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Representation of Sheaves of Metric Lattices by Sections

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The results of [2] have been extended to the case of sheaves of $C(X, Q) - K$-areas (see Def. 2.2) to say that the sheaf of sections of the bundle belonging to a given sheaf of complete $C(X, Q) - K$-areas of becoming sort over a hereditarily paracompact base is isomorphic to the latter.

Výsledky ze [2] jsou zobecněny na případ $C(X, Q) - K$ oblastí (def. 2.2). Ukazuje se, že svazek řezů bandlu daného svazku úplných $C(X, Q) - K$ oblastí vhodného druhu nad dědičně parakompaktní bázi je izomorfní původnímu svazku.

Результаты из [2] распространены на пучки $C(X, Q) - K$ областей (Деф. 2.2) и показывают что пучок резов накрывающего пространства данного пучка полных $C(X, Q) - K$ областей удобного сорта над наследственно паракомпактным базисом изоморфный данному пучку.

Introduction

In [1] K. H. Hofmann proved that the sheaf of sections of the bundle associated with a given sheaf of Banach $C(X)$-modules of suitable sort over a hereditarily paracompact base is isomorphic to the latter. This result has been brought over in [2] by the author to the sheaves of complete $C(X, P) - K$-areas to say that the sheaf of sections of the bundle associated with a given sheaf of complete $C(X, P) - K$-areas of suitable sort over a hereditarily paracompact base is isomorphic to the latter.

Denoting by $C(Y, P)\,(C(Y, Q))$ the set of all continuous functions on a topological space Y with values in $P = \langle -1, 1\rangle$ $(Q = \langle 0, 1\rangle)$. $C(Y, P) - K$-area is the structure $(X, d. +, \vee, \circ)$ where X is a set, d is a metric on X, $+$ is a commutative group operation in X, $\vee$ is an upper semilattice operation in X meaning that $\vee : X \times X \to X$ is a commutative and associative operation in X such that $a \vee a = a$ for all $a \in X$, and $\circ : C(Y, P) \times X \to X$ is a map such that for all $x, y, u, v \in X$

(1) $d(x \vee y, u \vee v) \leqq d(x, u) \vee_R d(y, v)$ (if a, b are real numbers then $a \vee_R b = \max(a, b)$),

(2) $d(x + y, u + v) \leqq d(x, u) + d(y, v)$,

(3) $(x \vee y) + u = (x + u) \vee (y + u)$.

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The map $\circ$ sending $(f, x) \in C(Y, P) \times X$ onto $f \circ x$ fulfils the conditions below for every $x, y \in X$, $f, g \in C(Y, P)$, $c, d \in Q$:

(4) $1 \circ x = x$,

(5) $(c \vee_R d) \circ x = (c \circ x) \vee (d \circ x)$ for any $x \in X^+ = \{x^+ = x \vee 0 \mid x \in X\}$,

(6) $(f + g) \circ x = f \circ x + g \circ x$ whenever $f + g \in C(Y, P)$.

(7) There is a constant K such that for all $x, y \in X, f \in C(Y, Q)$ we have $d(f \circ x, f \circ y) \leqq K\, d(x, y)$.

An important place in the theory is held by multiplying the elements by partitions of unity, but the functions which these partitions consist of have values only in Q and not in the whole of P, and though in [2] we need that the multiplication of elements should be by the functions from $C(Y, P)$, a question has arisen of whether there is a way round the requirement of the multiplication being by the functions from $C(Y, P)$, whether we can do only with $C(Y, Q)$. Also the seventh condition might seem being apt to be weakened and one is led to a question of whether the whole theory in [2] could be carried through under the only condition that $d(cx, cy) \leqq K\, d(x, y)$ for all $x, y \in X$, $c \in Q$.

The paper has originated from trying to find a way round the mentioned two conditions. The way has successfully been found and the results of [2] have been strengthened to hold for the sheaves of $C(Y, Q) - K$-areas.

A $C(Y, Q) - K$-area is a structure $(X, d, +, \vee, \circ)$, where $X, d, +, \vee$ keep the meaning which they have in case of $C(Y, P) - K$-areas, such that the conditions (1)–(3) of the definition of $C(Y, P) - K$-area hold, and $\circ : C(Y, Q) \times X \to X$ is a map sending $(f, x) \in C(Y, Q) \times X$ onto $f \circ x$ such that for every $x, y \in X$, $f, g \in C(X, Q)$, $c, d \in Q$ the conditions (4), (5) of the definition of $C(Y, P) - K$-area hold and

(6′) $(f + g) \circ x = f \circ x + g \circ x$ whenever $f + g \in (X, Q)$;

(7′) There is a constant K such that for all $x, y \in X$, $c \in Q$ we have $d(c \circ x, c \circ y) \leqq K\, d(x, y)$.

Therefore, it has been shown in this paper that the sheaf of sections of the bundle belonging to a given sheaf of complete $C(X, Q) - K$-areas of becoming sort over a hereditarily paracompact base is isomorphic to the latter.

1. Presheaves of metric spaces with contractions

The means listed in this section, and proven in [2, sec. 1] were originally developed by K. H. Hofmann in [1] for the presheaves of Banach spaces and later adopted and extended for presheaves of metric lattices in [2] by the author. In the latter form they will be needed here, therefore they have been taken over from [2, sec. 1] without change to endow us with the necessary tools for further use.

1.1. Notation. A map f of a metric space (X_1, d_1) into another (X_2, d_2) is called contraction if $d_2(f(x), f(y)) \leqq d_1(x, y)$ for all $x, y \in X$.

The category of all metric (complete metric) spaces with contractions as morphisms shall be denoted by $\mathfrak{M}(\mathfrak{MC})$.

A category $\mathfrak{K}$ is called inductive if for every presheaf $\mathscr{S} = \{X_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from $\mathfrak{K}$ there is its inductive limit $\varinjlim \mathscr{S} = \{I \mid \{\xi_\alpha \mid \alpha \in A\}\}$ in $\mathfrak{K}$ (here $\xi_\alpha : X_\alpha \to I$ are the natural $\mathfrak{K}$-morphisms).

1.2. Lemma. Both $\mathfrak{M}$ and $\mathfrak{MC}$ are inductive. Let $\mathscr{S} = \{(X_\alpha, d_\alpha) | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be a presheaf from $\mathfrak{MC}$, let $\langle (I^0, D) \mid \{\xi_\sigma \mid \alpha \in A\} \rangle$ be its inductive limit in $\mathfrak{M}$, and let (I, D) be the completion of (I^0, D). Then $\langle (I, D) \mid \{\xi_\alpha \mid \alpha \in A\} \rangle$ is inductive limit of $\mathscr{S}$ in $\mathfrak{MC}$. Moreover, the following holds:

A. If $\alpha, \beta \in A$, $a \in X_\alpha$, $b \in X_\beta$, then a, b represent the same element in I^0 (meaning $\xi_\alpha(a) = \xi_\alpha(b)$) iff there is $\gamma \geqq \alpha, \beta$ such that for $a' = \varrho_{\alpha\gamma}(a)$, $b' = \varrho_{\beta\gamma}(b)$ we have – setting $A(\gamma) = \{\delta \in A \mid \delta \geqq \gamma\}$:

$$\lim \{d_\delta(\varrho_{\gamma\delta}(a'), \varrho_{\gamma\delta}(b')) \mid \delta \in A(\gamma)\} = 0 .$$

B. If $p, q \in I$ such that there are representatives a, b of p, q in an X_α (if it is the case then $p, q \in I^0$) then

$$D(p, q) = \lim \{d_\beta(\varrho_{\alpha\beta}(a), \varrho_{\alpha\beta}(b)) \mid \beta \in A(\alpha)\} = \inf \{\textit{the same set}\} .$$

It should be noticed that, by 1.2A, $a \in X_\alpha$, $b \in X_\beta$ represent the same element in I not only when $\varrho_{\alpha\gamma}(a) = \varrho_{\beta\gamma}(b)$ for a $\gamma \geqq \alpha, \beta$ as it is in the usual categories.

1.3. Notation. Let $\mathscr{S} = \{(X_U, d_U) | \varrho_{UV} | X\}$ be a presheaf from $\mathfrak{M}$ ($\mathfrak{MC}$) over a topological space X.

A. For $x \in X$ let $\mathscr{B}(x) = \{U \subset X | U \text{ open}, x \in U\}$, let $\leqq$ be the partial order in $\mathscr{B}(x)$ defined as "$U \leqq V$ iff $V \subset U$", and let $\mathscr{S}_x = \{(X_U, d_U) | \varrho_{UV} | \langle \mathscr{B}(x) \leqq \rangle\}$. By 1.2, there is $\varinjlim \mathscr{S}_x = \langle (E_x^0, D_x) \mid \{\xi_{Ux} \mid U \in \mathscr{B}(x)\} \rangle$ in $\mathfrak{M}$ ($\varinjlim \mathscr{S}_x = \langle (E_x, D_x) \mid \{\xi_{Ux} \mid U \in \mathscr{B}(x)\} \rangle$ in $\mathfrak{MC}$). The metric space (E_x^0, D_x) ((E_x, D_x)) is called the stalk of $\mathscr{S}$ over x; it is thus a metric (complete metric) space with a metric D_x. If $\mathscr{S}$ is from $\mathfrak{MC}$ then (E_x, D_x) is just the completion of (E_x^0, D_x). If $r, s \in E_x$ such that there is $U \in \mathscr{B}(x)$ and some representatives $a, b \in X_U$ of r, s (in which case $r, s \in E_x^0$) then

$$D_x(r, s) = \lim \{d_V(\varrho_{UV}(a), \varrho_{UV}(b)) \mid V \in \mathscr{B}(x), V \subset U\} = \inf \{\textit{the same set}\} .$$

B. The set $E^0 = \bigcup\{E_x^0 | x \in X\}$ ($E = \bigcup\{E_x | x \in X\}$) with the projection $p : E^0 \to X$ ($E \to X$) defined as $p(r) = x$ for all $r \in E_x^0$ ($r \in E_x$) is called bundle of $\mathscr{S}$.

C. If $U \subset X$ is open, $a \in X_U$, we denote by $\hat{a}$ the map $\hat{a} : U \to E$ defined as $\hat{a}(x) = \xi_{Ux}(a)$ for $x \in U$, and set $A_U = \{\hat{a} \mid a \in X_U\}$.

D. Let $U \subset X$ be open. Any map $s : U \to E$ such that $ps =$ identity is called section over U. We say that s is bounded if there is $a \in X_U$ such that $\sup \{D_x(\hat{a}(x), s(x)) \mid x \in U\}$ is finite. The set of all bounded sections on U is denoted by $\tilde{\Gamma}(U)$. If $s, t \in \tilde{\Gamma}(U)$ we set $\tilde{d}_U(s, t) = \sup \{D_x(s(x), t(x)) \mid x \in U\}$.

1.4. Lemma. Under the conditions of 1.3 we have

(a): $\hat{a} \in \tilde{\Gamma}(U)$ for each $a \in X_U$, and if $a, b \in X_U$ then $\tilde{d}_U(\hat{a}, \hat{b}) \leqq d_U(a, b)$.

(b): The function $\tilde{d}_U$ defined on $\tilde{\Gamma}(U) \times \tilde{\Gamma}(U)$ is a metric; thus by (a), the map $p_U : (X_U, d_U) \to (\tilde{\Gamma}(U), \tilde{d}_U)$ sending any $a \in X_U$ onto $\hat{a} \in \tilde{\Gamma}(U)$ is a contraction.

1.5. Lemma. Let $\mathscr{S} = \{(X_U, d_U) \, |\varrho_{UV}| \, X\}$ be a presheaf from $\mathfrak{MC}$, E its bundle. If $U \subset X$ is open, $a \in X_U$, $\varepsilon > 0$, let $O(U, a, \varepsilon) = \{r \in E \mid x = p(r) \in U, D_x(\hat{a}(x), r) < < \varepsilon\}$. Then

(a): $\varphi(x) = D_x(\hat{a}(x), \hat{b}(x))$ is upper semicontinuous on U for any $a, b \in X_U$.

(b): $\mathscr{B} = \{O(U, a, \varepsilon) \mid U \subset X$ is open, $a \in X_U$, $\varepsilon > 0\}$ is a bases of a topology t in E which yields in the stalks E_x the same topology t_x as D_x.

1.6. Notation. Let $\mathscr{S} = \{(X_U, d_U) \, |\varrho_{UV}| \, X\}$ be a presheaf from $\mathfrak{MC}$, $U \subset X$ open, let E be the bundle of $\mathscr{S}$. If t is the topology defined in E by the sets $\mathscr{B}$ from the foregoing lemma, we denote by $\Gamma(U)$ the set of all continuous bounded sections on U.

1.7. Lemma. Under the conditions of 1.6 we have

(a): $\hat{a} \in \Gamma(U)$ for each $a \in X_U$; thus the map p_U from 1.4b sends X_U into $\Gamma(U)$ wherefore $A_U \subset \Gamma(U)$.

(b): If $r, s \in \Gamma(U)$ then $\varphi(x) = D_x(r(x), s(x))$ is upper semicontinuous on U.

1.8. Lemma. Let $\mathscr{S} = \{(X_U, d_U) \, |\varrho_{UV}| \, X\}$ be from $\mathfrak{MC}$. TFAE:

1) If $U \subset X$ is open, $a, b \in X_U$, and if $\mathscr{V}$ is an open cover of U then $d_U(a, b) = = \sup \{d_V(\varrho_{UV}(a), \varrho_{UV}(b)) \mid V \in \mathscr{V}\}$;

2) Given an open $U \subset X$, $a, b \in X_U$, an open cover $\mathscr{V}$ of U, and $\varepsilon > 0$, then there is $V \in \mathscr{V}$ such that $d_V(\varrho_{UV}(a), \varrho_{UV}(b)) > d_U(a, b) - \varepsilon$;

3) The natural map $p_U : (X_U, d_U) \to (\Gamma(U), \tilde{d}_U)$ is an isometry into $\Gamma(U)$ for any open $U \subset X$ (see 1.7a).

1.9. Definition. $\mathscr{S}$ is called a monopresheaf if it fulfils any of the conditions 1–3 of the foregoing lemma. Thus we have

1.10. Theorem. Let $\mathscr{S} = \{(X_U, d_U) \, |\varrho_{UV}| \, X\}$ be a monopresheaf from $\mathfrak{MC}$. Then for any open $U \subset X$ the natural map $p_U : (X_U, d_U) \to (\Gamma(U), \tilde{d}_U)$ is an isometry into $\Gamma(U)$.

1.11. Definition. A presheaf $\mathscr{S} = \{(X_U, d_U) | \varrho_{UV} | X\}$ from $\mathfrak{M}$ is called sheaf if it fulfils the following for any open $U \subset X$:

COND 1: If $a, b \in X_U$, and if for an open cover $\mathscr{V}$ of U we have $\varrho_{UV}(a) = \varrho_{UV}(b)$ for all $V \in \mathscr{V}$ then $a = b$.

COND 2: Given an open cover $\mathscr{V}$ of U and a family $\mathscr{F}_{\mathscr{V}} = \{a_V \in X_V \mid V \in \mathscr{V}\}$ such that $\varrho_{VV \cap W}(a_V) = \varrho_{WV \cap W}(a_W)$ whenever $V \cap W \neq \emptyset$ – we call such a family smooth – then

a) There is an $a \in X_U$ with $\varrho_{UV}(a) = a_V$ for all $V \in \mathscr{V}$;

b) If $\mathscr{G}_{\mathscr{V}} = \{b_V \in X_V \mid V \in \mathscr{V}\}$ is another smooth family and $b \in X_U$ such that $\varrho_{UV}(b) = b_V$ for all $V \in \mathscr{V}$, then $d_U(a, b) = \sup \{d_V(a_V, b_V) \mid V \in \mathscr{V}\}$.

1.12. Remark. It readily follows from COND 2b that every sheaf is a monopresheaf. Further, it is easy to see that COND 1 is equivalent to the 1-1 – ness of the natural map $p_U : X_U \to \Gamma(U)$. Also the element $a \in X_U$ determined by $\mathscr{F}_{\mathscr{V}}$ in COND 2 is unique because of COND 1.

2. $C(Y, Q)$ – K-areas

2.1. Definition. An upper semilattice is a pair $(S, \bigvee)$ where S is a set, and $\bigvee : S \times S \to S$ is a map such that for all $a, b, c \in S$ we have $(a \bigvee b) \bigvee c = a \bigvee (b \bigvee c)$, $a \bigvee b = b \bigvee a$, $a \bigvee a = a$.

Given two real numbers a, b, we set $a \bigvee_R b = \max (a, b)$.

2.2. Definition. The set of all continuous functions on a topological space Y with values in the interval $P = \langle -1, 1 \rangle$ ($Q = \langle 0, 1 \rangle$) is denoted by $C(Y, P)$ ($C(Y, Q)$).

A $C(Y, Q)$ – K-area is a structure $(X, d, +, \bigvee, \circ)$ where X is a set, d is a metric on X, $+$ is a commutative group operation on X, $\bigvee$ is an upper semillattice operation on X, and $\circ : C(Y, Q) \times X \to X$ is a map such that

A: $(X, d, \bigvee)$ is a $\bigvee_R$ – faithful upper semilattice, i.e.
(1): $d(x \bigvee y, u \bigvee v) \leqq d(x, u) \bigvee_R d(y, v)$ for any $x, y, u, v \in X$,

B: $(X, d, +)$ is a metric group meaning that for any $x, y, u, v \in X$
(2): 1) $d(x + y, u + v) \leqq d(x, u) + d(y, v)$
b) $d(-x, -y) \leqq d(x, y)$,

C: $(X, +, \bigvee)$ is a group upper semilattice meaning
(3): $(x \bigvee y) + z = (x + z) \bigvee (y + z)$ for any $x, y, z \in X$.

D: The map $\circ$ sending $(f, x) \in C(Y, Q) \times X$ onto $f \circ x$ fulfils the conditions below for every $x, y \in X$, $f, g \in C(Y, Q)$ and any $c, d \in Q$:

(4): $1 \circ x = x$,

(5): $(c \vee_R d) \circ x = (c \circ x) \vee (d \circ x)$ for any $x \in X^+ = \{x^+ = x \vee 0 \mid x \in X\}$,
(6): $(f + g) \circ x = f \circ x + g \circ x$ whenever $f + g \in C(Y, Q)$,
(7): There is a constant K such that $d(c \circ x,\ c \circ y) \leqq K\, d(x, y)$ for all $x, y \in X,\ c \in Q$.

We shall often write fx instead of $f \circ x$, for short.

If $\circ$ is only a map $\circ : Q \times X \to X$ instead of being defined on the whole of $C(Y, Q)$, such that the condition D is now fulfilled only for constant functions from $C(Y, Q)$, then $(X, d, +, \vee, \circ)$ is called a $Q - K$-area.

A map $F : (X_1, d_1, +_1, \vee_1, \circ_1) \to (X_2, d_2, +_4, \vee_2, \circ_2)$ between two $C(Y, Q) - K$-areas ($Q - K$-areas) is called A – homomorphism (A^Q – homomorphism) if for all $x, y \in X_1$ and any $f \in C(Y, Q)$ $(c \in Q)$

(1'): $F(x +_1 y) = F(x) +_2 F(y)$,
(2'): $F(x \vee_1 y) = F(x) \vee_2 F(y)$,
(3'): $F(f \circ_1 x) = f \circ F(x)\ (F(c \circ_1 x) = c \circ_2 F(x))$.

The category of all $C(Y, Q) - K$-areas (metric complete ones) with the contractive A – homomorphisms as morphisms is denoted by $\mathfrak{A}_Y^Q\mathfrak{M}(K)$ $(\mathfrak{A}_Y^Q\mathfrak{M}\mathfrak{C}(K))$. The category of all $Q - K$-areas (metric complete ones) with the contractive A^Q – homomorphisms as morphisms is denoted by $\mathfrak{Q}\mathfrak{M}(K)$ $(\mathfrak{Q}\mathfrak{M}\mathfrak{C}(K))$.

2.3. Lemma. A. Let $(X, +)$ be a commutative group such that there is a map $\circ : L = C(Y, Q) \times X \to X$ sending $(f, x) \in L$ onto $f \circ x$ such that the condition (6) of the foregoing definition is fulfilled for any $x \in X$ and any $f, g \in C(Y, Q)$ with $f + g \in C(Y, Q) : (f + g) \circ x = f \circ x + g \circ x$. If $x \in U$, $h \in C(Y, P)$, $f, g, p, q \in C(Y, Q)$, $h = f - g = p - q$ then $f \circ x - g \circ x = p \circ x - q \circ x$. Therefore, the map $\circ$ can be extended to the whole of $C(Y, P) \times X$ by setting $h \circ x = f \circ x - g \circ x$ for any $x \in X$, $h \in C(Y, P)$ and any decomposition $h = f - g$ with $f, g \in C(Y, Q)$. We then have $(-f)\, x = -fx$ and $0x = 0$ for $x \in X, f \in C(Y, Q)$.

B. Given a group upper semilattice $(G, +, \vee)$ meaning that $+$ is a commutative group operation and $\vee$ is an upper semilattice operation in G such that $(x \vee y) + z = (x + z) \vee (y + z)$ for any $x, y, z \in G$ – then for each $x \in G$ we have $x = x^+ - x^-$ where $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ ($-a$ is the inverse element of $a \in G$).

Proof. A. Let $h = f - g \in C(Y, P)$ with $f, g \in C(Y, Q)$. There is $r \in C(Y, Q)$ such that $f = h^+ + r$, $g = h^- + r$ (we have $f \geqq 0$ so $f \geqq h \vee 0 = h^+$ and set $r = f - h^+$). By (6), if $r \in X$ then $fx = h^+x + rx$, $gx = h^-x + rx$ so $fx - gx = h^+x - h^-x$ which settles the proof of A. For the proof of B see [2, Lemma 2.4].

Given a $C(Y, Q) - K$-area $(X, d, +, \vee, \circ)$ and a presheaf $\mathscr{S} \in \mathfrak{A}_Y^Q\, \mathfrak{M}(K)$. As the condition (7) in $C(Y, Q) - K$-areas is fulfilled only for $c \in Q$ and not for any $f \in C(Y, Q)$ we cannot extend the multiplication by $f \in C(Y, Q)$ to the completion $(\hat{X}, \hat{d})$ of (X, d), nor bring it over to the inductive limit of $\mathscr{S}$ as it was in [2, Prop. 2.7, 2.8]. But we have

2.4. Proposition. If $(X, d, +, \vee, \circ)$ is a $Q - K$-area, then there is a unique extension $\hat{+}, \hat{\vee}$ of $+, \vee$ to the completion $(\hat{X}, \hat{d})$ of (X, d), and a unique extension $\hat{\circ} : Q \times \times \hat{X} \to \hat{X}$ of the multiplications of elements of X by constants from $C(Y, Q)$ such that $(\hat{X}, \hat{d}, \hat{+}, \hat{\vee}, \hat{\circ})$ is a $Q - K$-area.

Proof. The same as [2, Prop. 2.7].

2.5. Proposition. For a fixed K, the category $\mathfrak{Q}\,\mathfrak{M}(K)$ ($\mathfrak{Q}\mathfrak{M}\,\mathfrak{C}(K)$) of all $Q - K$-areas (metric complete ones) is inductive. Namely let $\mathscr{S} = \{(X_\alpha, d_\alpha, +_\alpha, \vee_\alpha, \circ_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ be a presheaf from $\mathfrak{Q}\,\mathfrak{M}(K)$, let $\mathscr{S}_1 = \{(X_\alpha, d_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$, let $\langle (I^0, D) \{\xi_\alpha \mid \alpha \in \in A\} \rangle = \underrightarrow{\lim}\, \mathscr{S}_1$ in $\mathfrak{M}$, let $p, q \in I^0$, $c \in Q$, let $a, b \in X_\alpha$ be some representatives of p, q in X_α, and let $p \vee q$, $p + q$, $c \circ p$ be the element represented by $a \vee_\alpha b$, $a +_\alpha b$, $c \circ_\alpha a$. Then $p \vee q$, $p + q$, $c \circ p$ does not depend on the choice of α and $a, b \in \in X$, and $(I^0, D, +, \vee, \circ)$ is a $Q - K$-area. If $\mathscr{S}$ is from $\mathfrak{Q}\mathfrak{M}\,\mathfrak{C}(K)$, $\langle (I^0, D, +, \vee, \circ) \mid \mid \{\xi_\alpha \mid \alpha \in A\} \rangle = \underrightarrow{\lim}\, \mathscr{S}$ in $\mathfrak{Q}\,\mathfrak{M}(K)$, then $\underrightarrow{\lim}\, \mathscr{S}$ in $\mathfrak{Q}\mathfrak{M}\,\mathfrak{C}(K)$ is just the completion of the $Q - K$-area $(I^0, D, +, \vee, \circ)$ by Prop. 2.4.

Proof. The as that of [2, Prop. 2. 8.].

2.6. Corollary. Let $\mathscr{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) \,|\varrho_{UV}|\, X\}$ be a presheaf from $\mathfrak{A}\mathfrak{L}_Y\mathfrak{M}\,\mathfrak{C}(K)$ over a topological space X, let E be its bundle .Then

(a): For every $x \in X$ the stalk E_x over x is a $Q - K$-area with the operations $+_x, \vee_x, \circ_x$ defined as the natural bringover of these from $\mathscr{S}_x$ (see 2.4, 2.5). Further, we have for $c \in Q$, $a \in X_U$, $x \in U : (ca)^\wedge\, (x) = c\, \hat{a}(x)$.

(a'): If $Y = X$ then E_x can be made into a $C(X, Q) - K$-area by setting foı $f \in C(X, Q)$, $r \in E_x : fr = f(x)\, r$. (Sure enough, we need not have now $(fa)^\wedge\, (x) = = f\, \hat{a}(x)$ for any $f \in C(X, Q)$, $a \in X_U$ as it is when $f \in Q$ because fa need not repıesent the germ $(fa)^\wedge\, (x) = f\, \hat{a}(x)$ in E_x.)

(b): If $U \subset X$ is open, then the set $\tilde{\Gamma}(U)$ of all bounded sections over U in E with its natural metric $\tilde{d}_U$ (see 1.3D), and with the operations $\tilde{+}_U, \tilde{\vee}_U, \tilde{\circ}_U$ pointwise defined by $(r\, \tilde{\vee}_U\, s)(x) = r(x) \vee_x s(x)$ for $x \in U$ – and likewise for $\tilde{+}_U, \tilde{\circ}_U$ – is a $Q - K$-area. If $a \in X_U$, $c \in Q$, $x \in U$ then by 2.5 we have $(ca)^\wedge\, (x) = c\, \hat{a}(x)$.

(b'): If $Y = X$ then $\tilde{\Gamma}(U)$ can be made into a $C(X, Q) - K$-area by setting (for $f \in C(X, Q)$, $\sigma \in \tilde{\Gamma}(U)) f\, \tilde{\circ}_U\, \sigma \in \tilde{\Gamma}(U)$ to be $(f\, \tilde{\circ}_U\, \sigma)(x) = f(x) \circ_x \sigma(x)$ (we need not have now $(fa)^\wedge = f\hat{a}$ for any $f \in C(X, Q)$, $a \in X_U$).

Proof. (a) readily follows from 2.4, 2.5, (a'), (b), (b') are an easy matter of checking.

2.7. Proposition. Under the conditions of 2.6

(a): The operations $\vee$, $+$ can be stalkwise defined in E.

More precisely, if $p : E \to X$ is the natural projection (see 1.3B), we denote by $E \times_X E = \{(r, s) \in E \times E \mid p(r) = p(s)\}$ the pullback of $E \times E$ over X. If $(r, s) \in E \times_X E$, $x = p(r) = p(s)$, we set $r \bigvee s = r \bigvee_x s$, $r + s = r +_x s$ to get two maps $\bigvee, + : E \times_X E \to E$. Let t be the natural topology in E by 1.5b. Then, under this topology, $\bigvee, +$ are continuous.

(b): The set $\Gamma(U)$ of all continuous bounded sections over U is closed under the operations $\tilde{\bigvee}, \tilde{+}$ meaning that $r \tilde{\bigvee} s$, $r \tilde{+} s \in \Gamma(U)$ if $r, s \in \Gamma(U)$.

(c): The natural map $p_U : (X_U, d_U, +_U, \bigvee_U, \circ_U) \to (\Gamma(U), \tilde{d}_U, \tilde{+}_U, \tilde{\bigvee}_U, \tilde{\circ}_U)$ (see 1.4b) is an A^Q – homomorphism (see 2.2) meaning that for any open $U \subset X$ and any $a, b \in X_U$, $c \in Q$ we have $p_U(a \bigvee_U b) = p_U(a) \tilde{\bigvee}_U p_U(b) = \hat{a} \tilde{\bigvee}_U \hat{b}$, $p_U(a +_U b) = = p_U(a) \tilde{+}_U p_U(b) = \hat{a} \tilde{+}_U \hat{b}$, $p_U(ca) = c\, p_U(a) = c\hat{a}$.

Proof: It is an easy matter of checking (see also [2, Prop. 2.10]).

2.8. Definition. Let $\mathscr{S} = \{(X_U, d_U) |\varrho_{UV}| X\}$ be a presheaf from $\mathfrak{MC}$, $U \subset X$ open.

A. A subset $M \subset \Gamma(U)$ is called locally finite if for every $x \in U$ there is an open nbd $V \subset U$ of x and a finite set $F \subset M$ such that for each $r \in M$ there is $s \in F$ with $r(y) = s(y)$ for any $y \in V$.

B. Let $\mathscr{S}$ be from $\mathfrak{QM}\,\mathfrak{C}(K)$. A set $M \subset \Gamma(U)$ is called $\tilde{\bigvee}_U$ – closed if for every locally finite $N \subset M$ such that $r = \tilde{\bigvee}_U N = \tilde{\bigvee}_U \{s \mid s \in N\} \in \tilde{\Gamma}(U)$ (i.e. r is bounded; r is defined as $r(x) = \bigvee_x\{s(x) \mid s \in N\}$ for $x \in U$) we have $r \in M$.

Following K. H. Hofmann we get in our case

2.9. Lemma. Let $\mathscr{S} = \{(X_U, d_U, +_U, \bigvee_U, \circ_U) |\varrho_{UV}| X\}$ be a sheaf (see 1.11) from $\mathfrak{QM}\,\mathfrak{C}(K)$, (E, t) its bundle (see 1.3A, 1.5b), let $p_U : (X_U, d_U) \to (\Gamma(U), \tilde{d}_U)$ (see 1.4b, 1.7a) be the natural map sending X_U onto $\{\hat{a} \mid a \in X_U\} = A_U \subset \Gamma(U)$. Then for any locally finite $N \subset A_U$ we have $\tilde{\bigvee}_U N = \tilde{\bigvee}_U\{n \mid n \in N\} \in A_U$ wherefore A_U is $\tilde{\bigvee}_U$ – closed.

Proof. It is in [2, Lemma 2.12].

2.10. Lemma. Let $\mathscr{S} = \{(X_U, d_U, +_U, \bigvee_U, \circ_U) |\varrho_{UV}| X\}$ be a presheaf from $\mathfrak{QM}\,\mathfrak{C}(K)$, X regular, let $U \subset X$ be open and paracompact, let $M \subset \Gamma(U)$ such that

(1) M is $\tilde{\bigvee}_U$ – closed,

(2) M is a subgroup of $\Gamma(U)$ with respect to $\tilde{+}_U$, and $fm \in M$ for any $f \in \in C(X, Q)$, $m \in M$,

(3) $M(x) = \{m(x) \mid m \in M\}$ is dense in $\Gamma(U)(x) = \{\sigma(x) \mid \sigma \in \Gamma(U)\}$ for all $x \in U$. Then M is dense in $(\Gamma(U), \tilde{d}_U)$.

Proof. It goes precisely the same way as that of [2, Lemma 2.13] with the only difference that now the stalks $(E_z, D_z, +_z, \bigvee_z, \circ_z)$ are only Q – K-areas while they were $C(X, P)$ – K-areas in [2, 2.13]. Nonetheless, the proof holds also in this case because the fourth condition of [2, 2.13], which required that the multiplication of the sections $\sigma \in \Gamma(U)$ by the functions from $C(X, Q)$ be pointwise meaning that $(f \tilde{\circ}_U \sigma)(x) = f(x) \circ_x \sigma(x)$, is fulfilled here owing to the way of our definition of multiplication of sections from $\Gamma(U)$ by the functions from $C(X, Q)$ – see 2.6b′, and also the inequality $D_z(fa, fb) \leqq K\, D_z(a, b)$ is not needed here for any $f \in C(X, Q)$, $a, b \in E_z$, it is needed only that $D_z(f(z)\, a, f(z)\, b) \leqq K\, D_z(a, b)$ for any $f \in C(X, Q)$, $a, b \in E_z$, which is fulfilled as $f(z) \in Q$ and $D_z(ca, cb) \leqq K\, D_z(a, b)$ for any $c \in Q$ because the stalk $(E_z, D_z, +_z, \bigvee_z, \circ_z)$ is a $Q - K$-area where the inequality holds by 2.2(7). Finaly, the inequality $D(-a, -b) \leqq D(a, b)$, which is needed in the proof, is ensured by 2.2(2b).

2.11. Definition. Given a presheaf $\mathcal{S} = \{(X_U, d_U, +_U, \bigvee_U, \circ_U)\, |\varrho_{UV}|\, X\}$ from $\mathfrak{A}^{\mathfrak{Q}}_X\mathfrak{M}\,\mathfrak{C}(K)$, $M \subset X$, we set $I_M = \{f \in C(X, Q) \mid f = 0 \text{ on } M\}$. $\mathcal{S}$ is called "well supported" if for any open $U \subset X$, $f \in I_U$, $a \in X_U$ we have $f \circ_U a = 0$ (see [1, 2.14, p. 12]).

2.12. Lemma. Let $\mathcal{S} = \{(X_U, d_U, +_U, \bigvee_U, \circ_U)\, |\varrho_{UV}|\, X\}$ be a well supported sheaf from $\mathfrak{A}^{\mathfrak{Q}}_X\mathfrak{M}\,\mathfrak{C}(K)$, X normal, $U, V \subset X$ open, $\bar{V} \subset U$, $a \in X_U$. Then there is $b \in X_X$ with $\varrho_{XV}(b) = \varrho_{UV}(a)$.

Proof. The same as that of [2, 2.15].

2.13. Lemma. Let $\mathcal{S} = \{(X_U, d_U, +_U, \bigvee_U, \circ_U)\, |\varrho_{UV}|\, X\}$ be a well supported sheaf from $\mathfrak{A}^{\mathfrak{Q}}_X\mathfrak{M}\,\mathfrak{C}(K)$ over a normal X, $x \in X$, $r \in E^0_x$ (see 1.3A, B). Then there is $b \in X_X$ such that $\hat{b}(x) = r$.

Proof. The same as that of [2, 2.16].

2.14. Lemma. Let $\mathcal{S} = \{(X_U, d_U, +_U, \bigvee_U, \circ_U)\, |\varrho_{UV}|\, X\}$ be a sheaf from $\mathfrak{A}^{\mathfrak{Q}}_X\mathfrak{M}\,\mathfrak{C}(K)$ such that

a) $\mathcal{S}$ is well supported.

b) For every $a \in X_X$ the map $M_a : C(X, Q) \to X_X$ sending $f \in C(X, Q)$ onto fa is continuous at zero with respect to the sup-norm meaning: For every $a \in X_X$, $\varepsilon > 0$ there is $\delta > 0$ such that $0 \leqq f \leqq \delta$ yields $d_X(fa, 0) < \varepsilon$.

Let $U \subset X$ be open, $a \in X_U$, $x \in U$, $\varphi \in I_x$. Then $(\varphi a)^\wedge (x) = 0$.

Proof. There is an open V with $x \in V \subset \bar{V} \subset U$ and an $a, b \in X_X$ such that $\varrho_{XV}(b) = \varrho_{UV}(a)$ – see 2.12. Further, given $\varepsilon > 0$, there is $\delta > 0$, $\delta \leqq 1$ such that $d_X(gb, 0) < \varepsilon$ whenever $g \in C(X, Q)$, $0 \leqq g \leqq \delta$. There is an open W with $x \in W \subset V$ such that $0 \leqq \varphi < \delta$ on W. Set $h = \min(\varphi, \delta)$. Then $h \in C(X, Q)$, $0 \leqq h \leqq \delta$ hence $d_X(hb, 0) < \varepsilon$. Further, $\varrho_{XW}(hb) = h\varrho_{XW}(b) = h\varrho_{UW}(a) = \varphi\, \varrho_{UW}(a)$

as $h = \varphi$ on W and $\mathscr{S}$ is well supported. Thus $d_W(\varrho_{UW}(\varphi a), 0) = d_W(\varphi \varrho_{UW}(a), 0) = = d_W(\varrho_{XW}(hb), 0) \leqq d_X(hb, 0) < \varepsilon$ hence $\lim_V \{d_V(\varrho_{UV}(\varphi a), 0) \mid x \in V \subset U \text{ open}\} = = 0$, which by 1.3A shows that φa and 0 represent the same germ in E_x^0. We are done.

For sake of the next lemma let us recall that by 2.3A, if $f \in C(X, P)$, $g, h, k, l \in \in C(X, Q)$, $f = g - h = k - l$, $a \in X_U$ then $ga - ha = ka - la$.

2.15. Lemma. If $\mathscr{S} = \{(X_U, d_U, +_U, \bigvee_U, \circ_U) \,|\varrho_{UV}|\, X\}$ is a sheaf from $\mathfrak{A}_X^Q\mathfrak{M}\,\mathfrak{C}(K)$, X normal, then (1) $\Rightarrow$ (2) below:

(1) a) $\mathscr{S}$ is well supported.

b) For every $a \in X$ the map $M_a : C(X, Q) \to X_X$ sending $f \in C(X, Q)$ onto fa is continuous at zero with respect to the sup-norm (see 2.14).

(2) For every $a \in X_U$, $f \in C(X, Q)$, $x \in U$ we have $(fa)^\wedge (x) = f(x)\, \hat{a}(x)$.

Proof. Let $a \in X_U$, $f \in C(X, Q)$, $x \in U$. Then $h = f - f(x) \in C(X, P)$, $f, f(x)$, $h^+, h^- \in C(X, Q)$ hence by 2.3A, $h^+a - h^-a = fa - f(x)\, a$. Further, $h^+, h^- \in I_x$ hence $(h^+a)^\wedge (x) = (h^-a)^\wedge (x) = 0$ by the foregoing lemma, and thus $(fa)^\wedge (x) - - (f(x)\, a)^\wedge (x) = (fa - f(x)\, a)^\wedge (x) = (h^+a - h^-a)^\wedge (x) = (h^+a)^\wedge (x) - (h^-a)^\wedge \,. \,. (x) = 0$, which we have wanted.

2.16. Remark. Let $\mathscr{S} = \{(X_U, d_U, +_U, \bigvee_U, \circ_U) \,|\varrho_{UV}|\, X\}$ be a presheaf from $\mathfrak{A}_X^Q\mathfrak{M}\,\mathfrak{C}(K)$, E its bundle.

A. By 2.4, 2.5, 2.6(a′), the stalks $(E_x, D_x, +_x, \bigvee_x, \circ_x)$ are $Q - K$-areas with the operations $+_x, \bigvee_x, \circ_x$ defined as the natural bringover of those from the terms of $\mathscr{S}$. In 2.6(a′) we made the stalks into $C(X, Q) - K$-areas by setting $fp = f(x)\, p$ for $p \in E_x$. We could not bring these operations over from the terms of $\mathscr{S}$ as we lacked the inequality $d_U(fa, fb) \leqq K\, d_U(a, b)$ for $f \in C(X, Q)$, which caused that, given $x \in X$, $r \in E_x^0$, $U \subset X$ open with $x \in U$, and $a \in X_U$ with $\hat{a}(x) = r$, the germ $(fa)^\wedge (x)$ of fa in E_x which should represent fr might depend on the choice of U and of the representative $a \in X_U$ meaning that there might be an open $V \subset U$ with $x \in V$ and a $b \in X_V$ with $\hat{b}(x) = r$ such that $(fb)^\wedge (x) \neq (fa)^\wedge (x)$. But the foregoing lemma shows that if $\mathscr{S}$ is a sheaf which fulfils (1) of 2.15, then the multiplication by the functions from $C(X, Q)$ can be brought over to the stalks from the terms of the sheaf and that it agrees with the mentioned definition because fa represents $f(x)\, \hat{a}(x) = f(x)\, r = fr$ in E_x for any representative $a \in X_U$ of r. This also shows that the natural A^Q – morphisms $\xi_{Ux} : (X_U, d_U, +_U, \bigvee_U, \circ_U) \to (E_x, D_x, +_x, \bigvee_x, \circ_x)$ are A – homomorphisms (see 2.2) as $\xi_{Ux}(fa) = (fa)^\wedge (x) = f(x)\, \hat{a}(x) = f\, \hat{a}(x) = = f\, \xi_{Ux}(a)$.

B. It can be readily seen from A, that under the same conditions the $Q - K$-area $(\tilde{\Gamma}(U), \tilde{d}_U, \tilde{+}_U, \tilde{\bigvee}_U, \tilde{\circ}_U)$ defined in 2.6b and made into $C(X, Q) - K$-areas by 2.6b′ can be now made into $C(X, Q) - K$-areas naturally by setting (for $\sigma \in \tilde{\Gamma}(U)$, $f \in C(X, Q)$) $f\sigma$ to be the section defined as $(f\sigma)(x) = f\, \sigma(x)$ for $x \in U$ because the latter term is just $f(x)\,\sigma(x)$ which agrees with the definition of $f\sigma$ in 2.6b′. Clearly the natural map $p_U : (X_U, d_U, +_U, \vee_U, \circ_U) \to (\Gamma(U), \tilde{d}_U, \tilde{+}_U, \tilde{\vee}_U, \tilde{\circ}_U)$ is now an A – homomorphism as well because now we have $p_U(fa) = (fa)^\wedge = f\hat{a} = f\, p_U(a)$ for $f \in C(X, Q)$. From this we get that if $\sigma \in A_U = p_U(X_U)$, $f \in C(X, Q)$ then $f\sigma \in A_U$. Indeed, we have $\sigma = \hat{a}$ for an $a \in X_U$, and $f\hat{a} = (fa)^\wedge$, and $fa \in X_U$ so $(fa)^\wedge \in A_U$.

2.17. Theorem. Let $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U)\,|\varrho_{UV}|\,X\}$ be a well supported sheaf from $\mathfrak{A}\mathfrak{l}^{Q}_{X}\mathfrak{M}\,\mathfrak{C}(K)$, X locally paracompact, let for each $a \in X_\alpha$ the multiplication $M_a : C(X, Q) \to (X_X, d_X)$ sending $f \in C(X, Q)$ onto fa be continuous at zero (see 2.14b). Let t be the topology in the bundle E of $\mathcal{S}$ defined in 1.5b, let $\Gamma(U)$ for open $U \subset X$ be the set of all continuous bounded sections on U (see 1.6). Then for every open $U \subset X$ the natural map $p_U : (X_U, d_U) \to (\Gamma(U), \tilde{d}_U)$ (see 1.4b) is an isometric isomorphism onto $\Gamma(U)$.

Proof. $\mathcal{S}$ is a sheaf hence it is a monopresheaf by 1.12. By 1.10, $p_U : (X_U, d_U) \to \to (\Gamma(U), \tilde{d}_U)$ is an isometry into $\Gamma(U)$. Let $U \subset X$ be open and paracompact. By 2.9, the p_U – image A_U of X_U is $\tilde{\vee}$ – closed hence A_U fulfils the condition (1) of 2.10. Clearly A_U is a $\tilde{+}_U$ – subgroup of $\Gamma(U)$. If $m \in A_U$, $f \in C(X, Q)$ then by 2.16B $fm \in A_U$ hence A_U fulfils also the condition (2) of 2.10. By 2.13, $\{\sigma(x) \mid \sigma \in A_U\} = E_x^0$ for any $x \in U$, and as E_x^0 is dense in E_x, the condition (3) of 2.10 is fulfilled by A_U. By 2.10, A_U is dense in $\Gamma(U)$. Since p_U is an isometry and (X_U, d_U) is complete, we have $A_U = \Gamma(U)$, which finishes the proof for paracompact U. Now, the way of extending the proof to any open U has been shown in [2, added in proof].

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