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Cyclicity in a Special Class of Hypergroups

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Let \( \langle H, \ast \rangle \) be a multiplicative hypergroup as defined in [1], [2] i.e. the non-empty set \( H \) equipped with a non-degenerate hyperoperation

\[ \ast : H \times H \to \mathcal{P}(H) : (x, y) \mapsto x \ast y \subseteq H, \quad x \ast y \neq \emptyset \]

(If \( A, B \subseteq H \), we set \( A \ast B = \bigcup_{a \in A, b \in B} a \ast b \). If \( A = \{a\} \), we write \( A \ast B = a \ast B \).) which is associative: \( x \ast (y \ast z) = (x \ast y) \ast z, \forall x, y, z \in H \), and the condition \( a \ast H = H \ast a = H, \forall a \in H \), is valid.

For every integer \( \nu > 0 \), and \( \forall s \in H \), we get the powers of \( s \) : \( s^1 = \{s\}, s^{\nu+1} = s^\nu \ast s \subseteq H \).

Now, using the original definition of cyclic hypergroup as we can see in [3] as well, we give the following definitions.

Definitions. A hypergroup \( H \) is called cyclic, if

\[ H = h^1 \cup h^2 \cup \ldots \cup h^n \cup \ldots, \text{ for some } h \in H. \tag{1} \]

If there exists an integer \( n > 0 \), the minimum one with the following property

\[ H = h^1 \cup h^2 \cup \ldots \cup h^n, \tag{2} \]

then we call \( H \) cyclic hypergroup with finite period and we call \( h \) generator of \( H \) with period \( n \). If there is no number \( n \) for which (2) is valid, but (1) is valid, then we say that \( H \) has infinite period for \( h \). If all generators of \( H \) have the same period, then we call \( H \) cyclic with period.

If there exists an integer \( n > 0 \), the minimum one with the following property

\[ H = h^n, \tag{3} \]

then we call \( H \) single-power cyclic hypergroup and \( h \) generator of \( H \) with period \( n \). If (1) is valid and also \( \forall n \in \mathbb{N}_0 \) and \( n \geq n_0 \), for constant \( n_0 \in \mathbb{N}_0 \), the following condition is valid

\[ h^1 \cup h^2 \cup \ldots \cup h^{n-1} \subseteq h^n, \tag{4} \]
then we call $H$ single-power cyclic hypergroup with infinite period for $h$.

Obviously we can prove the following proposition.

Proposition 1. Let $(H, \cdot)$ be a commutative group and $P$ a subset of $H$. Then \( \langle H, P \rangle \) is a hypergroup, where the hyperoperation \( P \) is defined by the relation

\[
P : H \times H \to \mathcal{P}(H) : (x, y) \mapsto x^P y = xy(\{e\} \cup P),
\]

where $e$ is the unit element of $(H, \cdot)$.

We shall call the above hypergroup $P$-hypergroup.

Proposition 2. Let $(H, \cdot)$ be a finite cyclic group $\# H = n$ and $P \subset H$. Then \( \langle H, P \rangle \), where \( P \) is defined by (5), is a cyclic hypergroup which we shall call $P$-cyclic hypergroup.

Proof. From now on we denote the powers of the elements of $H_n$ for the hyperoperation in square brackets.

We can easily see that:

\[
x^{\nu v} = x^\nu(\{e\} \cup P \cup P^2 \cup \ldots \cup P^{\nu - 1}), \quad \forall \nu \in \mathbb{N}_0.
\]

So if $a \in H_n$ is a generator of $(H_n, \cdot)$, all over in this paper, then

\[
a^{[1]} \cup a^{[2]} \cup \ldots \cup a^{[n]} = H_n,
\]

so $a$ is a generator of \( \langle H_n, P \rangle \) with period at most $n$.

In the following, we shall prove some theorems which are valid in the special case of $P$-cyclic hypergroups, where $P = \{p\}$ is a set with only one element. We write it as \( \langle H_n, P \rangle \).

Theorem 1. In the $P$-cyclic hypergroup \( \langle H_n, a^x \rangle \), the element $a^x$ is a generator iff $(\lambda, \chi, n) = 1$, i.e. $\lambda, \chi, n$ are relatively prime.

Proof. The $\mu$-th power of the element $a^x$ under the hyperoperation $a^x$, using the relation (6), is

\[
a^{\mu x} = \{a^{\lambda \mu}, a^{\lambda \mu + x}, \ldots, a^{\lambda \mu + (\mu - 1)x}\}.
\]

Therefore the elements of the powers of $a^x$ have the form

\[
a^{sk + tx}, \quad \text{where} \quad s \in \mathbb{N}_0 \quad \text{and} \quad t = 0, 1, \ldots, s - 1.
\]

Also we have

\[
\lambda s + t \chi \equiv 1 \mod n \quad \text{iff} \quad \exists q \in \mathbb{Z} : \lambda s + t \chi - q^\nu = 1 \quad \text{iff} \quad (\lambda, \chi, n) = 1.
\]
So if we choose appropriate \( s, t, Q \mod n \), as we need above, the relation \( a^{s+tx} = a^1 = a \) is valid iff \( (\lambda, x, n) = 1 \). Therefore the element \( a \in H_n \) belongs to some power of \( a^\lambda \) iff \( (\lambda, x, n) = 1 \).

Now, if \( a \) belongs to some power of \( a^\lambda \), then \( \forall v \in \mathbb{N}_0 \) the element \( a^v \in H_n \) belongs to some power of \( a^\lambda \), because
\[
a^{\lambda(v s)+(vt) \times} = a^v.
\]

From the above, we obtain that the element \( a^\lambda \) is a generator of \( \langle H_n, a^x \rangle \) iff
\[ (\lambda, x, n) = 1. \]

**Theorem 2.** In the \( P \)-cyclic hypergroup \( \langle H_n, a^x \rangle \), \( a^\lambda \neq a^n = e \),

(i) the element \( a^\lambda \) is a generator with period \( \mu = \lceil n/2 \rceil + 1 \) (where \( \lceil n/2 \rceil = z \), when \( n = 2z \) or \( n = 2z + 1 \)),

(ii) the element \( a^{\lambda-x} \) is a generator with period \( n \) iff \( (n, x) = 1 \).

**Proof (i)** From (7) \( \forall \lambda \in \mathbb{N}_0 \), we get
\[
a^{\lambda[A]} = \{a^{x \lambda}, a^{x(\lambda + 1)}, \ldots, a^{x(2\lambda - 1)}\}
\]
and
\[
a^{\lambda[A+1]} = \{a^{x(\lambda + 1)}, a^{x(\lambda + 2)}, \ldots, a^{x(2\lambda - 1)}, a^{x2\lambda}, a^{x(2\lambda + 1)}\}.
\]
Therefore, increasing the power of \( a^\lambda \) from \( \lambda \) to \( \lambda + 1 \), there appear at most two new elements, i.e. \( a^{x2\lambda} \) and \( a^{x(2\lambda + 1)} \). Since \( a^{\lambda[A]} = \{a^\lambda\} \) is a set with only one element, to cover \( H_n \) we need at least \( \lceil n/2 \rceil \) other successive powers of \( a^\lambda \). In either case, if \( n \) is odd or even, for \( \mu = \lceil n/2 \rceil + 1 \) we get
\[
a^{\lambda[A]} \cup a^{\lambda[2]} \cup \ldots \cup a^{\lambda[A]} = \{a^\lambda, a^{x2}, \ldots, a^{x(n-1)}, e\}
\]
and in every higher power of \( a^\lambda \) the same elements are appearing.

If \( (n, x) = 1 \), then the elements of the set (8) are different, so \( a^\lambda \) is a generator with period \( \lceil n/2 \rceil + 1 \).

If \( (n, x) \neq 1 \), then \( (x, x, n) \neq 1 \); so from theorem 1 we get that \( a^\lambda \) is not a generator.

(ii) From (7), \( \forall \lambda \in \mathbb{N}_0 \) and \( \lambda < n \), we get
\[
a^{(n-x)[\lambda]} = \{a^{(n-x)\lambda}, a^{(n-x)\lambda + x}, \ldots, a^{(n-x)(\lambda + 1)x}\} \quad \text{and}
\]
\[
a^{(n-x)[\lambda+1]} = \{a^{(n-x)(\lambda + 1)}, a^{(n-x)(\lambda + 1) + x}, \ldots, a^{(n-x)(\lambda + 1) + 2x}\}
\]
from where we can see easily that
\[
a^{(n-x)[\lambda+1]} = \{a^{(n-x)(\lambda + 1)} \cup a^{(n-x)[\lambda]} \}, \quad \lambda < n.
\]
Let \( (n, x) = 1 \), then
\[
a^{(n-x)(\lambda + 1)} \notin a^{(n-x)[\lambda]},
\]
because, if there exists \( t \in \{0, 1, \ldots, \lambda - 1\} \) such that \( a^{(\lambda - \lambda)(\lambda + 1)} = a^{(\lambda - \lambda)\lambda + \lambda \cdot \lambda} \), then \( \lambda(t + 1) \equiv 0 \pmod{n} \), which is a contradiction. Therefore the sequence of sets

\[
{a^{(\lambda - \lambda)[1]}, a^{(\lambda - \lambda)[2]}, \ldots, a^{(\lambda - \lambda)[\lambda]}}
\]

is strictly increasing and also the set \( a^{(\lambda - \lambda)[\lambda]} \) has exactly \( n \) different elements of \( H_n \), i.e. \( a^{(\lambda - \lambda)[\lambda]} = H_n \).

So the element \( a^{(\lambda - \lambda)} \) is a generator with period \( n \) of \( \langle H_n, a^\lambda \rangle \), when \( (n, \lambda) = 1 \).

Let now \( (n, \lambda) \neq 1 \), then \( (\lambda, n - \lambda, n) \neq 1 \). Hence from theorem 1 we get that \( a^{(\lambda - \lambda)} \) is not a generator. Q.E.D.

The above theorem states that from \( n \) P-cyclic hypergroups \( \langle H_n, a^\lambda \rangle \), \( \varphi(n) \) elements \( a^\lambda \) and \( \varphi(n) \) elements \( a^{(\lambda - \lambda)} \) are generators, where \( \varphi(n) \) is the Euler's phi-function.

Theorem 3. The P-cyclic hypergroup \( \langle H_n, a^\lambda \rangle \), \( a^\lambda \neq e \), is a single-power cyclic hypergroup iff \( (\lambda, n) = 1 \) and in this case every element of \( H_n \) is a generator of \( \langle H_n, a^\lambda \rangle \) with period \( n \).

Proof. In the relation (7) we have at most \( \mu \) different elements, so in order \( \langle H_n, a^\lambda \rangle \) to be a P-cyclic hypergroup we must have \( \mu \geq n \).

For \( \mu = n \), we have

\[
A^{[\lambda]} = \{a^{\lambda n}, a^{\lambda n+\lambda}, \ldots, a^{\lambda n+(n-1)\lambda}\} = \{e, a^\lambda, \ldots, a^{(n-1)\lambda}\},
\]

while, for every \( \sigma \in \mathbb{N} \), we get

\[
a^{[\lambda + \sigma]} = a^{\lambda \sigma} \cdot a^{[\lambda]}.
\]

Therefore \( \langle H_n, a^\lambda \rangle \), \( a^\lambda \neq e \), is a single-power P-cyclic hypergroup with generator \( a^\lambda \) iff exactly the \( n \)-th power of \( a^\lambda \) is equal to \( H_n \).

The \( n \) elements of \( y^{[\lambda]} \) are different iff \( (\lambda, n) = 1 \), independently of \( \lambda \), and the period of \( a^\lambda \) is \( n \).

References