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Medial Division Groupoids

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The paper is devoted to an extensive study of medial division groupoids. A special attention is paid to subdirectly irreducible medial division groupoids.

Статья посвящена изучению медиальных группоидов с делением. В частности, исследуются подпрямо неразложимые медиальные группоиды с делением.

Článek je věnován rozsáhlému studiu mediálních grupoidů s dělením. Zvláštní pozornost je věnována subdirektně ireducibilním mediálním grupoidům s dělením.

1. Introduction

Let G be a groupoid. For every $a \in G$, define two mappings L_a and R_a of G into G by $L_a(x) = ax$ and $R_a(x) = xa$. Further, let r be an equivalence on G. We shall say that r is

- left compatible if $xa\,r\,xb$ for all $x, a, b \in G$, $a\,r\,b$,
- right compatible if $ax\,r\,bx$ for all $x, a, b \in G$, $a\,r\,b$,
- left cancellative if $a\,r\,b$, whenever $a, b, c \in G$ and $ca\,r\,cb$,
- right cancellative if $a\,r\,b$, whenever $a, b, c \in G$, $ac\,r\,bc$,
- cancellative if it is both left and right cancellative. A groupoid G is said to be
- cr-simple if $G \times G$ is the only right cancellative congruence of G,
- cl-simple if $G \times G$ is the only left cancellative congruence of G,
- crl-simple if G is both cr and cl-simple,
- c-simple if $G \times G$ is the only cancellative congruence of G,
- left regular if $R_a = R_b$, whenever $a, b, c \in G$ and $ca = cb$,
- right regular if $L_a = L_b$, whenever $a, b, c \in G$ and $ac = bc$,
- regular if it is both left and right regular,
- medial if $ab \,.\, cd = ac \,.\, bd$ for all $a, b, c, d \in G$,
- unipotent if $aa = bb$ for all $a, b \in G$,

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- left distributive if $a \,.\, bc = ab \,.\, ac$ for all $a, b, c \in G$,
- right distributive if $bc \,.\, a = ba \,.\, ca$ for all $a, b, c \in G$,
- distributive if it is both left and right distributive.

Clearly, every idempotent medial groupoid is distributive. Further, every groupoid satisfying the identity $x \,.\, yz = z \,.\, yx$ is medial.

Let f be an endomorphism of an abelian group $G(+)$. Then we put $\operatorname{Ker} f = \{x \in G \mid f(x) = 0\}$.

Let S be a set. Then $|S|$ is the cardinal number corresponding to S and d_S is the identical relation on S. If f is a mapping of S into T then $\ker f$ is the equivalence on S defined by $a \ker f\, b$ iff $f(a) = f(b)$.

Some informations concerning division groupoids may be found in [1], [2], [3] and [4].

2. Medial Division Groupoids

Throughout this paragraph, let G be a medial division groupoid.

For every natural number $0 \leq n$, we shall define two relations $p_{G,n}$ and $q_{G,n}$ on G as follows: $p_{G,0} = d_G = q_{G,0}$; if $1 \leq n$ then $a\, p_{G,n}\, b$ and $c\, q_{G,n}\, d$ iff $((ax_1) \ldots) x_n = ((bx_1) \ldots) x_n$ and $x_n(\ldots(x_1c)) = x_n(\ldots(x_1d))$ for all $x_1, \ldots, x_n \in G$. It is visible that $p_{G,0} \subseteq p_{G,1} \subseteq p_{G,2} \subseteq \ldots$, $q_{G,0} \subseteq q_{G,1} \subseteq q_{G,2} \subseteq \ldots$ and we put $\bar{p}_G = \bigcup p_{G,n}$, $\bar{q}_G = \bigcup q_{G,n}$. Further, we put $p_G = p_{G,1}$ and $q_G = q_{G,1}$.

The groupoid G is said to be right (left) faithful if $p_G = d_G$ ($q_G = d_G$). It is said to be faithful if it is both left and right faithful.

2.1 Lemma. (i) For every natural n, $p_{G,n}$ and $q_{G,n}$ are congruences of G and $p_{G,n+1}/p_{G,n} = p_{G/p_n}$, $q_{G,n+1}/q_{G,n} = q_{G,q_n}$.
(ii) $\bar{p}_G(\bar{q}_G)$ is the least congruence of G such that the corresponding factorgroupoid is right (left) faithful.
(iii) $p_G = \bigcap \ker R_x$, $x \in G$ and $a\, p_G\, b$ iff $L_a = L_b$.
(iv) If G is right faithful then $\bar{p}_G = d_G$.

Proof. (i) It is obvious that p is a right compatible equivalence. It remains to show that p is left compatible. For, let $a, b, c \in G$ and $a\, p\, b$. Then $ca \,.\, xy = cx \,.\, ay = cx \,.\, by = cb \,.\, xy$ for all $x, y \in G$. However $G = GG$, hence $ca\, p\, cb$ and we have proved that p is a congruence. The rest is clear.
(ii) It suffices to show that $G/\bar{p}$ is right faithful. For, let $a, b \in G$ be such that $ax\, \bar{p}\, bx$ for every $x \in G$. Then $aa\, \bar{p}\, ba$ and $aa\, p_n\, ba$ for some $1 \leq n$. From this we see that $(((ax \,.\, ay)x_2)\ldots)x_n = (((aa \,.\, xy)\, x_2)\ldots)\, x_n = (((bx \,.\, ay)\, x_2)\ldots)\, x_n$ for all $x, y, x_2, \ldots, x_n \in G$. Taking into account that G is a division groupoid, we see that $a\, p_{n+1}\, b$, and hence $a\, \bar{p}\, b$.
(iii) and (iv). These assertions are obvious.

2.2 Proposition. (i) For every natural number $1 \leq n$, the factorgroupoid G/p_n is regular.
(ii) $G/\bar{p}$ is a regular right cancellation groupoid.
(iii) $\bar{p}$ is the least right cancellative congruence of G.

Proof. (i) With respect to 2.1(i), it is enough to show that G/p is regular. First, we prove that G/p is right regular. Let $a, b, c \in G$ and $ac\, p\, bc$. Then $ac \,.\, x = bc \,.\, x$ for every $x \in G$. In particular, $ay \,.\, cz = by \,.\, cz$, and so $ay \,.\, u = by \,.\, u$ for all y, z, u. Hence $ay\, p\, by$ for every y and G/p is right regular. Indeed if $a, b, c \in G$ are such that $ca\, p\, cb$, then $ca \,.\, x = cb \,.\, x$, and so $cy \,.\, az = cy \,.\, bz$ for all $x, y, z \in G$. Consequently, $u \,.\, az = u \,.\, bz$ and $vw \,.\, az = vw \,.\, bz$ for all $u, v, w, z \in G$. Thus $va \,.\, wz = vb \,.\, wz$, i.e., $va\, p\, vb$ for every v.
(ii) Using (i), it is easy to show that $G/\bar{p}$ is regular. On the other hand, $G/\bar{p}$ is right faithful, and hence it is a right cancellation groupoid.
(iii) Apply (ii) and 2.1(ii).

2.3 Corollary. The following conditions are equivalent:
(i) G id cr-simple.
(ii) $\bar{p}_G = G \times G$.
(iii) For all $a, b \in G$, there is a natural number $1 \leq n$ such that $((ax_1)\dots)\, x_n = ((bx_1)\dots)\, x_n$ for all $x_1, \dots, x_n \in G$.

2.4 Proposition. The following conditions are equivalent:
(i) There is a natural number $0 \leq n$ such that $p_{G,n} = p_{G,n+1}$.
(ii) There is a natural number $0 \leq m$ such that $p_{G,n} = \bar{p}_G$.
(iii) There is a natural number $0 \leq k$ such that $p_{G,k}$ is right cancellative.
(iv) G is right faithful.
(v) G is a right cancellation groupoid.
(vi) $\bar{p}_G = d_G$.

Proof. Only the implication (ii) implies (iii) needs a proof. Let $a, b \in G$ and $a\, p_k\, b$. There are $c, d \in G$ with $ca = a$ and $da = b$. Since p_k is right cancellative, $c\, p_k\, d$, and so $a = ca\, p_{k-1}\, da = b$. The rest is clear.

2.5 Corollary. G is a cancellation groupoid iff it is faithful.

2.6 Lemma. The following conditions are equivalent for $a, b \in G$:
(i) $x \,.\, ay = x \,.\, by$ for all $x, y \in G$.
(ii) $xa \,.\, y = xb \,.\, y$ for all $x, y \in G$.

Proof. Obvious.

For every natural number $0 \leq n$, define a relation $o_{G,n}$ as follows: $o_{G,0} = d_G$; $a\, o_{G,n+1}\, b$ iff $x \,.\, ay\; o_{G,n}\; x \,.\, by$ for all $x, y \in G$. Further, put $o_G = o_{G,1}$ and $\bar{o}_G = \bigcup o_{G,n}$.

2.7 Lemma. (i) $a\, o_G\, b$ iff $xa\,.\,y = xb\,.\,y$ for all $x, y \in G$.
(ii) o_G is a congruence of G, p_G, $q_G \subsetneqq o_G$, $o_G/p_G = q_{G/p}$ and $o_G/q_G = p_{G/q}$.
(iii) For every natural $0 \leq n$, $o_{G,n}$ is a congruence of G and $o_{G,n+1}/o_{G,n} = o_{G,o_n}$.

Proof. Easy.

2.8 Lemma. Let $1 \leq n$ and $a, b \in G$. The following conditions are equivalent:
(i) $a\, o_{G,n}\, b$.
(ii) $x_1(\dots(x_n(((ay_1)\dots)y_n))) = x_1(\dots(x_n(((by_1)\dots)y_n)))$ for all $x_1, \dots, x_n, y_1, \dots, y_n \in G$.

Proof. (i) implies (ii). By induction on n. If $n = 1$, the assertion is obvious. Let $2 \leq n$ and $x, y \in G$. Put $c = x\,.\,ay$ and $d = x\,.\,by$. Then $c\, o_{G,n-1}\, d$, and so we have the equality $x_1(\dots(x_{n-1}(((cy_1)\dots)y_{n-1}))) = x_1(\dots(x_{n-1}(((dy_1)\dots)y_{n-1})))$ for all $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1} \in G$. Let $u_1, \dots, u_{n-1}, v_1, \dots, v_{n-1}$ be arbitrary elements from G and $y_1 = u_1v_1, \dots, y_{n-1} = u_{n-1}v_{n-1}$. Then $cy_1 = (x\,.\,ay)(u_1v_1) = (xu_1)(ay\,.\,v_1), \dots, ((cy_1)\dots)y_{n-1} = (((xu_1)\dots)u_{n-1})(((ay\,.\,v_1)\dots)v_{n-1})$. The rest as well as the converse implication are clear.

2.9 Lemma. (i) For every $0 \leq n$, $p_{G,n}$, $q_{G,n} \subseteq o_{G,n}$ and $o_{G,n}/p_{G,n} = q_{G/p_n,n}$, $o_{G,n}/q_{G,n} = p_{G/q_n,n}$.
(ii) For every $1 \leq n$, G/o_n is regular.

Proof. (i) This follows easily from 2.8.
(ii) Let $H = G/p_n$. By (i), G/o_n is isomorphic to $H/q_{H,n}$. According to the left hand form of 2.2(i), G/o_n is regular.

2.10 Proposition. (i) $\bar{o}_G$ is the least cancellative congruence of G.
(ii) $\bar{p}_G$, $\bar{q}_G \subseteq \bar{o}_G$ and $\bar{o}_G/\bar{p}_G = \bar{q}_{G/\bar{p}}$, $\bar{o}_G/\bar{q}_G = \bar{p}_{G/\bar{q}}$.

Proof. It follows from 2.7(iii) and 2.9(i) that $\bar{o}_G$ is a congruence containing $\bar{p}_G$ and $\bar{q}_G$. Further, we show that $G/\bar{o}$ is faithful. For, let $a, b \in G$ and $ax\, \bar{o}\, bx$ for every $x \in G$. Then $aa\, o_n\, ba$ for some $1 \leq n$. By 2.9(ii), G/o_n is regular, and so $ax\, o_n\, bx$ for every x. However, o_n is a congruence, and hence $y\,.\,ax\, o_n\, y\,.\,bx$ for all $x, y \in G$, i.e., $a\, \bar{o}\, b$. We have proved that $G/\bar{o}$ is right faithful. Similarly the other case and $G/\bar{o}$ is a cancellation groupoid by 2.5. Further, let r be a cancellative congruence of G. It is an easy task to show by induction on m that $o_m \subseteqq r$. Thus $\bar{o}$ is the least cancellative congruence of G. Finally, let $H = G/\bar{p}$ and s be equal to $\bar{o}/\bar{p}$. It follows from 2.8 that $s \subseteqq \bar{q}_H$. On the other hand, H/s is isomorphic to $G/\bar{o}$, therefore it is a cancellation groupoid and $\bar{q}_H \subseteqq s$. Similarly, we can show that $\bar{o}/\bar{q} = \bar{p}_{G/\bar{q}}$.

2.11 Corollary. The following conditions are equivalent:
(i) G is c-simple.

(ii) $\bar{o}_G = G \times G$.
(iii) For all $a, b \in G$, there exists $1 \leq n$ such that $x_1(\dots(x_n(((ay_1)\dots)y_n))) = x_1(\dots(x_n(((by_1)\dots)y_n)))$ for all $x_1, \dots, x_n, y_1, \dots, y_n \in G$.

2.12 Proposition. The following conditions are equivalent:
(i) There is $0 \leq n$ with $o_{G,n} = o_{G,n+1}$.
(ii) There is $0 \leq m$ with $o_{G,m} = \bar{o}_G$.
(iii) There is $0 \leq k$ such that $o_{G,k}$ is cancellative.
(iv) G is faithful.
(v) G is a quasigroup.
(vi) $\bar{o}_G = d_G$.
(vii) $\bar{p}_G = d_G = \bar{q}_G$.

Proof. Similar to that of 2.5.

Put $t_{G,0} = d_G$ and for every $0 \leq n$, let $t_{G,n+1}$ be the congruence of G such that $t_{G,n} \subseteq t_{G,n+1}$ and the factorcongruence $t_{G,n+1}/t_{G,n}$ is equal to $p_{G/t_n} \cap q_{G/t_n}$. Further, put $\bar{t}_G = \bigcup t_{G,n}$ and $t_G = t_{G,1}$ (hence $t_G = p_G \cap q_G$).

We shall say that G is semifaithful if $t_G = d_G$.

2.13 Lemma. (i) The congruence t_G is equal to $p_G \cap q_G$.
(ii) Every equivalence contained in t_G is a congruence of G.
(iii) For every $0 \leq n$, $t_{G,n+1}/t_{G,n} = t_{G/t_n}$ and $a\, t_{G,n+1}\, b$ iff $ax\, t_{G,n}\, bx$ and $xa\, t_{G,n}\, xb$ for every $x \in G$.
(iv) For all $0 \leq n, m$ with $1 \leq n + m$, $p_{G,n} \cap q_{G,m} \subseteq t_{G,n+m-1}$.
(v) For every $0 \leq n$, $t_{G,n} \subseteqq p_{G,n} \cap q_{G,n}$.

Proof. Only (iv) needs be proved. We shall proceed by induction on $n + m$. If $n + m = 1$ then either $p_{G,n} = d_G$ or $q_{G,m} = d_G$, and so $p_{G,n} \cap q_{G,m} = d_G = t_{G,0}$. Let $2 \leq n + m$ and $a, b \in G$ be such that $a\, p_{G,n} \cap q_{G,m}\, b$. We can assume that $1 \leq n$, the other case being similar. Then $ax\, p_{G,n-1}\, bx$ for every x. However, $q_{G,m}$ is a congruence, and therefore $ax\, q_{G,m}\, bx$. Hence $ax\, t_{G,n+m-2}\, bx$ for every x. The rest is clear.

2.14 Proposition. (i) For every $1 \leq n$, G/t_n is regular.
(ii) $G/\bar{t}$ is regular.
(iii) $\bar{t}_G$ is the least congruence of G such that the corresponding factor is semifaithful.
(iv) $\bar{t}_G = \bar{p}_G \cap \bar{q}_G$.
(v) G is semifaithful iff $\bar{t}_G = d_G$.

Proof. (i) By 2.2(i) and its left hand form, $G/t = G/p \cap q$ is regular. The general case follows from the fact that G/t_{n+1} is isomorphic to $(G/t_n)/t$.
(ii) This is an easy consequence of (i).
(iii) First, we show that $G/\bar{t}$ is semifaithful. For, let $a, b \in G$ and $ax\, \bar{t}\, bx$, $xa\, \bar{t}\, xb$ for every $x \in G$. Then $aa\, t_n\, ba$ for some $1 \leq n$, and so $ax\, t_n\, bx$ for every x,

since G/t_n is regular. Similarly, $xa\, t_m\, xb$ for some $1 \leq m$ and every x. Now, $a\, t_k\, b$, where $k = \max(n, m)$, $a\, \bar{t}\, b$. Finally, let r be a congruence of G such that G/r is semifaithful. By induction on n, we can show that $t_n \subseteq r$.
(iv) Apply 2.13(iv), (v).
(v) This follows from (iii).

2.15 Corollary. Let G be semifaithful. Then G is regular and G is a subdirect product of a left quasigroup and a right quasigroup.

2.16 Lemma. $t_{G,2} = o_G \cap p_{G,2} \cap q_{G,2}$.

Proof. Obvious.

2.17 Lemma. Let $ac = bc$ for some $a, b, c \in G$. Then $a\, p_{G,2} \cap o_G\, b$.

Proof. We can write $ax \,.\, cy = ac \,.\, xy = bc \,.\, xy = bx \,.\, cy$ and $xa \,.\, yc = xy \,.\, ac = xy \,.\, bc = xb \,.\, yc$ for all $x, y \in G$. The rest is clear.

2.18 Lemma. Let $a, b, c, d \in G$ be such that $ac = bc$ and $da = db$. Then $a\, t_{G,2}\, b$.

Proof. Use 2.16 and 2.17.

2.19 Lemma. G is right regular, provided at least one of the following (equivalent) conditions is satisfied:
(i) $p_G = p_{G,2} \cap o_G$.
(ii) $p_{G,2} \cap o_G \subseteq p_G$.
(iii) G/p is semifaithful.
(iv) $p_G/t_G = p_{G/t}$.

Proof. Apply 2.17.

2.20 Corollary. Suppose that every regular factorgroupoid of G is semifaithful. Then G is regular.

2.21 Lemma. The congruences p_G and q_G commute.

Proof. Let $a, b, c \in G$, $a\, p\, b$ and $b\, q\, c$. There are $d, e, f \in G$ with $b = db$, $a = eb$ and $c = df$. We have $xd \,.\, a = xd \,.\, eb = xe \,.\, db = xe \,.\, b = xe \,.\, c = {}$ $= xe \,.\, df = xd \,.\, ef$ for every $x \in G$, and so $a\, q\, ef$. Similarly, $ef \,.\, bx = eb \,.\, fx = {}$ $= a \,.\, fx = b \,.\, fx = db \,.\, fx = df \,.\, bx = c \,.\, bx$ and $ef\, p\, c$. The rest is clear.

2.22 Lemma. The congruences $\bar{p}_G$ and $\bar{q}_G$ commute.

Proof. Let $a, b, c \in G$, $a\, \bar{p}\, b$ and $b\, \bar{q}\, c$. There are $d, e, f \in G$ with $a = ad$, $b = ed$ and $c = ef$. However, $\bar{p}$ is right and $\bar{q}$ is left cancellative. Therefore, we have $a\, \bar{p}\, e$, $d\, \bar{q}\, f$, $c = ef\, \bar{p}\, af\, \bar{q}\, ad = a$.

2.23 Theorem. (i) p_G, $\bar{p}_G$, q_G, $\bar{q}_G$, o_G, $\bar{o}_G$, t_G, $\bar{t}_G$ are congruences of G and the corresponding factorgroupoids are regular.
(ii) $\bar{p}_G$ is the least right cancellative congruence of G.
(iii) $\bar{o}_G$ is the least cancellative congruence of G.
(iv) $\bar{t}_G$ is the least congruence of G such that the corresponding factorgroupoid is semifaithful.
(v) The congruences p_G and q_G commute.
(vi) The congruences $\bar{p}_G$ and $\bar{q}_G$ commute and $\bar{t}_G = \bar{p}_G \cap \bar{q}_G$.

Proof. See 2.2, 2.10, 2.14, 2.21, 2.22.

2.24 Proposition. Let r be a congruence of G such that $r \cap t_G = d_G$. Then $r \cap \bar{t}_G = d_G$.

Proof. Suppose, on the contrary, that $r \cap \bar{t}_G \neq d_G$. Then there is a natural number n which is the least with the property $r \cap t_n \neq d_G$. Obviously, $2 \leq n$. There are $a, b \in G$ such that $a \neq b$ and $a\, r \cap t_n\, b$. Then $ax\, r \cap t_{n-1}\, bx$ and $xa\, r \cap t_{n-1}\, xb$ for every $x \in G$. Consequently, $ax = bx$ and $xa = xb$, i.e., $a\, r \cap t\, b$. Thus $n = 1$, a contradiction.

2.25 Lemma. The following conditions are equivalent:
(i) G is crl-simple.
(ii) $\bar{p}_G = G \times G = \bar{q}_G$.
(iii) $\bar{t}_G = G \times G$.
(iv) No non-trivial factorgroupoid of G is semifaithful.

Proof. Obvious.

2.26 Proposition. Suppose that G is crl-simple. If r is a congruence of G such that $r \cap t_G = d_G$ then $r = d_G$.

Proof. Use 2.24 and 2.25.

We shall say that G satisfies the condition (C1) if o_G is contained in $p_G \circ q_G$ (then $o_G = q_G \circ p_G = p_G \circ q_G$). Further, we shall say that G satisfies the condition (C2) if $a\, t_G\, b$, whenever $a, b \in G$, $a\, p_G \circ q_G\, b$ and $aa = bb$.

2.27 Lemma. Consider the following two conditions:
(i) G satisfies (C1).
(ii) If $a, b, c, d \in G$, $a\, p_G\, b$ and $cd = a$, then there exists $e \in G$ with $d\, p_G\, e$ and $ce = b$.
Then (i) implies (ii). Moreover, if G is left regular, then (ii) implies (i).

Proof. (i) implies (ii). There is $f \in G$ with $b = cf$. We have $cd = a\, p\, b = cf$, and so $cx \,.\, dy = cd \,.\, xy = cf \,.\, xy = cx \,.\, fy$ for all $x, y \in G$. From this, $d\, o\, f$ and there is $e \in G$ such that $d\, p\, e$ and $e\, q\, f$. Then $ce = cf = b$.
(ii) implies (i). Let $a, b \in G$, $a\, o\, b$. Then $xa\, p\, xb$ for every $x \in G$. In particular,

$aa\, p\, ab$ and there is $c \in G$ with $a\, p\, c$ and $ac = ab$. Since G is left regular, $c\, q\, b$. Now, we see that $a\, p \circ q\, b$.

2.28 Lemma. Let G/t satisfy (C1). Then G is semifaithful.

Proof. Let $a, b \in G$, $a\, t\, b$ and f be the natural homomorphism of G onto $H = G/t$. There are $x, y, u, v, z \in G$ such that $xy \,.\, uv = a$ and $z \,.\, uv = b$. Since $f(a) = f(b)$ and H is regular, $f(xy)\, p_H\, f(z)$. However, H satisfies (C1). According to 2.27, there is $w \in G$ such that $f(y)\, p_H\, f(w)$ and $f(z) = f(xw)$. Hence $z\, t\, xw$ and $yc\, t\, wc$ for every $c \in G$. Now, $b = z \,.\, uv = xw \,.\, uv = xu \,.\, wv = xu \,.\, yv = xy \,.\, uv = a$.

2.29 Lemma. The following conditions are equivalent:

(i) G satisfies (C2).

(ii) If $a, b, c \in G$, $a\, p_G\, c$, $c\, q_G\, b$ and $ca = bc$, then $a\, t_G\, b$.

Proof. (i) implies (ii). We have $a\, p \circ q\, b$ and $aa = ca = bc = bb$. Therefore $a\, t\, b$.

(ii) implies (i). Let $a, b, c \in G$, $a\, p\, c$, $c\, q\, b$ and $aa = bb$. Then $ca = aa = bb = bc$ and $a\, t\, b$.

2.30 Lemma. Suppose that G is commutative. Then:

(i) $p_G = q_G = t_G$, $o_G = p_{G,2}$ and $\bar{p}_G = \bar{q}_G = \bar{o}_G = \bar{t}_G$.

(ii) G satisfies (C1) iff it is semifaithful iff it is a quasigroup.

(iii) G satisfies (C2).

Proof. Easy.

2.31 Lemma. Let G satisfy the identity $x \,.\, yz = z \,.\, yx$. Then:

(i) $t_G = q_G \subseteq p_G = q_{G,2}$ and $\bar{p}_G = \bar{q}_G = \bar{o}_G = \bar{t}_G$.

(ii) G satisfies (C1) iff it is semifaithful iff it is a quasigroup.

(iii) G satisfies (C2), provided G is left regular.

Proof. (i) First, let $x, y \in G$ and $x\, q_2\, y$. Then $x \,.\, uv = v \,.\, ux = v \,.\, uy = y \,.\, uv$ and we see that $x\, p\, y$. Similarly the converse and we have $t = q \subseteq q_2 = p$. The rest is clear.

(ii) First, let G be semifaithful. Then $\bar{p}_G = d_G = \bar{q}_G$ by (i), and so G is a quasigroup by 2.12. Further, let G satisfy (C1). Then $o = p \circ q = p$ and hence $q_{G/p} = o/p = d_{G/p}$ and G/p is left faithful. Hence G/p is a quasigroup and p is cancellative. By 2.4, $p = d_G$ and G is semifaithful.

(iii) Let $a, b, c \in G$ be such that $a\, p\, b\, q\, c$ and $aa = cc$. Since $q \subseteq p$, $a\, p\, c$. On the other hand, $ca = aa = ac$. Hence $aa = ac$ and $a\, q\, c$, since G is left regular.

2.32 Lemma. Let G be unipotent. Then:

(i) $p_G = q_G = t_G$, provided G is regular.

(ii) $\bar{p}_G = \bar{q}_G = \bar{t}_G = \bar{o}_G$.

(iii) G satisfies (C1) iff it is semifaithful iff it is a quasigroup.
(iv) G satisfies (C2), provided G is regular.

Proof. Easy.

2.33 Proposition. Suppose that G satisfies the identity $x \, . \, yz = z \, . \, yx$ Then G is a quasigroup, provided G is either commutative or unipotent.

Proof. Taking into account 2.28 and 2.31(ii), we can assume that G is regular. Then, by 2.31 and 2.32, $q = q_2$, and so G is semifaithful. By 2.31, G is a quasigroup.

3. Regular Medial Division Groupoids

3.1 Lemma. Let f, g be two surjective endomorphisms of an abelian group $G(+)$ such that $fg = gf$. The following conditions are equivalent:
(i) If $x, y \in G$ and $f(x) + g(y) = 0$ then $x = g(z), y = f(-z)$ for some $z \in G$.
(ii) $f(\operatorname{Ker} g) = \operatorname{Ker} g$.
(iii) $g(\operatorname{Ker} f) = \operatorname{Ker} f$.
(iv) $\operatorname{Ker} fg = \operatorname{Ker} f + \operatorname{Ker} g$.

Proof. Easy.

3.2 Proposition. Let $G(+)$ be an abelian group, f and g be two surjective endomorphisms such that $fg = gf$ and let $a \in G$ be an element. Put $xy = f(x) + g(y) + a$ for all $x, y \in G$. Then:
(i) G is a regular medial division groupoid.
(ii) $p_G = \ker f$, $q_G = \ker g$ and $o_G = \ker fg$.
(iii) G is semifaithful iff $\operatorname{Ker} f \cap \operatorname{Ker} g = 0$.
(iv) G satisfies (C1) iff the equivalent conditions of 3.1 hold.
(v) G satisfies (C2) iff $g(x) = 0$, whenever $x \in \operatorname{Ker} f$, $y \in \operatorname{Ker} g$ and $g(x) = f(y)$.
(vi) G is commutative iff $f = g$.
(vii) G is left distributive iff $f + g = 1$ and $f(a) = 0$.
(viii) G is unipotent iff $g = -f$.
(ix) G satisfies the identity $x \, . \, yz = z \, . \, yx$ iff $f = g^2$.

Proof. Easy.

3.3 Example. Let $G(+)$ be a vector space (over a field) with basis $\{x_{2i,i+j}, x_{2i+1,i+j+1} \mid 0 \leq i, j\}$. Define two endomorphisms f, g of $G(+)$ as follows:

$$f(x_{0,j}) = 0 \quad \text{and} \quad f(x_{2i,i+j}) = x_{2i-2,i+j} \quad \text{for all} \quad 1 \leq i, \quad 0 \leq j,$$
$$f(x_{2i+1,i+1}) = x_{2i,i}, \quad f(x_{2i+1,i+j+1}) = x_{2i+1,i+j}, \quad 0 \leq i, \quad 1 \leq j,$$
$$g(x_{1,j+1}) = 0, \quad g(x_{2i+1,i+j+1}) = x_{2i-1,i+j+1}, \quad 1 \leq i, \quad 0 \leq j,$$
$$g(x_{0,0}) = 0 \quad \text{and} \quad g(x_{2i,i+j}) = x_{2i,i+j-1} \quad \text{for all } 0 \leq i, \quad 1 \leq j,$$
$$g(x_{2i,i}) = x_{2i-1,i} \quad \text{for every} \quad 1 \leq i.$$

It is easy to check that f, g are surjective, $fg = gf$ and $\mathrm{Ker}\, fg = \mathrm{Ker}\, f + \mathrm{Ker}\, g$, $\mathrm{Ker}\, f \cap \mathrm{Ker}\, g \neq 0$. Hence the corresponding regular medial division groupoid $G(xy = f(x) + g(y))$ satisfies (C1) and is not semifaithful. Moreover, G does not satisfy (C2) and G is crl-simple.

3.4 Example. Let $G(+)$ be a vector space with basis $\{x_{i,j}\}$, where i, j are integers such that either $0 \leq i$ or $0 \leq j$. Define f, g as follows: $f(x_{0,j}) = 0$ for $\cdot < 0$ and $f(x_{i,j}) = x_{i-1,j}$ otherwise; $g(x_{i,0}) = 0$ for $i < 0$ and $g(x_{i,j}) = x_{i,j-1}$ otherwise. It is easy to see that the corresponding gruopoid is semifaithful but does not satisfy (C1).

3.5 Proposition. Let G be a regular medial division groupoid. Let $b \in G$ be an element and $a = bb$. Then there exist an abelian group $G(+)$ and two surjective endomorphisms f, g of $G(+)$ such that $fg = gf$, $b = 0$ is the zero of $G(+)$ and $xy = f(x) + g(y) + a$ for all $x, y \in G$.

Proof. See [1].

3.6 Lemma. Let G be a regular medial division groupoid. Then:
(i) If G is semifaithful then G satisfies (C2).
(ii) If G satisfies (C2) then either G is semifaithful or G does not satisfy (C1).
(iii) Every factor of G is semifaithful iff every factor of G satisfies (C1).

Proof. By 3.5, there are an abelian group $G(+)$, two surjective endomorphisms f, g of $G(+)$ and an element $a \in G$ such that $fg = gf$ and $xy = f(x) + g(y) + a$ for all $x, y \in G$.
(i) Since G is semifaithful, $\mathrm{Ker}\, f \cap \mathrm{Ker}\, g = 0$. Further, let $x \in \mathrm{Ker}\, f$, $y \in \mathrm{Ker}\, g$ and $g(x) = f(y)$. Then $fg(x) = gf(x) = 0$ and $gg(x) = gf(y) = fg(y) = 0$. Hence $g(x) = 0$ and G satisfies (C2).
(ii) Let G be not semifaithful and let G satisfy (C1). There are $x, y, z \in G$ such that $x, y \in \mathrm{Ker}\, f$, $x, z \in \mathrm{Ker}\, g$, $x \neq 0$ and $g(y) = x = f(z)$. Hence G does not satisfy (C2).
(iii) First, let every factor of G be semifaithful. Let $z \in \mathrm{Ker}\, g$ and $K(+)$ be the subgroup of $G(+)$ generated by $\{f^n(z) \mid 1 \leq n\}$. Obviously, $f(K) \subsetneq K \subseteq f(\mathrm{Ker}\, g)$ and $g(K) = 0 \subseteq K \subseteq \mathrm{Ker}\, g$. Hence the relation r defined by $x\, r\, y$ iff $x - y \in K$ is a congruence of G. Let h be the natural homomorphism of G onto $H = G/r$. It is easy to see that $h(z)\, t_H\, h(0)$. Consequently, $h(z) = h(0)$ and $z \in K$. In particular, $z = f(u)$ for some $u \in \mathrm{Ker}\, g$. Thus G satisfiies (C1). Now, let every factor of G satisfy (C1). Let $w \in \mathrm{Ker}\, f \cap \mathrm{Ker}\, g$ and $L(+)$ be the subgroup generated by w. Then $L \subseteq \mathrm{Ker}\, f \cap \mathrm{Ker}\, g$ and the relation s, $x\, s\, y$ iff $x - y \in L$, is a congruence of G. Further, let $u \in G$ be such that $w = g(u)$. Since G/s satisfies (C1), there is $v \in G$ with $u - f(v) \in L$ and $g(v) \in L$. Now, $w = g(u) =$ $= g(u - f(v)) = 0$.

3.7 Corollary. Every semifaithful medial division groupoid satisfies (C2).

3.8 Proposition. There exists a cardinal number α such that $|G| \leq \alpha$ for every subdirectly irreducible regular medial division groupoid G.

Proof. Let G be a subdirectly irreducible regular medial division groupoid. Denote by r the least non-trivial congruence of G. By 3.5, there are an abelian group $G(+)$, two surjective endomorphisms f, g of $G(+)$ and an element $a \in G$ such that $fg = gf$ and $xy = f(x) + g(y) + a$ for all $x, y \in G$. Let R be the ring of polynomials with two commuting indeterminates λ, ϱ over the ring of integers. We can define an R-module structure on $G(+)$ by $\lambda x = f(x)$ and $\varrho x = g(x)$ for every $x \in G$. Let $H(+)$ be a non-zero submodule of $G(+)$. The relation s defined by xsy iff $x - y \in H$ is obviously a congruence of the groupoid G. Hence $r \subseteq s$ and we see that the R-module $G(+)$ is cocyclic (with respect to 3.5, we can assume that $0\, r\, c$ for some $0 \neq c$).

3.9 Lemma. Let G be a left (right) regular medial division groupoid. Let A, B be two blocks of t_G. Then $|A| = |B|$.

Proof. It suffices to show that there is an injective mapping h of A into B. Let $a \in A$, $b \in B$ be arbitratry. There are $c \in G$ and two transformations f, g of G such that $a = ca$, $R_a f = 1 = L_c g$. Put $h(x) = f(b)g(x)$ for every $x \in A$. We have $cy \,.\, g(x)\, z = cg(x) \,.\, yz = x \,.\, yz = a \,.\, yz = ca \,.\, yz = cy \,.\, az$, and so $g(x)\; o\; a$. In particular, $f(b)g(x) \,.\, az = f(b)\, a \,.\, g(x)\, z = b \,.\, g(x)\, z = b \,.\, az$ for every $z \in G$. From this, $f(b)g(x)\; p\; b$. Further, $zc \,.\, f(b)g(x) = zf(b) \,.\, cg(x) = zf(b) \,.\, x = zf(b) \,.\, a = zf(b) \,.\, ca = zc \,.\, f(b)\, a = zc \,.\, b$ for every $z \in G$ and we have proved that $f(b)g(x)\; t\; b$. Hence h is a mapping of A into B. It remains to show that h is injective. For, let $x, y \in A$ and $h(x) = h(y)$. Then $f(b)g(x) = f(b)g(y)$, and so $g(x)\; q\; g(y)$, since G is left regular. In particular, $x = cg(x) = cg(y) = y$.

4. Primitive Medial Division Groupoids

Let G be a medial division groupoid. We shall say that G is primitive if there are two different elements $a, b \in G$ such that $t_G = \{\langle a, b\rangle, \langle b, a\rangle\} \cup d_G$.

4.1 Lemma. The following conditions are equivalent for a medial division groupoid G:
(i) G is primitive.
(ii) $t_G \neq d_G$ and t_G is a minimal congruence of G.

Proof. Obvious.

4.2 Proposition. Every subdirectly irreducible medial division groupoid is either semifaithful or primitive.

Proof. Apply 2.13(ii).

4.3 Proposition. A primitive medial division groupoid is neither left nor right regular.

Proof. This is an immediate consequence of 3.9.

4.4 Proposition. A non-trivial crl-simple medial division groupoid is subdirectly irreducible iff it is primitive.

Proof. The direct implication follows from 2.25 and 4.2. The converse implication is an easy consequence of 2.25, 2.26 and 4.1.

4.5 Lemma. Let G be a primitive medial division groupoid and $a, b \in G$ be such that $a \neq b$ and $a \, t_G \, b$. Let $r \neq d_G$ be a congruence of G. Then there is $c \in G$ with $a \neq c$ and $a \, r \, c$.

Proof. If $t \subseteq r$, we can put $c = b$. Suppose that t is not contained in r. Then $t \cap r = d_G$ and there are $x, y \in G$ with $x \neq y$, $x \, r \, y$ and $\langle x, y \rangle \notin t$. Hence $\{x, y\} \neq \{a, b\}$ and we can assume that $\langle x, y \rangle \notin q$ (the other case is similar). Further, $a = zx$ for some $z \in G$. We have $a = zx \, r \, zy$ and we can put $zy = c$, provided $zx \neq zy$. In the opposite case, $vx \, t \, vy$ for every $v \in G$ (since G/t is regular). But, $vx \, r \, vy$, too, and so $vx = vy$, i.e., $x \, q \, y$, a contradiction.

4.6 Proposition. The following conditions are equivalent for a medial division groupoid G:
(i) Every factorgroupoid of G is semifaithful.
(ii) No factorgroupoid of G is primitive.

Proof. (i) implies (ii). Apply 4.1.
(ii) implies (i). By 4.2, every subdirectly irreducible factor of G is semifaihful. However, semifaithful groupoids are closed under subdirect products.

4.7 Proposition. Let G be a regular medial division groupoid. Then there exists a congruence $r \subseteq t_G$ of G such that the factorgroupoid G/r is primitive.

Proof. There is a block A of t containing at least two elements. Let $a \in A$, $B = A \setminus \{a\}$ and $r = (t \setminus (A \times A)) \cup (B \times B) \cup d_G$. Then r is a congruence of G and we denote by f the natural homomorphism of G onto $H = G/r$. Let $x, y \in G$ and $f(x) \, t_H \, f(y)$. Then $xz \, r \, yz$ and $zx \, r \, zy$ for every $z \in G$. Let $u \in G$ be such that $a = xu$. We have $a = xu \, r \, yu$, consequently $xu = yu$ and $x \, p_G \, y$, since G is regular. Similarly, $x \, q_G \, y$. Thus $x \, t_G \, y$ and the rest is clear.

4.8 Proposition. Let G be a primitive medial division groupoid. Then:
(i) G/t satisfies (C2) and does not satisfy (C1).
(ii) G/p and G/q are not semifaithful.

Proof. (i) Let $x, y, z \in G/t = H$, $x \, p_H \, y \, q_H \, z$, $xx = zz$ and let f be the natural homomorphism of G onto H. There are $c, d, e \in G$ with $f(c) = x$, $f(d) = y$, $f(e) = z$. From this, $yx = xx = zz = zy$ and $dc \, t_G \, cc \, t_G \, ee \, t_G \, ed$. Further, $cu \, t_G \, du$ and $ud \, t_G \, ue$ for every $u \in G$. Now, it is visible that $cu \, . \, cv = du \, . \, cv = dc \, . \, uv = ed \, . \, uv = eu \, . \, dv = eu \, . \, cv$ and $wv \, . \, cu = wv \, . \, eu$ for all

$u, v, w \in G$. Therefore $cu\, t_G\, eu$ for every u. Similarly, $uc\, t_G\, ue$, and consequently $x\, t_H\, z$. We have proved that H satisfies (C2). Finally, H does not satisfy (C1), as it follows from 2.28.
(ii) Apply 2.19 and 4.3.

4.9 Construction. Let G be a medial division groupoid, $a \in G$ be an element and $M = M(a, G) = \{\langle x, y\rangle \mid x, y \in G, xy = a\}$. Let N be a subset of M and α be an element not belonging to G. We shall define a groupoid $G(a, \alpha, N) = H(*)$ as follows: $H = G \cup \{\alpha\}$; $x * y = xy$ for all $x, y \in G$, $\langle x, y\rangle \notin M$; $x * y = a$ for every $\langle x, y\rangle \in N$; $x * y = \alpha$ for every $\langle x, y\rangle \in M \setminus N$; $x * \alpha = x * a$ and $\alpha * x = a * x$ for every $x \in G$; $\alpha * \alpha = a * a$.

4.9.1 Lemma. $a\, t_{H(*)}\, \alpha$ and G is isomorphic to $H(*)/r$ for a congruence $r \subseteq t_{H(*)}$.

Proof. Obvious.

4.9.2 Lemma. Let $x, y \in G$. Then $x\, p_{H(*)}\, y$ iff $x\, p_G\, y$ and for every $z \in G$, $\langle x, z\rangle \in N$ iff $\langle y, z\rangle \in N$.

Proof. Use 4.9.1.

4.9.3 Lemma. $H(*)$ is a division groupoid iff the following two conditions are satisfied:
(i) For every $x \in G$ there are $y, z \in G$ such that $\langle x, y\rangle \in N$ and $\langle x, z\rangle \in M \setminus N$.
(ii) For every $x \in G$ there are $y, z \in G$ such that $\langle y, x\rangle \in N$ and $\langle z, x\rangle \in M \setminus N$.

Proof. Easy.

4.9.4 Lemma. $H(*)$ is medial iff $\langle xu, yv\rangle \in N$, whenever x, y, u, v are from G and $\langle xy, uv\rangle \in N$.

Proof. Let $x, y, u, v \in G$ be arbitrary. Taking into account that $a\, t_{H(*)}\, \alpha$, it is easy to verify that $(x * y) * (u * v) = (xy) * (uv)$. Now, the assertion is evident.

4.9.5 Lemma. $H(*)$ is commutative iff G is and $\langle x, y\rangle \in N$ iff $\langle y, x\rangle \in N$.

Proof. Obvious.

4.9.6 Lemma. $H(*)$ satisfies the identity $x \,.\, yz = z \,.\, yx$ iff G satisfies the identity and $H(*)$ is medial.

Proof. The direct implication is clear. As for the converse implication, let $x, y, z \in G$ be such that $\langle x, yz\rangle \in N$. There are $u, v \in G$ with $x = uv$ and $z = uy$ Then $\langle uv, yz\rangle \in N$, and so $\langle uy, vz\rangle = \langle z, vz\rangle \in N$. But $vz = v \,.\, uy = y \,.\, uv = yx$. The rest is clear.

4.10 Proposition. Let H be a medial division groupoid, $a, b \in H$ be such that $a \neq b$, $a\, t_H\, b$. Put $r = \{\langle a, b\rangle, \langle b, a\rangle\} \cup d_H$ and $G = H/r$. Then there is a subset $N \subseteq M(a/r, G)$ such that H is isomorphic to $G(a/r, \alpha, N)$.

Proof. Easy.

4.11 Construction. Let $G(+)$ be an abelian group, f, g be two surjective endomorphisms of $G(+)$ such that $fg = gf$ and $a \in G$ be an element. Put $xy = f(x) + g(y) + a$ for all $x, y \in G$. Then G is a regular medial division groupoid. Further, let K be the set of all ordered pairs $\langle x, y\rangle$ with $x, y \in G$ and $f(x) + g(y) = 0$ and L be the set of all ordered pairs $\langle g(x), f(-x)\rangle$, $x \in G$. Obviously, $L \subseteq K$ and both $L(+)$ and $K(+)$ are subgroups of $G(+) \times G(+)$. Finally, let I be a subset of K and $b, c \in G$ be such that $f(b) + g(c) = -a$.

4.11.1 Lemma. $\langle b, c\rangle + K = M(0, G) = M$ and $N = \langle b, c\rangle + I \subseteq M$.

Proof. Obvious.

Let α be an element not belonging to G. Put $H = G \cup \{\alpha\}$ and define an operation $*$ on H as follows: $x * y = xy = f(x) + g(y) + a$ for all $x, y \in G$ with $\langle x - b, y - c\rangle \notin K$; $x * y = 0$ for all $x, y \in G$ with $\langle x - b, y - c\rangle \in I$; $x * y = \alpha$ for all $x, y \in G$ with $\langle x - b, y - c\rangle \in K \setminus I$; $\alpha * x = 0 * x$ and $x * \alpha = x * 0$ for every $x \in G$; $\alpha * \alpha = 0 * 0$.

4.11.2 Lemma. $H(*) = G(0, \alpha, N)$.

Proof. Obvious.

4.11.3 Lemma. $H(*)$ is medial iff $I = J + L$ for a subset J of K.

Proof. First, let $H(*)$ be medial, $\langle x, y\rangle \in I$ and $\langle u, v\rangle \in L$. Then $f(x) + g(y) = 0$ and $u = g(z), v = f(-z)$ for some $z \in G$. Since $\langle x, y\rangle \in I$, $\langle x + b, y + c\rangle \in N$. Further, there are $r, s \in G$ such that $f(r) + a = x + b$ and $f(z) + g(s) + a = y + c$. Now, $\langle r0, zs\rangle = \langle x + b, y + c\rangle \in N$, and so $\langle rz, 0s\rangle \in N$. Then $\langle x + u, y + v\rangle = \langle rz, 0s\rangle - \langle b, c\rangle \in I$. We have proved that $I + L$ is contained in I. Consequently, $I = J + L$ for a subset J of L. Conversely, let $I = J + L$ and $\langle xy, uv\rangle \in N$. Then $\langle xu, yv\rangle = \langle f(x) + g(y) + a, f(u) + g(v) + a\rangle + \langle g(u - y), f(y - u)\rangle \in N$.

4.11.4 Lemma. $H(*)$ is a medial division groupoid iff $I \neq K$ and $I = J + L$ for a non-empty subset J of K.

Proof. Easy (apply 4.11.3 and 4.9.3).

4.11.5 Lemma. Let $I = L \neq K$ and the groupoid G satisfy (C2). Then $H(*)$ is a primitive medial division groupoid.

Proof. $H(*)$ is a medial division groupoid by 4.11.4. Let $x, y \in G$ and $x\, t_{H(*)}\, y$. There is $u \in G$ with $g(u) = x$. Then the pair $\langle x, f(-u)\rangle$ belongs to L, and so $\langle y, f(-u)\rangle$ is contained in L. Then $y = g(v), f(-u) = f(-v)$ for some $v \in G$ and we have $f(v - u) = 0, y = g(v) = g(v - u) + g(u) = g(r) + x$, where $r = v - u$. Similarly, there is $s \in G$ such that $g(s) = 0$ and $y = f(s) + x$.

Thus $f(r) = 0 = g(s)$ and $g(r) = f(s)$. But G satisfies (C2), and therefore $g(r) = = 0$. From this, $x = y$ and the rest is clear.

4.12 Proposition. Every primitive medial division groupoid can be constructed in the way described in 4.11.

Proof. Apply 2.14(i), 3.5, 4.10 and 4.11.

4.13 Construction. Let f be a surjective endomorphism of an abelian group $G(+)$ and $a \in G$ be an element. Let $b \in G$ be such that $f(b) = -a$ and T be a non-empty subset of $\operatorname{Ker} f$ such that $T \neq \operatorname{Ker} f$. Let α be an element not belonging to G. Put $H = G \cup \{\alpha\}$, $xy = f(x + y - b)$ for all $x, y \in G$. Let N be the set of all ordered pairs $\langle x, y \rangle$ with $x, y \in G$ and $x + y - b \in T$. We shall define an operation $*$ on H as follows: $x * y = xy = f(x + y - b)$ for all $x, y \in G$ with $x + y - b \notin \operatorname{Ker} f$; $x * y = 0$ for all $x, y \in G$ with $x + y - b \in T$; $x * y = \alpha$ for all $x, y \in G$ with $x + y - b \in \operatorname{Ker} f \setminus T$; $x * \alpha = x * 0$ and $\alpha * x = 0 * x$ for every $x \in G$; $\alpha * \alpha = 0 * 0$.

4.13.1 Lemma. $H(*) = G(0, \alpha, N)$ is a commutative medial division groupoid.

Proof. Obvious.

4.13.2 Lemma. $H(*)$ is primitive iff $T \neq A + B$ for every non-zero subgroup A of $\operatorname{Ker} f$ and every non-empty subset B of $\operatorname{Ker} f$.

Proof. First, let $H(*)$ be primitive. Suppose that $T = A + B$, where A is a subgroup of $\operatorname{Ker} f$ and B is a non-empty subset of $\operatorname{Ker} f$. Then $S = \operatorname{Ker} f \setminus \setminus T = A + C$ for a non-empty subset C of $\operatorname{Ker} f$. Assume that $0 \in T$ (the other case, when $0 \in S$, is similar). Let $c \in A$ be arbitrary. If $c + x \in T$ for some $x \in G$ then $x \in T$. Conversely, if $x \in T$ then $c + x \in T$. From this, it is easy to verify that $0 \; t_{H(*)} \; c$, and so $c = 0$ and $A = 0$. Conversely, let the condition from the lemma be satisfied. Let $x, y \in G$ and $x \; t_{H(*)} \; y$. Then $z = x - y \in \operatorname{Ker} f$. For $u \in T, y + (u + b - y) - b \in T$, and hence $x + (u + b - y) - b = z + u$ is contained in T. Consequently, $z + T \subseteq T$. Similarly, if $z + u$ is in T, then $x + (u + b - y) - b \in T$, and so $u \in T$. Thus $T - z \subseteq T$. Denote by D the subgroup generated by z and assume that $0 \in T$ (the other case is similar). We have $z \in T$ and it is easy to check that $D \subseteq T$. On the other hand, if $u \in D$ and $v \in T$ then $u = nz$ for some integer n and we have $u + v = nz + v \in T$. Thus $T = D + E$ for a non-empty subset E of $\operatorname{Ker} f$ and $D = 0$, $x = y$.

4.13.3 Lemma. $H(*)$ is primitive, provided neither T nor $\operatorname{Ker} f \setminus T$ contains a non-zero subgroup.

Proof. Obvious.

4.13.4 Lemma. $H(*)$ is unipotent, provided $2x = 0 = a$ for every $x \in G$ and $0 \in T$.

Proof. Easy.

4.14 Proposition. Every primitive commutative medial division groupoid can be constructed in the way described in 4.13.

Proof. Evident.

4.15 Proposition. The following conditions are equivalent for a medial division groupoid G:

(i) G is regular, satisfies (C2), and does not satisfy (C1).

(ii) There exists a primitive medial division groupoid H such that G is isomorphic to H/t_H.

Proof. (i) implies (ii). G is regular, and hence there are an abelian group $G(+)$, two surjective endomorphisms f, g of $G(+)$ and an element $a \in G$ such that $fg = gf$ and $xy = f(x) + g(y) + a$ for all $x, y \in G$. Let L be the set of all ordered pairs $\langle g(x), f(-x)\rangle$, $x \in G$, and let $b \in G$ be such that $f(b) = -a$. Consider the groupoid $H(*) = G(0, \alpha, N)$, where $\alpha \notin G$ and $N = \langle b, 0\rangle + L$. Since G does not satisfy (C1), there are $x, y \in G$ with $\langle x, y\rangle \notin L$ and $f(x) + g(y) = 0$. By 4.11.5, $H(*)$ is a primitive medial division groupoid. Obviously, G is isomorphic to $H(*)/t$.

(ii) implies (i). See 2.14 and 4.8.

4.16 Proposition. Let G be a crl-simple regular medial division groupoid such that G satisfies (C2) and does not satisfy (C1). Then there exists a subdirectly irreducible primitive medial division groupoid H such that G is isomorphic to H/t_H.

Proof. By 4.15, G is isomorphic to H/t for a primitive medial division groupoid H. Since $t_H \subseteq \bar{p}_H, \bar{q}_H$ and G is crl-simple, H has the same property, and therefore H is subdirectly irreducible by 4.4.

4.17 Proposition. The following conditions are equivalent for a commutative medial (unipotent) division groupod G:

(i) G is regular and G is not a quasigroup.

(ii) There exists a primitive commutative medial (unipotent) division groupoid H such that G is isomorphic to H/t_H.

Proof. Similar to that of 4.15 (use 4.13 and 2.30).

4.18 Proposition. Let G be a division groupoid satisfying the identity $x \,.\, yz = z \,.\, yx$. The following conditions are equivalent:

(i) G is regular and G is not a quasigroup.

(ii) G is isomorphic to H/t_H for a primitive division groupoid H satisfying the identity $x \,.\, yz = z \,.\, yx$.

Proof. Apply 4.15, 4.9.6 and 2.31.

5. Main Results

5.1 Theorem. The following conditions are equivalent for a medial division groupoid G:
(i) Every factorgroupoid of G is regular.
(ii) Every factorgroupoid of G is right regular.
(iii) Every factorgroupoid of G is left regular.
(iv) No factorgroupoid of G is primitive.
(v) Every factorgroupoid of G is semifaithuful.
(vi) Every regular factorgroupoid of G is semifaithful.
(vii) Every factorgroupoid of G satisfies (C1).
(viii) Every regular factorgroupoid of G satisfies (C1).
Moreover, if G is commutative, then these conditions are equivalent to:
(ix) G is a quasigroup and every congruence of G is cancellative.

Proof. The implications (i) implies (ii), (i) implies (iii), (v) implies (vi), (vii) implies (viii) and (ix) implies (i) are trivial. The remaining implications follow from 2.2(i), 2.14(i), 2.19, 2.28, 2.30, 3.6(iii), 4.3 and 4.6.

5.2 Corollary. The following conditions are equivalent for a variety V of groupoids:
(i) Every medial division groupoid from V is regular.
(ii) Every regular medial division groupoid from V is semifaithful.
(iii) Every regular medial division groupoid from V satisfies (C1).

5.3 Proposition. Let G be a subdirectly irreducible medial division groupoid. Then at least one of the following conditions is satisfied:
(i) G is a left quasigroup (and hence regular).
(ii) G is a right quasigroup (and hence regular).
(iii) G is primitive (and hence neither left nor right regular).

Proof. Apply 2.2(i), (iii), 2.14(iv), 4.2, 4.3.

5.4 Theorem. (i) There exists a cardinal number α such that $|G| \leq \alpha$ for every subdirectly irreducible regular medial division groupoid G.
(ii) For every cardinal number β, there exists a commutative unipotent medial dividsion groupoid G such that G is subdirectly irreducible, primitive, c-simple and $|G| \geq \beta$.
(iii) For every cardinal nubmer β, there exists a division groupoid G satisfying the identity $x \,.\, yz = z \,.\, yx$ such that G is subdirectly irreducible, primitive, crl-simple and $|G| \geq \beta$.

Proof. (i) See 3.8
(ii) Let R be the ring of polynomials with one indeterminate λ over the two-element field. Then R is a commutative principal ideal domain. Further, let $M(+)$

be an abelian group such that M contains at least β elements and $2x = 0$ for every $x \in M$. Then $M(+)$ is an R-module and we can consider the injective hull $G(+)$ of $M(+)$. Then $|G| \geq \beta$, $\lambda G = G$ and for every $x \in G$ there is $1 \leq n$ with $\lambda^n x = 0$. Put $x * y = \lambda(x + y)$ for all $x, y \in G$. It is visible that $G(*)$ is a commutative unipotent medial division groupoid. Moreover, $G(*)$ is c-simple and it is not a quasigroup. By 4.17, $G(*)$ is isomorphic to H/t for a primitive commutative unipotent medial division groupoid H. Clearly, $|H| \geq \beta$ and H is c-simple. Hence H is subdirectly irreducible.
(iii) Let $M(+)$ be an abelian group such that $|M| \geq \beta$ and $2x = 0$ for every $x \in M$. Denote by $G(+)$ the divisible hull of $M(+)$ and put $x * y = 4x + 2y$ for all $x, y \in G$. In the rest, we can proceed similarly as in the proof of (ii).

5.5 Proposition. Every left distributive medial division groupoid is regular, semifaithful and satsifies (C1), (C2).

Proof. Let G be a left distributive medial division groupoid. If G is regular then G is semifaithful as it follows easily from 3.2(iii), (vii). Now, it remains to use 5.1.

5.6 Theorem. The following conditions are equivalent for a groupoid G:
(i) G is a left distributive medial division groupoid.
(ii) There exist an abelian group $G(+)$, a surjective endomorphism f of $G(+)$ and an element $a \in G$ such that $f(a) = 0$, the endomorphism $1 - f$ is surjective and $xy = f(x - y) + y + a$ for all $x, y \in G$.

Proof. Apply 5.5 and 3.5.

5.7 Proposition. The following conditions are equivalent for a medial division groupoid G:
(i) G is left distributive.
(ii) The factorgroupoid G/p is idempotent.
(iii) Every block of p_G is a subgroupoid.
(iv) For every $x \in G$, $x \, p_G \, xx$.

Proof. (iv) implies (i). Let $x, y, z \in G$. Since $x \, p \, xx$, $x \,.\, yz = xx \,.\, yz = xy \,.\, xz$.

5.8 Corollary. The following conditions are equivalent for a groupoid G:
(i) G is a distributive medial division groupoid.
(ii) There exist an abelian group $G(+)$ and a surjective endomorphism f of $G(+)$ such that $1 - f$ is surjective and $xy = f(x - y) + y$ for all $x, y \in G$.
(iii) G is an idempotent medial division groupoid.
(iv) G is a medial division groupoid and the factorgroupoid G/t is idempotent.

5.9 Theorem. A medial division groupoid G is regular, provided at least one of the following conditions is satisfied:

(i) G is semifaithful.
(ii) The factorgroupoid G/t satisfies (C1).
(iii) The factorgroupoids G/p and G/q are semifaithful.
(iv) G is a left quasigroup.
(v) G is a right quasigroup.
(vi) G is left distributive.
(vii) G is right distributive.
(viii) G is idempotent.

Proof. Apply 2.15, 2.19, 2.28, 5.5 and 5.8.

5.10 Lemma. Let G be a medial groupoid and $a, b, c, d, e, f \in G$. Then for all $x, y, z \in G$, the following equalities hold:
$((x \,.\, yd)(yb \,.\, z))((a \,.\, bc)(de \,.\, f)) = ((x \,.\, yd)(yb \,.\, z))((a \,.\, dc)(be \,.\, f))$,
$((a \,.\, bc)(de \,.\, f))((x \,.\, cy)(ey \,.\, z)) = ((a \,.\, dc)(be \,.\, f))((x \,.\, cy)(ey \,.\, z))$.

Proof. $((x \,.\, yd)(yb \,.\, z))((a \,.\, bc)(de \,.\, f)) = ((x \,.\, yd)(a \,.\, bc))((yb \,.\, z)(de \,.\, f)) =$
$= ((xa)(yd \,.\, bc))((yb \,.\, de)(zf)) = ((xa)(yb \,.\, dc))((yd \,.\, be)(zf)) =$
$= ((x \,.\, yb)(a \,.\, dc))((yd \,.\, z)(be \,.\, f)) = ((x \,.\, yb)(yd \,.\, z))((a \,.\, dc)(be \,.\, f)) =$
$= ((x \,.\, yd)(yb \,.\, z))((a \,.\, dc)(be \,.\, f))$ and $((a \,.\, bc)(de \,.\, f))((x \,.\, cy)(ey \,.\, z)) =$
$= ((a \,.\, de)(bc \,.\, f))((x \,.\, cy)(ey \,.\, z)) = \ldots = ((a \,.\, dc)(be \,.\, f))((x \,.\, cy)(ey \,.\, z))$.

5.11 Proposition. A medial groupoid G satisfies the identity $(x \,.\, yz)(uv \,.\, w) = (x \,.\, uz)(yv \,.\, w)$, provided at least one of the following conditions holds:
(i) G is regular and semifaithful.
(ii) G is a cancellation groupoid.
(iii) G is a division groupoid.

Proof. Apply 5.10, 3.5, 4.11, 4.12.

5.12 Corollary. (i) The variety of medial groupoids is not generated by the class of medial cancellation groupoids.
(ii) The variety of medial groupoids is not generated by the class of medial division groupoids.

5.13 Example. Let $Q(+)$ be the additive group of rational numbers. Put $x * y = 2x + 2y$ for all $x, y \in Q$. Then $Q(*)$ is a commutative medial quasigroup. Define a relation r on Q by $a\,r\,b$ iff $a - b$ is an integer. Obviously, r is a congruence of $Q(*)$. Denote by f the natural homomorphism of $Q(*)$ onto $Q(*)/r$. We are going to show that f is a monomorphism in the category of medial division groupoids. For, let G be a medial division groupoid and g, h be homomorphisms of G into $Q(*)$ such that $fg = fh$. We have $g(ab) = 2g(a) + 2g(b)$ for all $a, b \in G$ Similarly for h. Put $k(a) = g(a) - h(a)$ for every $a \in G$. It is easy to check that k is a homomorphism of G into $Q(*)$. Since $fg = fh$, $k(G) \subseteq Z$, Z being the set of integers. However, $k(G)$ is a subgroupoid of $Q(*)$ and $k(G)$ is a division groupoid. Consequently, $k(G) = 0$ and $g = h$.

5.14 Corollary. The category of medial division groupoids possesses non-injective monomorphisms.

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