Konrad Behnen
Adaptive rank tests

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 24 (1983), No. 1, 13--22

Persistent URL: http://dml.cz/dmlcz/142501

Adaptive Rank tests

K. BEHNEN
Department of Statistics, University of Hamburg*)

Received 22 December 1982

The test procedure based on ranks for the twosample testing problem:

$$H_0 : F = G \text{ versus } H_1 : F \leqq G , \quad F \neq G$$

where F, G are unknown distribution functions, is proposed. Namely, it is suggested to apply the usual rank statistic for this problem with the score-generating function (its choice generally depends on F and G) replaced by its estimator based on ranks. The asymptotic properties of the estimator are studied. The results of simulation study are presented.

Adaptivní postupy. V předložené práci je navržen test založený na pořadích pro dvouvýběrový problém:

$$H_0 : F = G \text{ versus } H_1 : F \leqq G , \quad F \neq G$$

kde F a G jsou neznámé distribuční funkce. Navržený test spočívá v použití pořadové statistiky užívané pro tento problém, kde skórová funkce (její volba závisí na F a G) je nahrazena odhadem založeným na pořadí. Autor studuje asymptotické vlastnosti tohoto odhadu a uvádí výsledky simulační studie.

Адаптивные методы. В статье предлагается критерий основанный на рангах для двухвыборочной проблемы, где функция скоров (зависящая ом F и G) заменена оценкой основанной на рангах. Авторы изучают асимптотические свойства этой оценки и обсуждают некоторые результаты на симуляциях.

1. Introduction

Everybody knows that there have to be assumptions on the underlying distributions of data, if we want to test hypotheses. For example, the classical t-test for comparing two treatments on the basis of m and n independent repetitions, respectively, is based on the special assumption of underlying normal distribution, whereas two-sample rank tests are based on the much more realistic assumption of m and n independent repetitions from two arbitrary continuous distributions F and G, respectively.

Especially the work of J. Hájek showed that in case of known type of alternative there is a linear rank test which is approximately optimal for this situation. For

*) D-2000, Hamburg 13, Bundesstrasse 55, West Germany.

example, this means that instead of using the classical t-test we should use the van der Waerden rank test, which is approximately as good as the t-test if the underlying distributions are normal but which is valid also in cases of arbitrary deviations from normality.

In order to have not only validity but also high power of a test, i.e., in order to use the approximately optimal rank test, we should know the type of underlying alternative, which is an unrealistic assumption for applications. Therefore, we may try to estimate the type of alternative from the data, i.e., we try to estimate the score function of the optimal linear rank test for the (unknown) underlying situation.

In case of the two-sample shift model $F(x) = G(x - \vartheta)$ this has been done by Hájek and Šidák (1967) and others by estimating the optimal shift score function

$$-f' \circ F^{-1}/f \circ F^{-1} \tag{1.1}$$

on the basis of the order statistic of the pooled sample, leading to an approximately optimal adaptive test in this situation.

In the more general two sample testing problem $F = G$ versus $F \leqq G$, $F \neq G$ it is shown in Behnen and Neuhaus (1981) that a good test should be based on the linear rank statistic with score function

$$b = \bar{f} - \bar{g}\,, \tag{1.2}$$

where $\bar{f} = d(F \circ H^{-1})/dx$, $\bar{g} = d(G \circ H^{-1})/dx$, $H = (mF + nG)/(m + n)$. The more general nonparametric score function (1.2) is quite different from the shift score function (1.1) if there is some deviation from shift model. This is the reason for the breakdown of adaptive tests based on an estimator of (1.1), if the shift model is not exactly true, cf. Behnen (1975).

Since b is invariant under strictly isotone transformations of the data estimators of b should be based on the ranks only, leading to adaptive tests which are (non-linear) rank tests.

2. *A kernel type rank estimator of b.* In order to be definite we have to fix the assumptions and notations:

Let $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be independent real valued random variables and suppose that the distribution of $X_i[Y_j]$ is given by a continuous (cumulative) distribution function $F[G]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Let $N = m + n$ be the size of the pooled sample and consider the testing problem

$$\mathrm{H}_0 : F = G \text{ versus } \mathrm{H}_1 : F \leqq G\,, \quad F \neq G\,. \tag{2.1}$$

As discussed in the introduction [cf. Behnen and Neuhaus (1981)] we are interested in rank estimators of the Lebesgue-densities on the unit interval [0, 1] (μ-densities) defined by

$$f_N = d(F \circ H_N^{-1})/d\mu\,, \quad g_N = d(G \circ H_N^{-1})/d\mu\,, \tag{2.2}$$

where

$$(2.3) \qquad H_N = (mF + nG)/N \quad \text{and} \quad (mf_N + ng_N)/N = 1\,.$$

Especially, we are interested in rank estimators of the *nonparametric score function*

$$(2.4) \qquad b_N = f_N - g_N\,,$$

which has the properties

$$(2.5) \qquad -\frac{N}{n} \leqq b_N \leqq \frac{N}{m}\,, \quad f_N = 1 + \frac{n}{N}\,b_N\,, \quad g_N = 1 - \frac{m}{N}\,b_N\,.$$

Since the i.i.d. random variables $H_N(X_1), \ldots, H_N(X_m)$ have μ-density f_N and the i.i.d. random variables $H_N(Y_1), \ldots, H_N(Y_n)$ have μ-density g_N, we may (formally) build estimators of f_N and g_N on the basis of $H_N(X_1), \ldots, H_N(X_m)$ and $H_N(Y_1), \ldots$ $\ldots, H_N(Y_N)$, respectively. The only problem is that H_N is unknown and that we want rank estimators. But fortunately on one hand the Kolmogorov-Smirnov theorem tells us (under hypothesis and under alternative)

$$(2.6) \qquad \|\hat{H}_N - H_N\| = O_P(N^{-1/2})\,, \quad \text{if} \quad N \to \infty\,,$$

where $\|\cdot\|$ denotes the supremum norm, where

$$(2.7) \qquad \hat{H}_N = (m\hat{F}_m + n\hat{G}_n)/N$$

is the empirical distribution function of the pooled sample, and where $\hat{F}_m$ and $\hat{G}_n$ are the empirical distribution functions of the X-sample and the Y-sample, respectively. On the other hand we have

$$(2.8) \qquad N\hat{H}_N(X_i) = R_{1i} = \text{rank of } X_i \text{ in the pooled sample}\,,$$

$$N\hat{H}_N(Y_j) = R_{2j} = \text{rank of } Y_j \text{ in the pooled sample.}$$

Therefore we may estimate f_N and g_N on the basis of the rank data

$$R_{11}/N; \ldots, R_{1m}/N \quad \text{and} \quad R_{21}/N, \ldots, R_{2n}/N\,,$$

respectively.

Since f_N and g_N are μ-densities on the compact interval $[0, 1]$ and since we want to construct consistent estimators of $b_N = f_N - g_N$ and its derivative, the usual kernel estimators won't work without modifications near zero and one. The modification is done by applying an usual kernel estimator to the modified rank data

$$-R_{11}/N, \ldots, -R_{1m}/N,\ R_{11}/N, \ldots, R_{1m}/N,\ 2 - R_{11}/N, \ldots, 2 - R_{1m}/N\,,$$

and

$$-R_{21}/N, \ldots, -R_{2n}/N,\ R_{21}/N, \ldots, R_{2n}/N,\ 2 - R_{21}/N, \ldots, 2 - R_{2n}/N\,,$$

respectively. This artificial enlargement of the original rank data by their reflections at the points zero and one will guarantee (uniformly in N) the boundedness (in probability) of the estimator and its first derivative and also the (uniform) consistency.

Formally the estimator is defined according to

$$\hat{b}_N = \hat{f}_N - \hat{g}_N , \tag{2.9}$$

where

$$\hat{f}_N(t) = \frac{1}{m} \sum_{i=1}^{m} K_N(t, R_{1i}/N) = \int K_N(t, \hat{H}_N)\, d\hat{F}_m , \tag{2.10}$$

$$\hat{g}_N(t) = \frac{1}{n} \sum_{j=1}^{n} K_N(t, R_{2j}/N) = \int K_N(t, \hat{H}_N)\, d\hat{G}_n ,$$

and

$$K_N(t, s) = \frac{1}{a_N} \left\{ K\left(\frac{t+s}{a_N}\right) + K\left(\frac{t-s}{a_N}\right) + K\left(\frac{t-2+s}{a_N}\right) \right\} . \tag{2.11}$$

Here $K : \mathbf{R} \to \mathbf{R}$ is a kernel with the following properties,

(2.12) K is a Lebesgue density on $\mathbf{R}$ with absolutely continuous derivative K' and essentially bounded second derivative K'', such that $K(x) = 0$, if $|x| \geqq 1$,

and a_N is a sequence in $\mathbf{R}$ such that

$$0 < a_N \leqq 1/2 , \quad a_N \xrightarrow[N\to\infty]{} 0 , \quad N a_N^6 \xrightarrow[N\to\infty]{} \infty . \tag{2.13}$$

Theorem 2.1. Assume $N \to \infty$ such that $m/N \to \lambda \in (0, 1)$. Then, for each fixed (F,G) such that b according to

$$b = d((F - G) \circ H^{-1})/d\mu , \quad H = \lambda F + (1 - \lambda)\, G , \tag{2.14}$$

has bounded continuous derivative b' throughout $[0, 1]$, we have under the above assumptions and notations in (F, G)-probability

$$\|\hat{b}_N - b_N\| \to 0 , \quad \int |\hat{b}'_N - b'_N|\, d\mu \to 0 , \tag{2.15}$$

$$P_{(F,G)}\{\|\hat{b}'_N\| \leqq \|b'\| + \varepsilon\} \to 1 \quad \forall \varepsilon > 0 .$$

Moreover, for each N the functions $\hat{b}_N$ and b_N have bounded continuous derivatives throughout $[0, 1]$ and

$$\|b_N - b\| \to 0 , \quad \|b'_N - b'\| \to 0 . \tag{2.16}$$

Proof. Slight modification of Behnen, Neuhaus, and Ruymgaart (1982).

Lemma 2.2. If, in addition, we assume

$$K(x) = K(-x) , \quad x \in \mathbf{R} , \tag{2.17}$$

then, for each N, we have

$$\int_0^1 \hat{f}_N d\mu = \int_0^1 \hat{g}_N d\mu = 1\,, \quad \int_0^1 \hat{b}_N d\mu = 0\,. \tag{2.18}$$

Proof. Immediate consequence of (2.17) and (2.10) to (2.12).

This means that for large classes of alternatives we have tractable consistent rank estimators $\hat{b}_N$ of the underlying nonparametric score function b_N. An approximately optimal rank statistic in case of underlying b_N is

$$S_{b_N} = \sum_{i=1}^{m} b_N\left(\frac{R_{1i} - 1/2}{N}\right). \tag{2.19}$$

Therefore, since b_N is unknown, we substitute b_N by its rank estimator $\hat{b}_N$ and get

$$\hat{S}_N = \sum_{i=1}^{m} \hat{b}_N\left(\frac{R_{1i} - 1/2}{N}\right) \tag{2.20}$$

as an adaptive rank statistic for the testing problem (2.1).

In this paper we discuss a simple algorithm for evaluating the estimated scores

$$\hat{b}_N\left(\frac{i - 1/2}{N}\right), \quad i = 1, \dots, n\,,$$

in case of a special kernel K_3 and report the results of some power simulations in this case. Some asymptotic results under H_1 of Chernoff-Savage type may be found in Behnen, Neuhaus, and Ruymgaart (1982). Some joint work together with Marie Hušková and Georg Neuhaus on better asymptotic results in case of special kernel type rank estimators is in progress.

Description of the algorithm:

(a) Choose $s \in \mathbf{N}$ (smoothing-number, e.g., $s = 3$) and $w \in \mathbf{N}$ (width of window, e.g., $w = 2, 3$) such that $s(w + 1/2)/N \leqq 1/2$.

(b) Given the ranks $R_{11}, \dots, R_{1m}$ of the X-sample in the pooled sample of size $N = m + n$, we put

$$f_{0i} = 1_{\{R_{11}, \dots, R_{1m}\}}(i)\,, \quad i = 1, \dots, N\,.$$

(c) For $r = 1, \dots, s$ define f_{ri}, $i = 1, \dots, N$, by iteration according to

$$f_{ri} = \begin{cases} \sum_{j=1}^{i+w} f_{r-1,j} + \sum_{j=1}^{w+1-i} f_{r-1,j}\,, & \text{if} \quad i = 1, \dots, w\,, \\ \sum_{j=i-w}^{i+w} f_{r-1,j}\,, & \text{if} \quad i = w + 1, \dots, N - w\,, \\ \sum_{j=i-w}^{N} f_{r-1,j} + \sum_{j=2N+1-i-w}^{N} f_{r-1,j}\,, & \text{if} \quad i = N - w + 1, \dots, N\,. \end{cases}$$

(d) Use

$$\hat{b}_{Ni} := \frac{N^2}{mn}\left(\frac{1}{2w+1}\right)^s f_{si} - \frac{N}{n}, \quad i = 1, \ldots, N, \tag{2.21}$$

as estimators of the (unknown) underlying scores

$$b_N\left(\frac{i-1/2}{N}\right), \quad i = 1, \ldots, N.$$

Properties of the algorithm:

$$-N/n \leqq \hat{b}_{Ni} \leqq N/m, \quad i = 1, \ldots, N, \quad \sum_{i=1}^{N} \hat{b}_{Ni} = 0 \tag{2.22}$$

(Proof by induction on the smoothing-number s.)

Theorem 2.3. For each N let $\hat{b}_{Ni}$, $i = 1, \ldots, N$, be given by (2.21) according to $s = 3$ and $w = w_N \in \mathbf{N}$ such that

$$a_N := 3(w_N + 1/2)/N \tag{2.23}$$

satisfies condition (2.13), and let $\hat{b}_N$ be the estimator defined in (2.9) to (2.11) with a_N from (2.23) and $K = K_3$ according to

$$K_3(x) = \begin{cases} 0, & \text{if} \quad x \leqq -1, \\ 27(x+1)^2/16, & \text{if} \quad -1 \leqq x \leqq -1/3, \\ 9(1-3x^2)/8, & \text{if} \quad -1/3 \leqq x \leqq 1/3, \\ 27(x-1)^2/16, & \text{if} \quad 1/3 \leqq x \leqq 1, \\ 0, & \text{if} \quad x \geqq 1. \end{cases} \tag{2.24}$$

Then we have

$$\max_{1 \leqq i \leqq N} \left| \hat{b}_N\left(\frac{i-1/2}{N}\right) - \hat{b}_{Ni} \right| \xrightarrow[N\to\infty]{} 0. \tag{2.25}$$

If the assumptions on (F, G) listed in Theorem 2.1 are fulfilled, we have in addition in (F, G)-probability

$$\left\| b - \hat{b}_{N1}\, 1_{[0,1/N]} - \sum_{i=2}^{N} \hat{b}_{Ni}\, 1_{((i-1)/N, i/N]} \right\| \xrightarrow[N\to\infty]{} 0. \tag{2.26}$$

Sketch of proof: Given

$$(Q_1, \ldots, Q_{3m}) = (-R_{1m}, \ldots, -R_{11}, R_{11}, \ldots, R_{1m}, 2N - R_{1m}, \ldots, 2N - R_{11})$$

we define

$$\hat{F}_0(x) = \frac{1}{m} \sum_{i=1}^{3m} 1_{[Q_i/N, \infty)}(x), \quad x \in \mathbf{R},$$

$$\hat{f}_1(x) = \frac{N}{2w+1}\left(\hat{F}_0\left(x + \frac{w+1}{N}\right) - \hat{F}_0\left(x - \frac{w}{N}\right)\right), \quad -1 + \frac{w}{N} < x < 2 - \frac{w+1}{N},$$

$$\hat{f}_2(x) = \frac{N}{2w+1}\int_{x-(w+1)/N}^{x+w/N} \hat{f}_1(y)\,dy, \quad -1 + \frac{2w+1}{N} < x < 2 - \frac{2w+1}{N},$$

$$\hat{f}_3(x) = \frac{N}{2w+1}\int_{x-(w+1/2)/N}^{x+(w+1/2)/N} \hat{f}_2(y)\,dy, \quad -1 + \frac{3(w+1/2)}{N} < x < 2 - \frac{3(w+1/2)}{N}.$$

Similarly, we define $\hat{g}_1$, $\hat{g}_2$, and $\hat{g}_3$ with respect to

$$(-R_{2n}, \ldots, -R_{21}, R_{21}, \ldots, R_{2n}, 2N - R_{2n}, \ldots, 2N - R_{21}).$$

By iteration from 1 to 3 we show on one hand

$$\hat{f}_3(x) = \frac{1}{m}\sum_{i=1}^{3m} \frac{N}{3(w+1/2)} K_3\left(\frac{Nx - Q_i}{3(w+1/2)}\right),$$

which has the form (2.10), (2.11) with K_3 instead of K and $3(w + 1/2)/N$ instead of a_N. By symmetry we get a similar representation of $\hat{g}_3$.

On the other hand we show for $i = -N + (w + 1), \ldots, 2N - (w + 1)$,

$$\hat{f}_1\left(\frac{i-1/2}{N}\right) = \frac{1}{2w+1}\sum_{j=i-w}^{i+w} \frac{N}{m} 1_{\{Q_1,\ldots,Q_{3m}\}}(j) = \frac{1}{2w+1}\frac{N}{m}\tilde{f}_{1i},$$

where [cf. part (c) of the algorithm]

$$(\tilde{f}_{1i},\ i = -N, \ldots, 2N) = (f_{1N}, \ldots, f_{11}, f_{11}, \ldots, f_{1N}, f_{1N}, \ldots, f_{11}).$$

and for $i = -N + (2w + 2), \ldots, 2N - (2w + 2)$,

$$\hat{f}_2\left(\frac{i}{N}\right) = \frac{1}{2w+1}\sum_{j=i-w}^{i+w} \hat{f}_1\left(\frac{j-1/2}{N}\right) = \left(\frac{1}{2w+1}\right)^2 \frac{N}{m}\tilde{f}_{2i},$$

where [cf. part (c) of the algorithm]

$$(\tilde{f}_{2i},\ i = -N, \ldots, 2N) = (f_{2N}, \ldots, f_{21}, f_{21}, \ldots, f_{2N}, f_{2N}, \ldots, f_{21}).$$

Moreover,

$$\hat{f}_1\left(\frac{i-1/2}{N}\right) \geqq 0, \quad \hat{g}_1\left(\frac{i-1/2}{N}\right) \geqq 0, \quad \frac{m}{N}\hat{f}_1\left(\frac{i-1/2}{N}\right) + \frac{n}{N}\hat{g}_1\left(\frac{i-1/2}{N}\right) = 1,$$

$$\hat{f}_2\left(\frac{i}{N}\right) \geqq 0, \quad \hat{g}_2\left(\frac{i}{N}\right) \geqq 0, \quad \frac{m}{N}\hat{f}_2\left(\frac{i}{N}\right) + \frac{n}{N}\hat{g}_2\left(\frac{i}{N}\right) = 1,$$

$$\frac{1}{N}\sum_{i=1}^{N} \hat{f}_1\left(\frac{i-1/2}{N}\right) = 1, \quad \frac{1}{N}\sum_{i=1}^{N} \hat{f}_2\left(\frac{i}{N}\right) = 1.$$

Finally, we prove for $i = 1, \ldots, N$,

$$\left|\hat{f}_3\left(\frac{i-1/2}{N}\right) - \left(\frac{1}{2w+1}\right)^3 \frac{N}{m} f_{3i}\right| \leqq \frac{27}{4} \frac{a_N^4}{N a_N^6} \to 0\,,$$

$$\left|\hat{g}_3\left(\frac{i-1/2}{N}\right) - \left(\frac{1}{2w+1}\right)^3 \frac{N}{n} g_{3i}\right| \leqq \frac{27}{4} \frac{a_N^4}{N a_N^6} \to 0\,,$$

where

$$f_{3i} \geqq 0\,, \quad g_{3i} \geqq 0\,, \quad f_{3i} + g_{3i} = (2w+1)^3\,.$$

Thus,

$$\left|\hat{b}_N\left(\frac{i-1/2}{N}\right) - \hat{b}_{Ni}\right| = \left|\hat{b}_N\left(\frac{i-1/2}{N}\right) - \left(\frac{1}{2w+1}\right)^3 \frac{N}{n}\left(\frac{N}{m} f_{3i} - (2w+1)^3\right)\right| =$$

$$= \left|\hat{f}_3\left(\frac{i-1/2}{N}\right) - \hat{g}_3\left(\frac{i-1/2}{N}\right) - \left(\frac{1}{2w+1}\right)^3 \left(\frac{N}{m} f_{3i} - \frac{N}{n} g_{3i}\right)\right| \leqq \frac{27}{2} \frac{a_N^4}{N a_N^6} \to 0\,.$$

3. *Adaption of the estimator to $F \leqq G$ and some Monte Carlo results.*

Because of $H_0 : F = G$ and $H_1 : F \leqq G,\ F \neq G$ we have (under H_0 and H_1),

$$\int_0^t b_N(x)\,dx \leqq 0 \quad \forall t \in [0, 1]\,, \int_0^1 b_N(x)\,dx = 0\,. \tag{3.1}$$

An adaption of estimators to this type of alternatives should increase the power of the corresponding test in finite situations. Moreover, the test should become more specific for $H_1 : F \leqq G,\ F \neq G$. The adaption of $\hat{b}_{Ni}$, $i = 1, \ldots, N$, to H_1 is done in the following way: Use

$$\hat{\hat{b}}_{Ni} := \begin{cases} \hat{b}_{Ni}\,, & \text{if } \sum_{j=1}^{i-1} \hat{b}_{Nj} + \frac{1}{2}\hat{b}_{Ni} \leqq 0\,, \\ 0\,, & \text{elsewhere}\,, \quad i = 1, \ldots, N\,, \end{cases} \tag{3.2}$$

as estimators of the (unknown) underlying scores $b_N((i - 1/2))/N)$, $i = 1, \ldots, N$.

A Monte Carlo study of the power of the corresponding tests was done under seven types of nonparametric alternatives (A. 1 – A. 7) with sample sizes $m = n = 20, 40$. The Monte Carlo sample size was 3000. The alternatives A. 1 to A. 7 are the same as in Behnen and Neuhaus (1981). They were designed to bring out some special features of Galton's test against Wilcoxon's test. Since the power of rank tests under alternatives (2.2) is independent of the special H_N in (2.2) and since we assume $m = n$, i.e., $m/N = n/N = 1/2$, the alternatives are given by Lebesgue densities on $[0, 1]$ of the form (cf. formula (2.5))

$$f = 1 + b/2\,, \quad g = 1 - b/2 \tag{3.3}$$

with b according to A. 1 – A. 7:

A. 1: $b(t) = 1.3(2t - 1)$, $0 \leqq t \leqq 1$ (Wilcoxon type),

A. 2: $(5/4)\, b = -1_{[0,0.5)} + 1_{[0.5,1]}$ (rank median type) ,
A. 3: $b = (-0.3)\, 1_{[0,0.3)} - (1.2)\, 1_{[0.3,0.5)} + (1.2)\, 1_{[0.5,0.7)} + (0.3)\, 1_{[0.7,1]}$,
A. 4: $(5/6)\, b = -1_{[0.3,0.5)} + 1_{[0.5,0.7)}$,
A. 5: $b = -1_{[0,0.25)} + 1_{[0,25,0.5)}$,
A. 6: $(4/3)\, b = -1_{[0,0.2)} + 1_{[0.2,05)} - 1_{[0.5,0.8)} + 1_{[0.8,1]}$,
A. 7: $b = (0.3)\, 1_{[0,0.3)} - (0.9)\, 1_{[0.3,0.7)} + (0.9)\, 1_{[0.7,1)}$.

The types A. 6 and A. 7 do not correspond to alternatives from $H_1 : F \leqq G, F \neq G$, since (3.1) is not fulfilled. They were included in order to find out whether the tests are specific for H_1.

From *Table* 1 we may conclude that we should use the $\hat{\hat{b}}_N$-test (adaption to H_1) instead of the $\hat{b}_N$-test (general estimation of b_N). Moreover, the $\hat{\hat{b}}_N$-test shows good adaptive behavior for quite different types of alternatives. In cases where the Wilcoxon test is nearly optimal (A. 1, A. 2) the power of the suitable $\hat{\hat{b}}_N$-test is comparable to the power of the Wilcoxon test, whereas in other cases (especially in case of A. 4) the power of the $\hat{\hat{b}}_N$-test is much higher. The width of window should increase rather slowly with sample sizes, i.e., $w = 2$, if $m = n = 20$, and $w = 3$, if $m = n = 40$. For "difficult" alternatives (A. 4) it seems to be hard to come close to the optimal power by general adaption, at least with sample sizes up to $m = n = 40$. Finally, it should be mentioned that the adaptive $\hat{\hat{b}}_N$-test is not very specific for H_1 (cf. A. 6, A. 7). In order to get a more specific test for H_1 we have to modify the $\hat{\hat{b}}_N$-test on the basis of some (empirical) measure of deviation from H_1, for example.

References

[1] Behnen K.: The Randles-Hogg test and an alternative proposal. *Comm. Statist. 4*, 1975, 203—238.

[2] Behnen K. and Neuhaus G.: Two-sample rank tests with estimated scores and the Galton test. Preprint Nr. *81—10, Universität Hamburg,* 1981.

[3] Behnen K., Neuhaus G. and Ruymgaart F.: A Chernoff-Savage theorem for rank statistics with estimated scores, and rank estimators of score-functions. Preprint Nr. *82—2, Universität Hamburg,* 1982.

[4] Hájek J. and Šidák Z.: Theory of Rank Tests. *Academic Press,* New York, 1967.

Table 1. Monte Carlo simulation of power of $\hat{b}_N$-test, $\hat{\hat{b}}_N$-test, (optimal) b_N-test, Wilcoxon's test, and Galton's test, respectively, under null hypothesis H_0 und seven types of fixed alternatives (A.1—A.7) for sample sizes $m = n = 20, 40$.

Type	$\hat{b}_N$-test ($s = 3,\ w = 1$)	$\hat{b}_N$-test ($s = 3,\ w = 2$)	$\hat{\hat{b}}_N$-test ($s = 3,\ w = 2$)	$\hat{\hat{b}}_N$-test ($s = 3,\ w = 3$)	b_N-test power/level	Wilcoxon	Galton
				$m = n = 20$			
H_0	9.6	9.9	9.8	9.9		10.7	9.9
A.1	45.2	57.5	79.8	80.9	88/10.4	88.6	77.2
A.2	48.9	59.6	78.8	80.2	78/6.0	83.3	58.4
A.3	43.2	48.1	60.1	60.4	67/11.3	47.6	26.5
A.4	38.4	41.2	41.3	40.2	57/11.5	23.4	12.5
A.5	36.8	41.8	47.8	31.0	73/12.9	28.1	19.3
A.6	38.0	43.7	44.8	45.3	54/5.6	24.9	21.6
A.7	44.5	42.7	49.6	52.3	74/9.3	29.0	12.6
				$m = n = 40$			
H_0		9.7	10.1	9.7		10.1	9.9
A.1		76.5	93.9	94.7	99/10.1	98.9	94.6
A.2		79.0	94.0	95.2	99/13.2	97.5	78.7
A.3		74.1	83.8	85.8	90/9.3	67.5	35.6
A.4		68.4	59.5	64.3	84/10.6	30.0	12.7
A.5		64.3	64.5	70.6	88/7.9	38.2	21.7
A.6		69.6	63.7	67.2	93/13.5	32.3	24.0
A.7		75.5	65.3	72.2	95/9.6	32.2	10.9