasigroup if Q has no proper subquasigroup containing at least two elements.

The following lemma is obvious.

Lemma 1. (i) Every 2-simple quasigroup is 3-simple.

(ii) Every 3-simple quasigroup containing at least one idempotent is 1-simple.

(iii) Every 3-simple quasigroup is countable.

Let Q be a left loop with left unit j. Suppose that Q is 3-simple and contains at least three elements. Let $j \neq x \in Q$, $g(j) = x$, $g(x) = j$ and $g(a) = a$ for every $a \in Q$, $a \neq x, j$. Put $a * b = a \, . \, g(b)$ for all $a, b \in Q$. Finally, we shall assume that $xj \neq x$.

Lemma 2. $Q(*)$ is a 2-simple quasigroup.

Proof. Let $P(*)$ be a subquasigroup of $Q(*)$. If $x \in P$ then $xj = x * x \in P$. However, as it is easy to see, $xj \neq x$ and $xj \neq j$. *If* $c \in P$ and $a \neq x, j$ then $b \in P$, where $b * a = a$. But $b * a = ba$ and $b = j$. Finally, if $j \in P$ then $j * j = x$ is contained in P. We have proved that $j, x \in P$. Now it is easy to check that P is a subquasigroup of Q, and consequently $P = Q$.

Proposition 3. Let Q be a 3-simple countable left loop such that Q is not a right loop. Then Q is isotopic to a 2-simple quasigroup.

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Proof. It is evident that Q contains at least three elements and there is $x \in Q$ with $x \neq j$ and $xj \neq x$. Now we can apply Lemma 2.

Let Q be a countable loop containing at least three elements. Let j be the unit of Q and $P = \{ a \in Q \mid a \neq j\}$. We shall define a permutation f of the set Q.

First, let Q be finite. Then there are an integer $n \geq 2$ and a biunique mapping h of $\{1, 2, \ldots, n\}$ onto P. Put $f(j) = j$, $f(a) = h(h^{-1}(a) + 1)$ if $a \in P$ and $h^{-1}(a) < n$ and $f(a) = h(1)$ if $a \in P$ and $a = h(n)$.

Next, let Q be infinite and $P = \{a_1, a_2, \ldots\}$. We shall define a biunique mapping h of the set of all integers onto P. Put $h(0) = a_1$ and $h(1) = a_2$. Since P is infinite, there is a natural number $i \geq 3$ such that $a_1 a_i \notin \{j, a_1, a_2, a_i\}$. Then we put $h(2) = a_i$ and $h(-1) = a_1 a_i$. Further, $h(3) = a_j$, where j is the least natural number with $a_j \notin \{a_1, a_2, a_i, a_1 a_i\}$. Similarly, there is a natural number k such that $k \neq 1, 2, i, j$, $a_k \neq a_1 a_i$, $a_i a_k \notin$ $\notin \{j, a_1 a_i, a_1, a_2, a_i, a_k\}$ and we put $h(4) = a_k$, $h(-2) = a_i a_k$. Further, $h(5) = a_m$, where m is the least natural number with $a_m \notin \{a_1, a_2, a_i, a_j, a_k, a_1 a_i, a_i a_k\}$. We can proceed further in a similar way and we get a biunique mapping h. Now $f(j) = j$ and $f(a) =$ $= h(h^{-1}(a) + 1)$ for every $a \in P$.

We shall define a new binary operation on Q by $a \circ b = f(a) . b$ for all $a, b \in Q$. The following lemma is obvious.

Lemma 4. $Q(\circ)$ is a left loop and $Q(\circ)$ is not s right loop.

Lemma 5. If $K(\circ)$ is a subquasigroup of $Q(\circ)$ then $f(K) \subseteq K$.

Proof. $Q(\circ)$ is a left loop, and hence $j \in K$. If $a \in K$ then $f(a) = a \circ j$ is contained in K.

Lemma 6. $Q(\circ)$ is a 3-simple quasigroup.

Proof. Let $K(\circ)$ be a proper subquasigroup of $Q(\circ)$ such that $K(\circ)$ contains at least two elements. With respect to Lemma 5 and the definition of f, we can assume that Q is infinite. Similarly we can assume that there exists $x \in K \cap P$ such that $h^{-1}(x) \leq h^{-1}(a)$ for every $a \in K \cap P$. However, this is contradiction with the construction of h.

Corollary 7. Every countable quasigroup containing at least three elements is isotopic to a 2-simple quasigroup.

Remark. The preceding corollary gives a positive solution of the problem 1.7 formulated in [1].

Corollary 8. Every countable quasigroup is isotopic to a 3-simple quasigroup.

Corollary 9. Every countable quasigroup is isotopic to a 1-simple left loop.

Remark. As it is easy to see, every quasigroup isotopic to a 1-simple loop is 1-simple. On the other hand, the author does not know whether the preceding corollary remains true for arbitrary quasigroups.

Reference

[1] J. Dénes, A. D. Keedwell: Latin Squares and their Applications, Akadémiai Kiadó, Budapest 1974.