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Remarks on Tolerance Algebras

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This note is intended to be a short introduction to a general theory of tolerance algebras. It is an adaptation of an author's summer school lecture. The results belong to the author.

Эти замечания предназначены как краткое введение в общую теорию толеранционных алгебр. Они возникли из доклада автора на летней школе. Результаты принадлежат автору.

Tyto poznámky jsou myšleny jako krátký úvod do obecné teorie tolerančních algeber. Vznikly z autorovy přednášky na letní škole. Obsahují autorovy původní výsledky.

1. Introduction

Tolerance spaces have apparently been introduced by E. C. ZEEMAN [61]. Soon they were used in the automata theory [2], [3]. Several papers have been published on tolerances on algebraic systems in the recent years (see [1], [4]–[24], [48]–[55], [58], [62]–[67]). Some unpublished results of the author concern the homotopy theory of tolerance spaces. Of great value are the fixed-point theorems for tolerance spaces obtained by A. PULTR [56], [57].

The basic notions are as follows. By a tolerance t on a set A we mean any binary relation on A which is reflexive and symmetric. A *tolerance space* is a set together with a tolerance on it. A *tolerance mapping* (or a *continuous mapping*) from one tolerance space to another tolerance space is any mapping between the underlying sets which preserves the tolerance relation. It is obvious that tolerance spaces and tolerance mappings make together a category, the *category of tolerance spaces*. Tolerance relation is called a "pseudoequivalence" by A. I. MAL'CEV [44].

2. Tolerance groupoids

Having a *tolerance groupoid* G we first have a binary operation $G \times G \to G$ on a non-empty set G, then we have a tolerance t on G and, finally, we suppose

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that the groupoid operation is a tolerance mapping. The tolerance on $G \times G$ which we denote again by t is defined as follows: for any a, b, c, d in G we set $(a, b)\, t(c, d)$ if and only if atc and btd.

Call any subset A of G a *simplex* if and only if $A \times A \subset t$. This means that we have atb for all a, b in A. Clearly, we have maximal simplices in G (Zorn lemma). They form a non-empty set $M \subset \exp G$. This set M is obviously a covering of G which satisfies the following condition.

(1) If $A, B \in M$ then there is always some $C \in M$ with $AB \subset C$.

The tolerance t on G is uniquely determined by the covering M. There is an obvious correspondence between all tolerance relations t on the groupoid G making G a tolerance groupoid in the above sense and all coverings M of G satisfying (1). Moreover, this correspondence can be made easily to a bijection by adding a second condition

(2) If $A \in M$, $x \in G$, $x \notin A$ then there is some $a \in A$ such that there is no $B \in M$ with $x \in B$ and $a \in B$.

Consider any covering M of G with (1). Then we can define for all $A, B \in M$ their *star-product* by $A * B = \{C \in M \mid AB \subset C\}$. In this way M becomes a *multigroupoid*, a set with a multioperation on it: for all $A, B \in M$, $\emptyset \neq A * B \subset M$. Now, we have a *representation theorem*: Every multigroupoid is isomorphic to the multigroupoid M of all maximal simplices of some tolerance groupoid.

To prove this look at any abstract multigroupoid N with a binary multioperation $\emptyset \neq ab \subset N$, $(a, b \in N)$. Make the power set $\exp N$ to a groupoid by setting $AB = \bigcup ab$ where $a \in A$, $b \in B$, $A \in \exp N$, $B \in \exp N$. Let G be any subgroupoid of $\exp N$ with $\emptyset \notin G$ which contains all $\{a\}$, $a \in N$. For any $a \in N$ set $G(a) = \{A \in G \mid a \in A\}$ and let $M = \{G(a) \mid a \in N\}$. Then it is easy to see that M is a covering of G. Really, every $G(a) \in M$ is non-empty as $\{a\} \in G(a)$ and the union $\bigcup M$ equals G. Next we see that M satisfies (1). Really, having some $G(a)$ and $G(b)$ in M we take any $c \in ab$ and then, for all $A \in G(a)$ and for all $B \in G(b)$, we get $c \in ab \subset AB$, $AB \in G(c)$ and, finally, $G(a) \,.\, G(b) \subset G(c)$.

On the other hand, if $G(a) \,.\, G(b) \subset G(c)$ for some $a, b, c \in N$ then $\{a\} \,.\, \{b\} \in G(c)$, $ab \in G(c)$ $c \in ab$. We have proved that for any $a, b, c \in N$, $c \in ab$ is equivalent to $G(a) \,.\, G(b) \subset G(c)$. Now, defining a multioperation (the star-product) on M by setting $G(a) * G(b) = \{G(c) \mid G(a) \,.\, G(b) \subset G(c)\}$ we obtain that $c \in ab$ if and only if $G(c) \in G(a) * G(b)$. The mapping $a \mapsto G(a)$ is an injection as if $G(a) = G(b)$ then $\{a\} \in G(b)$, $b \in \{a\}$ and $b = a$. Thus, M is isomorphic to N. We shall prove yet that M satisfies (2). Let $G(a) \in M$, $B \in G$, $B \notin G(a)$. Then $a \notin B$. On the other hand $\{a\} \in G(a)$. If there were some $G(c)$ in M with $\{a\} \in G(c)$ and $B \in G(c)$ then it would follow that $c \in B$, $c \in \{a\}$, $c = a$ and $a \in B$, a contradiction. M satisfies (2) and it is the multigroupoid of all maximal simplices of G with respect to the corresponding tolerance t on G.

Remarks: Multigroupoids (hypergroupoids) have been investigated many years ago. The earliest papers on this subject seem to be [60], [36], [37], [40]–[43], [59], [34]. A number of papers reflecting new ideas in multigroupoid theory have appeared recently ([25]–[33], [45]–[47]).

All we have said about tolerance groupoids and multigroupoids can be generalized to tolerance algebras and multialgebras.

3. Tolerance Algebras and Their Classes

A *tolerance algebra* of type Δ is, by definition, an algebra of type Δ with the following additional properties. The underlying set has a tolerance on it inducing tolerances on all cartesian powers of the underlying set. With this convention all algebraic operations of the algebra are supposed to be tolerance mappings. A *tolerance homomorphism* is any homomorphism which is a tolerance mapping at the same time. All tolerance algebras of a fixed type and all tolerance homomorphisms form a category. *Tolerance subalgebras* are subalgebras for which the embedding is a tolerance mapping. A tolerance relation t_A of a tolerance algebra A can be viewed as a subalgebra of the algebra $A \times A$ containing the diagonal and being symmetric in an obvious way. A tolerance homomorphism $f : A \to B$ takes always t_A into t_B. If f maps t_A onto t_B then f will be called *superjective*. A superjective homomorphism is always surjective. If A is a tolerance algebra and η a congruence relation on A then A/η is supposed to be equipped with the smallest possible tolerance such that the canonical map $A \to A/\eta$ is a tolerance mapping. With this convention A/η becomes a tolerance algebra and the canonical map is superjective.

If X is a tolerance space we may form the *Peano* (*absolutely free*) algebra $W = W(X)$ over the set X with respect to some fixed type and then we can take the smallest possible tolerance on W containing that of X for which W is a tolerance algebra. With this convention W will be called the *Peano tolerance algebra* of type Δ over X and it has the following characteristic property: Let $j : X \to W$ denote the embedding. For any tolerance algebra A of type Δ and for any tolerance mapping $f : X \to A$ there is exactly one tolerance homomorphism $g : W \to A$ such that $f = g \circ j$.

Products and *coproducts* do exist in the category of tolerance algebras of a given type Δ. We get them easily when taking products and coproducts in the category of algebras of that type and then taking the greatest possible tolerance on the product and the smallest possible tolerance on the coproduct with respect to the condition that the mappings defining products and coproducts respectively should be tolerance mappings.

For any set X take the set $\bar{X} = \{0, 1\} \times X$ with a tolerance t defined as follows: we set $(i, x)\, t\, (j, y)$ if and only if $x = y$ $(i, j \in \{0, 1\},\ x, y \in X)$. Then $P = W(\bar{X})$ is a *projective* tolerance algebra in the following sense: Given any tolerance homomorphisms $f : P \to A$, $h : B \to A$ where h is superjective there is always some tolerance homomorphism $g : P \to B$ such that $f = h \circ g$. It follows that the category of all tolerance algebras of a given type has enough projective objects.

Take $W(\overline{X})$ as before. Any superhomomorphism $s : W(\overline{X}) \to Q$ defines a class V_s consisting of all tolerance algebras A of type Δ such that for every tolerance homomorphism $f : W(\overline{X}) \to A$ there is some tolerance homomorphism $g : Q \to A$ with $f = g \circ s$. The class V_s is then closed under taking tolerance subalgebras, superhomomorphic images and products. Conversely, if X is infinite, any class V closed under the above operations is of the form V_s for some superhomomorphism s. This is simply a version of the *Birkhoff theorem* (see also [35]). Classes with the above properties will be called *tolerance varieties.*

The main tool for constructions in tolerance varieties is the *modification*: Given a tolerance variety V of tolerance algebras of type Δ and given any tolerance algebra A of type Δ we can always find some A' in V and a tolerance homomorphism $m : A \to A'$ with the following property. If $B \in V$ and $f : A \to B$ a tolerance homomorphism then there is exactly one tolerance homomorphism $g : A' \to B$ such that $f = g \circ m$. By this property A' is uniquely determined up to a tolerance isomorphism. A' is called the modification of A in V. It follows that m must be superjective.

Free tolerance algebras $F(\overline{X})$ in V are obtained from Peano algebras $W(\overline{X})$ through modification. They have obvious characteristic properties. Free tolerance algebras in a given tolerance variety V are projective in the sense restricted to V. Coproducts in V can also be easily obtained through modification as well as many other frequent universal algebraic constructions.

For free tolerance algebras we have, for example, the following theorem. Assume V a tolerance variety containing some $C \in V$ with $1 < |C| < \infty$. Let free tolerance algebras $F(\overline{X})$ and $F(\overline{Y})$ in V be tolerance isomorphic. Then $|X| = |Y|$.

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