Le Van Hot

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On the Extremal Points of the Closed Unit Balls In Some Abstract Spaces

LE VAN HOT
Mathematical Institute of the Charles University, Prague*)

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This note is concerned with the extremal points of the closed unit balls in the Banach spaces of abstract measures and in the spaces $L_1(S, \mu, X)$.

V této práci jsou vyšetřeny extremální body uzavřené jednotkové koule v Banachových prostorech abstraktních měr a v prostoru $L_1(S, \mu, X)$.

В этой заметке исследуются экстремальные точки замкнутого единичного шара в Банаховых пространствах абстрактных мер и в пространстве $L_1(S, \mu, X)$.

1. Introduction

This note is concerned with the extremal points of the closed unit balls in the Banach spaces of abstract measures defined on the σ-field of subsets of a set S; having values in a Banach space X, and in the spaces $L_1(S, \mu, X)$, where μ is either a complex measure or a positive measure defined on the σ-field of subsets of set S. Our main results are following: 1) The measure μ belongs to the closed convex hull of the set of the extremal points of a unit closed ball in the space $M(S, \Sigma, X)$, where X is a strictly convex Banach space, if and only if μ is a discrete meesure.

2) The function $f \in L_1(S, \mu, X)$, where X is strictly convex Banach space (μ is either a complex measure or a positive measure) belongs to the closed convex hull of the set of all extremal points of the unit closed ball in $L_1(S, \mu, X)$ if and only if $\|f\| \leqq 1$ and the set $\{s\colon s \in S;\ f(s) \neq 0\}$ is contained in the union of the countable family of the atoms for measure μ.

Throughout this note, S denotes a fixed set; Σ denotes a σ-field of subsets of a set S; and $M(S, \Sigma, X)$, where X is a Banach space, denotes the space of all vector measures defined on Σ with values in X with bounded absolute variation; i.e. $M(S, \Sigma, X)$ is a set of all σ-additive set's functions μ defined on Σ with values in X and

$$\|\mu\| = |\mu|\,(S) = \sup \Sigma_i \|\mu(E_i)\| < +\infty\,,$$

where the suppremum is taken over the set of all finite families $\{E_i\}$ of pairwise disjoint sets from Σ.

*) 186 00 Praha 8, Sokolovská 83, Czechoslovakia.

We recall some definitions from the theory of measures. The set $E \in \Sigma$ is said to be an atom for the measure μ if $\mu(E) \neq 0$ (and $\mu(E) < +\infty$ for positive measure μ), and for each $F \in \Sigma$, $F \subseteq E$ there is either $\mu(F) = 0$ or $\mu(E \setminus F) = 0$.

It follows immediately that the set E is an atom for the measure μ if and only if E is an atom for the positive measure $|\mu|$. The measure μ is said to be atomic if $\|\mu\| = 1$ and if S is an atom for the μ.

The following results are known (see, for instance [1]):

1) If μ is the measures defined on Σ with the values in the finite dimensional space X, then the range of μ is compact. If μ has no atom, then its range is also convex.

2) If μ is a vector measure or a σ-finite positive measure, then S can be partioned into a countable family of atoms for μ and atomless part (the set of atoms, or the atomless part may be empty).

The measure μ is sad to be a discrete, measure, if for each $E \in \Sigma$, $0 < |\mu|(E) < +\infty$ there exists an atom $A \subseteq E$ for μ. Then it follows that if μ is a vector measure or finite positive measure then μ is a discrete measure, if and only if S can be pationed into a countable family of atoms for μ and a μ-null set.

The point x is said to be an extremal point of the convex set A of the linear space X if $x \in A$ and if $x = tx_1 + (1-t)x_2$, where $x_1, x_2 \in A$, $0 < t < 1$, then $x = x_1 = x_2$. Denote by $B_M = \{\mu : \mu \in M(S, \Sigma, X), \|\mu\| \leqq 1\}$, $B_X = \{x \in X : \|x\| \leqq 1\}$ the unit closed ball in $M(S, \Sigma, X)$ and B_X, respectively.

2. Some results

Lemma 1. *Let B_M be a closed unit ball if the space $M(S, \Sigma, X)$. The measure $\mu \in B_M$ is an extremal point of the unit closed ball B_M if and only if μ is atomic and $\mu(S)$ is an extremal point of the unit closed ball B_X in the space X.*

Proof: 1) Let μ be an extremal point of B_M, then it is clear that $\|\mu\| = |\mu|(S) = 1$. Suppose that μ is not atomic. There exists an $A \in \Sigma$: Such that $|\mu|(A) = t > 0$ and $1 - t = |\mu|(S \setminus A) > 0$.

We define

$$\lambda(K) = t^{-1} \mu(K \cap A) \quad \text{for} \quad K \in \Sigma,$$

$$\nu(K) = (1-t)^{-1} \mu(K \cap (S \setminus A)) \quad \text{for} \quad K \in \Sigma.$$

Then we obtain two measures $\lambda, \nu \in M(S, \Sigma, X)$ and

$$\|\lambda\| = |\lambda|(S) = t^{-1}|\mu|(A) = 1;$$

$$\|\nu\| = |\nu|(S) = (1-t)^{-1}|\mu|(S \setminus A) = 1$$

$\lambda \neq \nu$. It is easily to see that $\mu = t\lambda + (1-t)\nu$. But it is a contradiction with the assumption that μ is an extremal point of B_M. That means, μ is atomic.

Suppose now that $x_0 = \mu(S)$ is not an extremal point of B_X. Then there exist two points $x_1, x_2 \in B_X$; $x_1 \neq x_2$ and $t : 0 < t < 1$ such that $x_0 = tx_1 + (1 - t)\, x_2$.

We define

$$\lambda(K) = x_1\,; \quad \nu(K) = x_2 \quad \text{for} \quad K \in \Sigma \quad \text{and} \quad \mu(K) = x_0\,,$$

$$\lambda(K) = 0\,; \quad \nu(K) = 0 \quad \text{for} \quad K \in \Sigma \quad \text{and} \quad \mu(K) = 0\,.$$

Then λ, ν are atomic from $M(S, \Sigma, X)$ and $\|\lambda\| = \|\nu\| = \|x_i\| \leqq 1$. It is clear $\mu = t\lambda + (1 - t)\,\nu$. This again contradicts the assumption that μ is an extremal point of B_M.

That means that μ is atomic and $\mu(S)$ is an extremal point of B_X.

2) Let μ be atomic and $\mu(S)$ be an extremal point of B_X. Then of course $\|\mu\| = \|\mu(S)\| = 1$. We suppose $\mu = t\lambda + (1 - t)\,\nu$, where $0 < t < 1$ and $\lambda, \nu \in B_M$. For each $A \in \Sigma$ either $\mu(A) = 0$ or $\mu(A) = \mu(S)$, as μ is atomic. If $\mu(A) = \mu(S)$, then $\mu(A) = t\,\lambda(A) + (1 - t)\,\nu(A)$ and $\|\lambda(A)\| \leqq \|\lambda\| \leqq 1$; $\|\nu(A)\| \leqq \|\nu\| \leqq 1$, hence $\lambda(A) \in B_X$, $\nu(A) \in B_X$. Then it follows that $\mu(A) = \lambda(A) = \nu(A)$ because $\mu(A)$ is an extremal point of B_X.

If $\mu(A) = 0$, then $\mu(S \setminus A) = \mu(S)$; and we see that $\lambda(S \setminus A) = \nu(S \setminus A) = \mu(S \setminus A)$. Hence $\|\lambda(S \setminus A)\| = \|\nu(S \setminus A)\| = \|\mu(S \setminus A)\| = 1 = \|\lambda\| = \|\nu\|$. This shows that $\lambda(A) = \nu(A) = \mu(A) = 0$ and we obtain $\lambda = \nu = \mu$. This shows that μ is an extremal point of B_M. This completes the proof.

Corollary 1. *Let X be a strictly convex Banach space. Then μ is an extremal point of B_M if and only if μ is atomic.*

Theorem 1. *Let X be a strictly convex Banach space, $\mu \in M(S, \Sigma, X)$. Then μ belongs to the closed convex hull of the set of extremal points of B_M (i.e. $\mu \in \overline{conv}\,(\text{Ext } B_M)$) if and only if μ is a discrete measure and $\|\mu\| \leqq 1$.*

Proof. First of all we prove that, the set of all extremal points of the unit closed ball in $M(S, \Sigma, X)$ is not empty, i.e. Ext $B_M \neq \Phi$. Let s be a fixed point of S and x be a fixed point in X such that $\|x\| = 1$. We define $\mu(A) = 0$ for $A \in \Sigma$ and $s \notin A$; and $\mu(A) = x$ for all $A \in \Sigma$ and $s \in A$. Then μ is atomic and by the Corollary 1, μ is an extremal point of B_M.

1) Let μ be a discrete measure, then there exists a countable family of disjont atoms and a null set N such that $S = \bigcup_n A_n \cup N$. We shall prove that $\mu \in \overline{\text{conv}}$. (Ext B_M). Without loss of generality, one can suppose $\|\mu\| = 1$; as $0 \in$ conv (Ext B_M). Let $\varepsilon > 0$ be an arbitrary positive number. We set $t_i = \|\mu(A_i)\| = |\mu|\,(A_i)$ for all $i = 1, 2, \ldots$. Then

$$\|\mu\| = \sum_i \|\mu(A_i)\| = \sum_i t_i = 1\,.$$

We define $\mu_i(K) = t_i^{-1}\,\mu(K \cap A_i)$ and $\tilde{\mu}_i(K) = -t_i^{-1}\,\mu(K \cap A_i)$ for $i = 1, 2, \dots$. By Lemma 1 we obtain a sequence of atomic measures μ_i, $\tilde{\mu}_i$ and $\mu_i, \tilde{\mu}_i \in \operatorname{Ext} B_M$. Let n_0 be a positive integer such that

$$t = \sum_{i=n_0+1}^{\infty} t_i < \varepsilon .$$

If we put

$$\lambda = \sum_{i=1}^{n_0} t_i\mu_i + \frac{t}{2}\mu_{n_0+1} + \frac{t}{2}\tilde{\mu}_{n_0+1} = \sum_{i=1}^{n_0} t_i\mu_i ,$$

then $\lambda \in \operatorname{conv}(\operatorname{Ext} B_M)$ and for all $K \in \Sigma$ we have:

$$(\mu - \lambda)(K) = (\mu - \lambda)(K \cap \bigcup_{i=n_0+1}^{\infty} A_i) = \mu(K \cap \bigcup_{i=n_0+1}^{\infty} A_i),$$

$$\|\mu - \lambda\| = |\mu - \lambda|(S) = |\mu|(\bigcup_{i=n_0+1}^{\infty} A_i) = \sum_{i=n_0+1}^{\infty} \|\mu(A_i)\| < \varepsilon .$$

This means that $\mu \in \overline{\operatorname{conv}}(\operatorname{Ext} B_M)$.

2) Suppose that μ is not discrete measure. There exists $P \in \Sigma$ such that $r = |\mu|(P) > 0$ and the measure μ_P defined by $\mu_P(K) = \mu(K \cap P)$ has no atom. We shall prove that $\|\mu - \lambda\| > r/2$ for all $\lambda \in \operatorname{conv}(\operatorname{Ext} B_M)$ and this will complete our proof. Let $\lambda \in \overline{\operatorname{conv}}(\operatorname{Ext} B_M)$, then there exist atomic measures μ_i $i = 1, 2, \dots, n$ and $t_i \geqq 0$ $\sum_{i=1}^{n} t_i = 1$ such that $\lambda = \sum_{i=1}^{n} t_i\mu_i$.

From 1) it easily follows that, P can be devided into 2n disjoint sets $P_i \in \Sigma$ such that $|\mu_P|(P_i) = |\mu|(P_i) = r/2n$ and $P = \bigcup_{1}^{2n} P_i$; because μ_P has not atom.

We set $I = \{i;\ 1 \leqq i \leqq 2n;\ \mu_j(P_i) = 0 \text{ for all } j = 1, 2, \dots, n\}$. The set I has at least n elements, since μ_j $(j = 1, 2, \dots, n)$ is atomic. For $K \in \Sigma$ we have:

$$|\mu - \lambda|(K) \geqq |\mu - \lambda|(K \cap \bigcup_{i\in I} P_i) = |\mu|(K \cap \bigcup_{i} P_i).$$

Then $\|\mu - \lambda\| \geqq |\mu|(\bigcup_{i\in I} P_i) = \sum_{i\in I} |\mu|(P_i) = \sum_{i\in I} r/2n \geqq r/2$; which concludes the proof.

In the remainder the measure μ is assumed to be either a complex measure or positive measure.

$L_1(S, \mu, X)$ will denot the space of all μ-integrable functions of S into a Banach space X (X is a complex space if μ is a complex measure; X is real, if μ is real) with norm:

$$\|f\|_1 = \int \|f(s)\|\, d|\mu|(s) . \qquad \text{(See [2])} .$$

If f and g are measurable functions, then $f =_\mu g$ denotes that, $f = g$ μ-almost every where.

Lemma 2. *If $f \in L_1(S, \mu, X)$ and A is an atom for μ then there exists an atom A' for μ and $A' \subseteq A$ and f is a constant on A'.*

Proof. Since $f \in L_1(S, \mu, X)$, there exists a sequence of simple integrable functions, $f_n \in L_1(S, \mu, X)$ and f_n converges μ-a.e. to f, priori $\chi_A f_n$ converges a.e. to $\chi_A f$. Let:

$$\chi_A f_n = \sum_{j=1}^{K_n} x_j^n \chi_{A_j^n}, \quad \text{where} \quad A_j^n \in \Sigma ; \quad A_j^n \subseteq A$$

$x_j^n \in X$; $A_j^n \cap A_i^n = \emptyset$ for $j \neq i$. For each n, there exists a unique set $A_{j_n}^n$, $1 \leqq j_n \leqq k_n$ such that $\mu(A_{j_n}^n) = \mu(A)$; $\mu(A_j^n) = 0$ for all $j \neq j_n$, since A is an atom for μ. Set $B = \bigcap_n A_{j_n}^n$, then $\mu(B) = \mu(A)$ and f_n is a constant on B for each n. It implies that there exists a null set $N \subseteq B$ such that f is constant on $A' = B \setminus N$. Our lemma is proved.

We know (see [2]) that, there exists an isometric map of $L_1(S, \mu, X)$ into $M(S, \Sigma, X) : f \to \mu_f$, where μ_f is defined by

$$\mu_f(E) = \int_E f(s)\, d\mu(s) \quad \text{for all} \quad E \in \Sigma$$

and

$$|\mu_f|(E) = \int_E \|f(s)\|\, d|\mu|(s).$$

Lemma 3. *If $f \in L_1(S, \mu, X)$, then the following three conditions are equivalent:*

1) *f is an extremal point of the unit closed ball B_L in $L_1(S, \mu, X)$;*

2) *μ_f is an extremal point of B_M;*

3) *there exists an atom $A \in \Sigma$ for μ such that $f = 0$ μ-a.e. on $S \setminus A$, $f(s) = (\mu(A))^{-1} x$, where x is an extremal point of B_X.*

Proof. 1) implies 2) and 3). If we prove that μ_f is atomic and $\mu_f(S)$ is an extremal point of B_X, then by Lemma 1; μ is an extremal point of B_M.

Suppose that μ_f is not atomic. Then there exists a set $E \in \Sigma$ such that

$$t = |\mu_f|(E) = \int_E \|f(s)\|\, d|\mu|(s) > 0,$$

$$1 - t = |\mu_f|(S \setminus E) = \int_{S \setminus E} f(s)\, d|\mu|(s) > 0.$$

We define

$$g(s) = t^{-1} \chi_E(s) f(s),$$

$$h(s) = (1 - t)^{-1} \chi_{S \setminus E}(s) f(s),$$

where χ_E is a characteristic function of the set E. It is easy to see $g, h \in L_1(S, \mu, X)$ and $\|g\|_1 = \|h\|_1$ and

$$f = tg + (1 - t) h, \quad g \neq_\mu h .$$

It contradicts the assumption, that f is an extremal point of B_L. This implies μ_f is atomic. We claim that the set $B = \{s \in S; f(s) \neq 0\}$ is an atom for μ. Suppose that, it is not true, then there exists a set $E \in \Sigma$, such that $E \subseteq B$ $|\mu|(E) > 0$ and $|\mu|(B \setminus E) > 0$ and then $|\mu_f|(E) = \int_E \|f(s)\| \, d|\mu|(s) > 0$ and $|\mu_f|(B \setminus E) = \int_{B \setminus E} \|f(s)\| \, d|\mu|(s) > 0$. But it is impossible, for μ_f is atomic. By the Lemma 2, there exists an atom $A \subseteq B$ such that f is constant x_0 on A and $f = 0$ μ-almost every where on $S \setminus A$. To prove 2) and 3) it is sufficient to prove that $x = \mu_f(S) = x_0 \, \mu(A)$ is an extremal point of B_X. Suppose that, this is not true. Then there exist z_1, z_2 from B_X and t $0 < t < 1$ such that $x = tz_1 + (1 - t) z_2$. We define $g(s) = (\mu(A))^{-1} z_1$ for $s \in A$; $g(s) = 0$ for $s \notin A$; $h(s) = (\mu(A))^{-1} z_2$ for $s \in A$; $h(s) = 0$ for $s \notin A$. Then $g, h \in L_1(S, \mu, X)$ and

$$\|\mu_g\| = \|g\|_1 = \|z_1\| \leqq 1,$$

$$\|\mu_h\| = \|h\|_1 = \|z_2\| \leqq 1.$$

$$g \neq_\mu h \quad \text{and} \quad f =_\mu tg + (1 - t) h .$$

It contradicts the assumption, that f is an extremal point of B_L.

3) $\Rightarrow$ 2) It is obvious.

2) $\Rightarrow$ 1). Suppose, f is not an extremal point of B_L, then there exist g and h; $g \neq_\mu h$ and t $0 < t < 1$ such that $f =_\mu tg + (1 - t) h$.

Then $\mu_g \neq \mu_h$ and $\mu_f = t\mu_g + (1 - t) \mu_h$.

It is an contradiction with the assumption, that μ_f is an extremal point of B_M, which finishes the proof.

Theorem 2. *Let X be a strictly convex Banach space and μ be either a complex measure or positive measure, which has at least one atom. Then $f \in L_1(S, \mu, X)$; $\|f\| \leqq 1$ belongs to the closed convex hull of the set of all extremal points of B_L if and only if there exists a countable family of disjoint atoms $\{A_i\}$ for μ such that $f = 0$ a.e. on $S \setminus \bigcup_i A_i$.*

Proof. It easy to see that $\operatorname{Ext} B_L = \phi$; $0 \in \operatorname{conv} \operatorname{Ext} B_L$. Let $f \in L_1(S, \mu, X)$, then $Q = \{s \in S; f(s) \neq 0\}$ is a σ-finite set, and by 2) there exists a countable family of atoms for μ contained in Q such that $Q \setminus \bigcap_n A_n$ has no atom.

1) Let $f \in L_1(S, \mu, X)$ and $|\mu| (Q \setminus \bigcup_n A_n) > 0$, then for each $g \in \text{conv}(\text{Ext } B_L)$ $g = 0$ a.e. on $P = Q \setminus \bigcup_n A_n$ and

$$\|f - g\|_1 \geqq \int_P \|f(s) - g(s)\| \, d|\mu| (s) = \int_P \|f(s)\| \, d|\mu| (s) = r > 0 ,$$

which means that $f \notin \overline{\text{conv}} (\text{Ext } B_L)$.

2) Let $f \in L_1(S, \mu, X)$; $\|f\| \leqq 1$ and $|\mu| (P) = 0$. By Lemma 2, one can suppose that f is constant on A_n for all r. Let

$$f(s) = x_n \quad \text{for all} \quad s \in A_n .$$

$$\int f(s) \, d\mu(s) = \sum_{n=1}^{\infty} x_n \, \mu(A_n) ,$$

$$\|f\|_1 = \int \|f(s)\| \, d|\mu| (s) = \sum_n \|x_n\| \, |\mu| \, (A_n) = \sum_n \|x_n\| \, |\mu(A_n)| \leqq 1 .$$

We define:

$$f_n(s) = \begin{cases} \dfrac{x_n}{\|x_n\| \, |\mu(A_n)|} = \dfrac{(\mu(A_n) \, x_n)}{(\|x_n\| \, |\mu(A_n)|) \, \mu(A_n)} , & \text{for} \quad s \in A_n \\ 0 , \quad \text{for} \quad s \notin A_n . \end{cases}$$

Then f_n and $-f_n$ are extremal points of B_L for all n, since

$$\left| \frac{\mu(A_n) \, x_n}{\|x_n\| \, |\mu(A_n)|} \right| = 1$$

and X is trictly convex space. Let $\varepsilon > 0$ be an arbitrary positive number and let n_0 be a positive integer such that:

$$\sum_{n_0+1} \|x_n\| \, |\mu(A_n)| < \varepsilon .$$

We set $t_i = \|x_i\| \, |\mu(A_i)|$ for $i = 1, 2, \ldots, n_0$,

$$t = 1 - \sum_{i=1}^{n_0} t_i \geqq 0 ,$$

$$g = \sum_{i=1}^{n_0} t_i f_i + t/2 \, f_{n_0+1} + t/2(-f_{n_0+1}) = \sum_{i=1}^{n_0} t_i f_i .$$

Then $g \in \text{conv}(\text{Ext } B_L)$ and

$$\|f - g\|_1 = \sum_{n_0+1}^{\infty} \|x_n\| \, |\mu(A_n)| < \varepsilon ,$$

This means that $f \in \overline{\text{conv}}(\text{Ext } B_L)$. Theorem is proved.

Corollary 2. *Let X be a strictly convex Banach space, $f \in L_1(S, \mu, X)$. Then $f \in \text{conv}(\text{ext } B_L)$ if and only if $\mu_f \in \text{conv}(\text{Ext } B_M)$.*

Proof. 1) Let $f \in \overline{\text{conv}}(\text{Ext } B_L)$ then for $\varepsilon > 0$ there exist $g_1, \ldots, g_n \in \text{Ext } B_L$ such that $\|f - \sum_{i=1}^{n} t_i g_i\|_1 < \varepsilon$ for some $t_1, \ldots, t_n > 0$ $\sum_1^n t_i = 1$. By Lemma 3, it implies $\mu_{g_i} \in \text{Ext } B_M$ and

$$\|\mu_f - \sum_{i=1}^{n} t_i \mu_{g_i}\| = \|f - \sum_1^n t_i g_i\|_1 < \varepsilon ,$$

which implies that $\mu_f \in \overline{\text{conv}}(\text{Ext } B_M)$.

2) Let $f \notin \overline{\text{conv}}(\text{Ext } B_L)$, then there exists an $E \in \Sigma$ such that $f(s) \neq 0$ for all $s \in E$; $|\mu|(E) > 0$ and E has no atom for μ. It is easy to verify that μ_f is not a discrete measure and that is, $\mu_f \notin \overline{\text{conv}}(\text{Ext } B_M)$.

Corollary 3. *Let X be a strictly convex Banach space, then $B_L = \overline{\text{conv}}(\text{Ext } B_L)$ if and only if μ is a discrete measure.*

References

[1] Debreu G.: Integration of correspondences. Proc. the Fifth. Berkeley symp. on Math. statistics an Probability, Vol. II. Univ. of California Press, 1967.

[2] Dunford N. and Schwartz J.: Linear operators. Part I. (Intersc. Publ. New York.

[3] Phelps R. R.: Lectures of Choquet's theorem. D. Van Nostrand Co., Inc. 1966.