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The Comparison of Spectrum of Normalizable Matrices

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The author studies a class of *normalizable operators* and proves the theorem about the comparison of spectrum between a normalizable operator A and a linear operator T in the finite dimensional space

$$\sigma(T) \subset V(\sigma(A), |A - T| \, \delta(A))$$

where by $\sigma(A)$ we denote the spectrum of operator A, $V(M, r)$ and $\delta(A)$ will be defined in § 2

Сравнение спектров нормализуемых матриц. Автор изучает здесь класс нормализуемых операторов и доказывает теорему о сравнении между спектром нормализуемого A и линейного оператора T в конечномерном пространстве.

$$\sigma(T) \subset V(\sigma(A), |A - T| \, \delta(A)) ,$$

где $\sigma(A)$ означает спектр оператора A, $V(M, r)$ и $\delta(A)$ будут определены в § 2

Porovnání spektra normalizovaných matic. Autor studuje třídu normalizovatelných operátorů a dokazuje větu o porovnání spektra mezi normalizovatelným operátorem a lineárním operátorem v konečně dimenzionálním prostoru

$$\sigma(T) \subset V(\sigma(A), |A - T| \, \delta(A)),$$

kde $\sigma(A)$ značí spektrum operátoru A, $V(M, r)$ a $\delta(A)$ budou definovány v § 2.

1. Introduction

In the paper [1] V. Pták and J. Zemánek considered the relation of the spectrum between two normal operators and between a normal operator and a linear operator in the Hilbert space. In the present paper we generalize the results of [1] in a wider range of the normalizable operators. The results are formulated for the matrices.

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2. Definitions and Notations

Let A be an $n \times n$ matrix. The matrix A is said to be a normalizable matrix if and only if there exists a non-singular matrix X_A such that

$$X_A A X_A^{-1} = N \tag{1}$$

where N is normal matrix.

where N is a normal matrix.

Lemma. *A is a normalizable matrix if and only if there exists a non-singular matrix X_A such that*

$$X_A A X_A^{-1} = D \tag{2}$$

where D is a diagonal matrix.

Proof. If A is normalizable then there exists a non-singular matrix Y_A for which

$$Y_A A Y_A^{-1} = N,$$

where N is a normal matrix. As N is normal, there exists a unitary matrix U such that

$$UNU^* = D,$$

where D is a diagonal matrix. Set $X_A = UY_A$.

Then $X_A A_A^{-1} X = UY_A A Y_A^{-1} U^* = UNU^* = D$. The part "only" is evident. The proof of the lemma is complete.

Put

$$\delta(A) = \min_{X_A} |X_A|\,|X_A^{-1}| \tag{3}$$

where the minimum is taken with respect to all matrices X_A satisfying (2).

It follows from the definition of the normalizable matrix that if A is a normal matrix then A is also a normalizable matrix and $\delta(A) = 1$.

Let M, M_1, M_2 be the sets in the complex plane x be a complex number, r be a non-negative real number we shall introduce the following notations

$$d(x, m) = \inf_{y \in M} d\,(y, x) \tag{4}$$

where $d(y, x)$ is the distance between x and y.

$$V(M, r) = \{y; d(y, M) \leq r\} \tag{5}$$

$$\text{dist}\,(M_1, M_2) = \inf\,\{r; M_1 \subset V(M_2, r) \text{ and } M_2 \subset V(M_1, r)\} \tag{6}$$

We shall denote by $\sigma(A)$ the spectrum of the matrix A and by $|A|$ we denote the norm. of A.

3. The Comparison of Spectrum

Theorem 1. *Let A and T be two $n \times n$ matrices, let A be a normalizable matrix. Then:*

$$\sigma(T) \subset V(\sigma(A), |A - T|\,\delta(A)) \tag{7}$$

If A Tand bother a normalizable, then

$$\text{dist}\,(\sigma(A), \sigma(T)) \leq |A-T|\, \max\,(\delta(A), \delta(T)) \tag{8}$$

where $\delta(A)$ *is defined in* (3).

Proof:

(1) Let A be normalizable and λ be a complex number such that doesn't belong to the right-hand side of (7), i.e.

$$d(\lambda\,.\,\sigma(A)) > |A - T|\,\delta(A) \tag{9}$$

According to the lemma there is a non-singular matrix X_A with $X_A A X_A^{-1} = D$ where D is a diagonal matrix. We shall write simply $(A - \lambda)$ for $(A - \lambda I)$ where I is the unit matrix.

Evidently,

$$|(A - \lambda)^{-1}| = |(X_A^{-1} D X_A - \lambda)^{-1}| = |X_A^{-1}\,(D - \lambda)^{-1} X_A\,| \leq |X_A|\,|X_A^{-1}|\,|(D - \lambda)^{-1}|.$$

This inequality holds for every matrix X_A satisfying (2). So it follows that

$$|(A - \lambda)^{-1}| \leq \delta(A)\,|(D - \lambda)^{-1}|$$

Since $(D - \lambda)^{-1}$ is a diagonal matrix, we have

$$|(D - \lambda)^{-1}| = d(\lambda, \sigma(D))^{-1} = d(\lambda, \sigma(A))^{-1}.\ \text{Hence}$$

$$|(A - \lambda)^{-1}| \leq \delta(A)\, d(\lambda, \sigma(A))^{-1} \tag{10}$$

By (9) and (10) we have

$$|(A - \lambda)^{-1}\,(T - A)| \leq d(\lambda, \sigma(A))^{-1}\,|A - T|\,\delta(A) < 1 \tag{11}$$

from (11) and the fact that

$$(\lambda - T) = (\lambda - A) - (T - A) = (\lambda - A)\,(I - (\lambda - A)^{-1}\,(T - A)$$

it follows that there exists $(\lambda - T)^{-1}$, i.e. $\lambda \bar{\in} \sigma(T)$.

(2) If both A and T are normalizable, according to the proof of the first part yields:

$$\sigma(T) \subset V(\sigma(A), |A - T|\,\delta(A)) \subset V(\sigma(A), |A - T|\,\bar{\delta}(A, T))$$
$$\sigma(A) \subset V(\sigma(T), |A - T|\,\delta(T)) \subset V(\sigma(T), |A - T|\,\bar{\delta}(A, T))$$

where $\bar{\delta}(A, T) = \max(\delta(A), \delta(T))$.

By the definition of the function dist we obtain

$$\text{dist}(\sigma(A), \sigma(T)) \leq |A - T|\,\bar{\delta}(A, T)$$

The proof is complete.

Remarks:

(1) If A is normal, then $\delta(A) = 1$ and we obtain, therefore, the Theorem 1 in [1].

(2) If A is normalizable, then for every μ

$$\sigma(T) \subset V(\sigma\,(A - \mu), |A - T - \mu|\,\delta(A)).$$

The proof follows from the fact that $(A - \mu)$ is normalizable and $\delta(A - \mu) = \delta(A)$ for every μ.

Theorem 2. *Let A be a normalizable $n \times n$ matrix partitioned in the form:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11}, A_{22} are square and the dimension of A_{11} is equal to m $(1 \leq \mathrm{m} \leq \mathrm{n})$. Ler P be a matrix of projector transforming an n-dimensional vector x with the coordinates x_i into the vector y with the coordinates $y_i = x_i$ for $i = 1, \ldots, \mathrm{m}$ and $y_j = 0$ *for* $\mathrm{j} = \mathrm{m} + 1, \ldots, \mathrm{n}$, $\mathrm{Q} = \mathrm{I} - \mathrm{P}$.

If λ belongs to $\sigma(A_{11}) \cup \sigma(A_{22})$ then the disk

$$K(\lambda, |PAQ + QAP|\, \delta(A)) = \{\alpha; |\alpha - \lambda| \leq |PAQ + QAP|\, \delta(A)\},$$

contains at least one proper value of A.

Proof. According to the theorem 1 we have

$$\sigma(PAP + QAQ) \subset V(\sigma(A), |A - PAP - QAQ|\, \delta(A)) = V(\sigma(A), |PAQ + QAP| \delta(A).).$$

From the fact that $\sigma(PAP + QAQ) = \sigma(A_{11}) \cup \sigma(A_{22})$, it follows that if $\lambda \in \sigma(A_{11}) \cup \cup\, \sigma(A_{22})$, then $K(\lambda, |PAQ + QAP|\, \delta(A))$ contains at least one proper value of A. The proof is complete.

Remark. If A is normal, A_{11} is a matrix of dimension 1 and of we use the Euclidean norm, then we obtain the Theorem 2 in [1]. The result of this theorem, when A is normal, was obtained in the paper [2].

Theorem 3. *Let A be an $n \times n$ matrix paritioned as in Theorem* 2

A_{11}, A_{22} and $PAQ + QAP$ be normalizable, then

$$\sigma(A) \subset V(\sigma(PAQ + QAP), \delta(PAP + QAQ)\, \delta(PAQ + QAP) \max |\lambda_j|) \tag{12}$$

where $\lambda_j \in \sigma(A_{11}) \cup \sigma(A_{22})$.

Proof. First, we shall prove that PAP, QAQ, $PAP + QAQ$ are normalizable. Indeed, since A_{11} and A_{22} are normalizable there are X_1 and X_2 such that

$$X_1 A_{11} X_1^{-1} = D_1$$
$$X_2 A_{22} X_2^{-1} = D_2$$

where D_1 and D_2 are diagonal. Put X, Y, Z the $n \, x \, n$ matrices for which

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & I_{m-n} \end{bmatrix} \quad Y = \begin{bmatrix} I_m & 0 \\ 0 & X_2 \end{bmatrix} \quad Z = X + Y - I_n$$

where by I_k we denote the unit matrix of the dimension k. It is not difficult to verify that:

$XPAPX^{-1}$, $YQAQY^{-1}$, $Z(PAP + QAQ)Z^{-1}$ are the diagonal matrices. Since $PAP + QAQ$ is normalizable, $PAP + QAQ = T^{-1} \Lambda\, T$ with some nonsingular matrix T and diagonal matrix Λ.

Hence $|PAP + QAQ| \leq |T|\,|T^{-1}|\,|\Lambda|$.
This inequality holds for every matrix T satisfying

$$PAP + QAQ = T^{-1}\Lambda T$$

We obtain, therefore:

$$|PAP + QAQ| \leq \delta(PAP + QAQ)\,|\Lambda| \leq \delta(PAP + QAQ)\max|\lambda_j|$$

where $\lambda_j \in \sigma(PAP + QAQ)$ i.e. $\lambda_j \in \sigma(A_{11}) \cup \sigma(A_{22})$.

By Theorem 1 we obtain

$$\begin{aligned}\sigma(A) &\subset V(\sigma(PAQ + QAP), |PAP + QAQ|\,\delta(PAQ + QAP)) \\ &\subset V(\sigma(PAQ + QAP), \delta(PAQ + QAP)\,\delta(PAP + QAQ)\max|\lambda_j|)\end{aligned}$$

Corollary. *Let A_{ij} be square and normalizable, A_{12} and A_{21} be regular and $A_{12}A_{21} = A_{21}A_{12}$ then* (12) *holds.*
Proof. Since A_{12}, A_{21} are normalizable and $A_{12}A_{21} = A_{21}A_{12}$ there exists (see [3]) a non-singular matrix X such that

$$XA_{12}X^{-1} = D_1,\ XA_{21}X^{-1} = D_2$$

where D_1 and D_2 are diagonal.

$$\text{Set } T = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \quad \text{then } T^{-1} = \begin{bmatrix} 0 & X^{-1} \\ X^{-1} & 0 \end{bmatrix} \text{ and}$$

$$T(PAQ + QAP)T^{-1} = \begin{bmatrix} 0 & D_2 \\ D_1 & 0 \end{bmatrix}$$

Since A_{12} and A_{21} are regular, there exists a diagonal nonsingular matrix M such that

$$M^2 = D_2^{-1}D_1.$$

$$\text{Set } Y = \begin{bmatrix} I & M^{-1} \\ I & -M^{-1} \end{bmatrix} \ ;\ Z = YT \text{ then } Y^{-1} = \frac{1}{2}\begin{bmatrix} I & I \\ M & -M \end{bmatrix}$$

and

$$Z(PAQ + QAP)Z^{-1} = Y\begin{bmatrix} 0 & D_2 \\ D_1 & 0 \end{bmatrix}Y^{-1} = \frac{1}{2}\begin{bmatrix} M^{-1}D_1 + D_2M & M^{-1}D_1 - D_2M \\ -(M^{-1}D_1 - D_2M) & -(M^{-1}D_1 + D_2M) \end{bmatrix}$$

Where evidently $M^{-1}D_1 + D_2M$ is a diagonal matrix; $M^{-1}D_1 - D_2M$ is a null matrix. Hence $Z(PAQ + QAP)Z^{-1}$ is a diagonal matrix. That means $PAQ + QAP$ is normalizable. We can, therefore, apply Theorem 3 to obtain (12).

Theorem 4. *Let $A = B + C$, B and C be normalizable and $BC = CB$ then*

$$\operatorname{dist}(\sigma(A), \sigma(B)) \leq \delta(C)\max(\delta(B), \delta(A))\max|\lambda_j(C)|$$

where by $\lambda_j(C)$ we denote the eigenvalues of C.

Proof. First we prove that A is normalizable. Indeed, from the fact $BC = CB$

and the fact B, C are normalizable, it follows that there exists a non-singular matrix X such that

$$XBX^{-1} = D_1$$
$$XCX^{-1} = D_2$$

where D_1 and D_2 are diagonal.
We have, therefore

$$XAX^{-1} = X(B + C)X^{-1} = D_1 + D_2$$

That means A is a normalizable matrix and by the Theorem 1 we obtain

$$\text{dist}\,(\sigma(A), \sigma(B)) \leq |A - B| \max\,(\delta(A), \delta(B)) = |C| \max\,(\delta(A), \delta(B))$$

Matrix C is normalizable, hence, there exists a non-singular matrix X_C such that

$$X_C C X_C^{-1} = D, \text{ or } C = X_C^{-1} D X_C$$

where D is a diagonal matrix, whose diagonal elements are eingevalues of C. So $|C| \leq$ $\leq \delta(C) \max |\lambda_j(C)|$

Finally, we have

$$\text{dist}\,(\sigma(A), \sigma(B)) \leq \delta(C) \max\,(\delta(A), \delta(B)) \max |\lambda_j(C)|$$

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