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Adaptive Procedures

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The aim of this paper is to explain the main ideas of adaptive procedures, to summarize their basic structure and to review procedures obtained by modification of those based on ranks. Some properties are stated. The results are presented for one- and two-sample model.

Cílem článku je vysvětlit hlavní ideje adaptivních metod, popsat jejich základní strukturu a poskytnout přehled adaptivních metod získaných modifikací postupů založených na pořadí. Dále jsou uvedeny některé jejich vlastnosti. Výsledky jsou presentovány pro jedno- a dvouvýběrový model.

Цель этой статьи — объяснить основные идеи адаптивных методов непараметрического оценивания, описать их структуру и дать обзор приемов, основанных на применении ранговых статистик. Обсуждаются результаты для одновыборочной и двухвыборочной модели.

1. Introduction

Consider the one-sample and the two-sample models in the following form:

Two-sample model: $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_m)$ are independent random samples from the distribution with the density $f(x)$ resp. $f(x - \theta)$, where f is absolutely continuous with nonzero finite Fisher's information

$$0 < I(f) = \int (f'(x))^2 f^{-1}(x)\,\mathrm{d}x < +\infty\,,$$

θ is a parameter.

One-sample model: $(X_1, \ldots, X_n)$ is the sample from the distribution with the density $f(x - \theta)$, where f is symmetric about zero, absolutely continuous and $0 < I(f) < +\infty$.

The general form of rank test statistics for testing hypothesis $H : \theta = 0$ versus $A : \theta > 0$ in the two-sample model is the following:

$$S_N(\varphi) = \sum_{i=1}^{n} \varphi(R_i/(N + 1))\,, \tag{1.1}$$

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where $N = n + m$, R_i is the rank of X_i in the sequence $X_1, \ldots, X_n, Y_1, \ldots, Y_m$, φ is a square integrable function on $(0, 1)$. Analogously, rank test statistics for $\mathrm{H} : \theta = 0$ versus $\mathrm{A} : \theta > 0$ in the one-sample model can be expressed as follows:

$$S_n^+(\varphi) = \sum_{i=1}^{n} \operatorname{sign} X_i \, \varphi(R_i^+/(n + 1)), \tag{1.2}$$

where R_i^+ is the rank of $|X_i|$ in the sequence $|X_1|, \ldots, |X_n|$, φ is as above, sign $x = 1$, $x \geqq 0$, sign $x = -1$, $x < 0$.

As for rank estimators of θ, the Hodges-Lehman rank estimators $\hat{\theta}(\varphi)$ and the linearized rank estimators $\hat{\theta}_L(\varphi)$ are used, where φ is a monotone square integrable functions on $(0, 1)$. The definitions will be done only for the two-sample model and φ nondecreasing; for other cases they can be easily modified. The estimator $\hat{\theta}(\varphi)$ is defined as follows:

$$\hat{\theta}(\varphi) = \alpha \, \hat{\theta}_1(\varphi) + (1 - \alpha) \, \hat{\theta}_2(\varphi), \tag{1.3}$$

where $0 \leqq \alpha \leqq 1$,

$$\hat{\theta}_1(\varphi) = \sup \left\{ b;\ \sum_{i=1}^{n} \varphi\left(\frac{R_i(b)}{N + 1}\right) > 0 \right\},$$

$$\theta_2(\varphi) = \inf \left\{ b;\ \sum_{i=1}^{n} \varphi\left(\frac{R_i(b)}{N + 1}\right) < 0 \right\},$$

with $R_i(b)$ being the rank of $X_i - b$ in the sequence $X_1 - b, \ldots, X_n - b, Y_1, \ldots, Y_m$.

The estimator $\hat{\theta}_L(\varphi)$ has the following form:

$$\hat{\theta}_L(\varphi) = \bar{\theta} + \left(\int_0^1 \varphi^2(u) \, \mathrm{d}u \right)^{-1} \frac{N}{mn} \sum_{i=1}^{n} \varphi\left(\frac{R_i(b)}{N + 1}\right), \tag{1.4}$$

where $\bar{\theta}$ is the preliminary translation and scale invariant estimator fulfilling

$$\sqrt{N} \, (\bar{\theta} - \theta) = O_p(1) \quad \text{as} \quad \min(m, n) \to \infty. \tag{1.5}$$

If the density f is known and satisfies some regularity conditions the test for H versus A based on $S_N(\varphi_f)$ and $S_n^+(\varphi_f^+)$ leads to the asymptotically most powerful test for contiguous alternatives in the two-sample resp. one-sample model. Here

$$\varphi_f(u) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}, \quad u \in (0, 1), \tag{1.6}$$

$$\varphi_f^+(u) = \varphi_f((u + 1)/2), \quad u \in (0, 1), \tag{1.7}$$

$$F^{-1}(u) = \inf \{x;\ F(x) \geqq u\}.$$

The functions φ_f and φ_f^+ generate also the asymptotically optimal estimators of θ (by asymptotically optimal estimator $\hat{\theta}$ of θ is understood the estimator with the property:

$$\mathscr{L}(\sqrt{(n)} \, (\hat{\theta} - \theta) \sqrt{I(f)}) \to_w N(0, 1) \quad \text{as} \quad n \to \infty).$$

Analogous situation arises in case of other types of test statistics and estimators (e.g. L-estimators, maximumlikelihood type estimators).

In practice, we usually do not know the true density f, nevertheless we are interested in having the asymptotically optimal (or at least reasonable) tests or estimators. The procedures solving this problem are called *adaptive procedures* (the known estimators and test statistics are adapted according to the data). These *procedures* are generally of two kinds, *restrictive* and *nonrestrictive*.

The basic struture of the restrictive procedure is:

1. choose a reasonable family $\mathscr{F}$ of distributions, a decision rule for selection of the distribution from $\mathscr{F}$;
2. choose according to the decision rule (based on the sample $X_1, \ldots, X_n$, preferably on the corresponding ordered sample $X_{(1)} \leqq \ldots \leqq X_{(n)}$) the density $f_0 \in \mathscr{F}$;
3. provide the test (or the estimator) of type, which is optimal for f_0,

The nonrestrictive adaptive procedure consists in

1. estimating of φ_f;
2. application of the test statistics (or estimator) with φ_f replaced by their estimators.

There were suggested several adapative procedure of both kinds and studied their properties. Generally, the restrictive procedures are usually simple, utilizing well known tests or estimators. However, the resulting tests and estimators are asymptotically optimal only if the true density belongs to the chosen family $\mathscr{F}$. The results of simulation studies support these procedures for small samples.

Both the tests estimators obtained by nonrestrictive procedures are mostly asymptotically optimal for a quite broad class of the densities, but the convergence is very slow and application is usually connected with long computations.

The most of adaptive procedures was developed by modification of the rank type test, rank estimators, L-estimators, maximumlikelihood estimators. The next two sections are devoted to rank type tests and estimators. Adaptive maximum-likelihood type estimators were suggested by Stone (1975) and Moberg et al. (1980). Adaptive L-type estimators were proposed by Takeuchi (1971), Johns (1974), Sacks (1975). The welldone survey of adaptive procedures till 1974 was done by Hogg (1974).

2. Restrictive procedures

The attention here is concentrated to the review of possible families $\mathscr{F}$ of distributions and decision rules for selection of the distribution in the one-sample model. In the following, we shall often work with the subfamily $\mathscr{F}(f)$ of densities generated by a density f as follows:

$$\mathscr{F}(f) = \{g;\, g(x) = \eta\, f(\eta x - u),\ -\infty < u < \infty,\ \eta > 0\}\,.$$

First, we mention the procedures with decision rules motivated by the behavior of the tails of distribution (e.g. Hájek (1970), Hogg and Randles (1973), Hogg, Fisher and Randles (1975), Jones (1979)). In such a case the family $\mathscr{F}$ contains densities ranging from the light-tailed (like uniform) to heavy-tailed (like Cauchy).

Randles and Hogg (1973) considered the family $\mathscr{F}$ consisting of three type densities $\mathscr{F}(f_1)$, $\mathscr{F}(f_2)$, $\mathscr{F}(f_3)$, where f_1 is double exponential density (heavy-tailed), f_2 is logistic one (medium-tailed), f_3 is uniform one (light-tailed). The decision rule is the following:

$$\text{choose } \mathscr{F}(f_1) \text{ if } Q > 2.96 - 5.5/n\,,$$

$$\text{choose } \mathscr{F}(f_2) \text{ if } 2.96 - 5.5/n \geqq Q \geqq 2.08 - 2/n\,,$$

$$\text{choose } \mathscr{F}(f_3) \text{ if } 2.08 - 2/n > Q\,,$$

where

$$Q = (X_{(n)} - X_{(1)})\, n\{\sum_{i=1}^{n} |X_{(i)}\text{-median of } X_i's|\, 2\}^{-1} \quad \text{for} \quad n \leqq 20\,, \tag{2.1}$$

$$Q = (\bar{U}_{0.05} - \bar{L}_{0.05})(\bar{U}_{0.5} - \bar{L}_{0.5})^{-1} \quad \text{for} \quad n > 20\,, \tag{2.2}$$

with $\bar{U}_\alpha$ and $\bar{L}_\alpha$ being the average of 100α % the largest and smallest order statistics, resp. The motivation for this decision rule, if $n \leqq 20$, comes from the fact that the optimal translation and scale invariant test that the considered sample is from the uniform distribution versus double exponential one can be (approximately) based on Q given by (2.1). As for $n > 20$, notice that

$$Q \to 3.3 \quad \text{in probability as } n \to \infty \text{ for } f \in \mathscr{F}(f_1)\,,$$

$$Q \to 2.6 \quad \text{in probability as } n \to \infty \text{ for } f \in \mathscr{F}(f_2)\,,$$

$$Q \to 1.96 \quad \text{in probability as } n \to \infty \text{ for } f \in \mathscr{F}(f_3)\,.$$

Then the test statistics are chosen according to the general rule except $\mathscr{F}(f_3)$ – the authors recommend to use some modified Wilcoxon one-sample test. This decision rule was slightly modified and used also for proposing of other adaptive procedures (e.g. Moberg et al. (1980)).

Another procedure based on tail behavior was suggested by Hájek (1970) for the family $\mathscr{F} = \{\mathscr{F}(f_1), \ldots, \mathscr{F}(f_k)\}$, where f_i are distinct symmetric densities. The decision rule consists, in fact, in choosing $\mathscr{F}(f_i)$ for which the quantile function corresponding to f_i is close to the sample quantile function. The procedure is very quick but no properties were studied.

Jones (1979) introduced the family $\mathscr{F} = \{f_\lambda, \lambda \in R_1\}$, where f_λ satisfies

$$F_\lambda^{-1}(u) = (u^\lambda - (1 - u)^\lambda)/\lambda$$

(i.e. $\varphi(u, f_\lambda) = (\lambda - 1)(u^{\lambda-2} - (1 - u)^{\lambda-2})(u^{\lambda-1} + (1 - u)^{\lambda-1})^{-2}$). This family contains densities ranging from light-tailed ones ($\lambda > 0$) to heavy-tailed ones ($\lambda < 0$).

Particularly, for $\lambda = 1$ and $\lambda = 2$ f_λ is uniform, for $\lambda = 0.135$ f_λ is approximately normal, for $\lambda = 0$ f_λ is logistic. The author proposed to estimate λ through the ordered sample as follows:

$$\hat{\lambda} = (\log 2)^{-1} \log \{[X_{(n-2M+1)} - X_{(n-4M+1)}] \cdot [X_{(n-M+1)} + X_{(n-2M+1)}]^{-1}\}$$

where M is chosen in some proper way reflecting the behavior of the tail. As the resulting φ-function is taken $\varphi(u, f_\lambda)$.

As examples of procedures not motivated by the behavior of tails, we shall sketch two procedures published by Hájek (1970) for a general family $\mathscr{F} = \{\mathscr{F}(f_1), \ldots, \mathscr{F}(f_k)\}$, where $f_1, \ldots, f_k$ are distinct densities and the procedure by Albers (1979). In the first procedure, the decision rule is the Bayesian translation and scale invariant rule and the second one is based on the asymptotic linearity of rank statistics, the third one utilizes the estimate of the kurtosis. In order to have the decision rule dependent only on the ordered sample $|X|_{(1)} \leqq \ldots \leqq |X|_{(n)}$ (corresponding to $|X_1|, \ldots, |X_n|$) we define new random variables $X_i^* = V_i|X|_{(Q_i)}$, $i = 1, \ldots, n$, where $(Q_1, \ldots, Q_n)$ is a random permutation of $(1, \ldots, n)$ and $(V_1, \ldots, V_n)$ are i.i.d. with $P(V_i = 1) = P(V_i = -1) = 1/2$ independent of $X_1, \ldots, X_n$. Then under H the random vector $(X_1^*, \ldots, X_n^*)$ is independent of $(R_1^+, \ldots, R_n^+)$ and (sign $X_1, \ldots,$ sing X_n) and are distributed as $(X_1, \ldots, X_n)$.

The Baysian translation and scale invariant rule (if all types are apriori equiprobable) yields the following:

$$\text{choose} \quad \mathscr{F}(f_i) \quad \text{if} \quad \max_{1 \leqq j \leqq k} p_{jn}(X_1^*, \ldots, X_n^*) = p_{in}(X_1^*, \ldots, X_n^*)$$

where

$$p_{jn}(X_1^*, \ldots, X_n^*) = \int_0^\infty \int_{-\infty}^{+\infty} \prod_{i=1}^{n} f(\lambda X_i^* - u)\, \lambda^{n-2}\, du\, d\lambda\,, \quad j = 1, \ldots, k\,.$$

Uthoff (1970) derived $p_n(X_1, \ldots, X_n)$ for some well known distributions (e.g. normal, uniform, exponential). Sometimes there are computational problems with evaluating $p_n(X_1, \ldots, X_n)$. For such cases Hogg et al. (1972) recommended to use

$$p_{jn}^*(X_1^*, \ldots, X_n^*) = \prod_{i=1}^{n} \hat{\sigma}_{jn}^{-1} f((X_i^* - \hat{\mu}_{jn})\, \hat{\sigma}_{jn}^{-1})\,,$$

where $\hat{\mu}_{jn}$ and $\hat{\sigma}_{jn}$ are maximum likelihood estimators of location and scale for the j-th distribution, instead of $p_{jn}(X_1^*, \ldots, X_n^*)$.

The decision rule based on the asymptotic linearity of rank statistics ($f_1, \ldots, f_k$ absolutely continuous and $I(f_j) < +\infty$, $j = 1, \ldots, k$) is defined as follows:

$$\text{choose} \quad \mathscr{F}(f_i) \quad \text{if} \quad \max_{1 \leqq j \leqq n} L_{jn} = L_{in}\,,$$

where

$$L_{jn} = [S_n^*(j, n^{-1/2}) - S_n^*(j, 0)] \,.\, [\text{var}_H\, S_n^*(j, 0)]^{-1/2}\,,$$

$$S_n^*(j,t) = \sum_{i=1}^{n} \operatorname{sign} X_i^* \varphi_{f_j}^+(R_i^+(t)\,(n+1)^{-1})$$

with $R_i^+(t)$ being the rank of $|X_i^* - t|$ in the sequence $|X_1^* - t|, \ldots, |X_n^* - t|$. Notice that by the asymptotic linearity (see van Eeden (1972))

$$L_{jn} \to \int_0^1 \varphi_{f_j}(u)\, \varphi_f^+(u)\, du \left(\int_0^1 \varphi_{f_j}^{+2}(u)\, du\right)^{-1/2}$$

in probability as $n \to \infty$, $j = 1, \ldots, k$, if the true density is f. By Schwarz inequality the right-hand side is smaller or equal $(\int_0^1 \varphi_f^{+2}(u)\, du)^{1/2}$ and the maximum is achieved if $f \in \mathscr{F}(f_i)$.

Albers (1979) considered instead of the family of distributions $\mathscr{F}$ the family (say J) of functions φ generating the test statistics $S_N^+(\varphi)$ given by (1.2), where $J = \{\varphi_r;\ \varphi_r = \varphi_0 + rh,\ -D_1 \leqq r \leqq D_2\}$, φ_0 and h are smooth functions on $(0, 1)$, $D_1 > 0$, $D_2 > 0$. He reccommended to choose such $\varphi_{\tilde{r}}$, where $\tilde{r}$ is minimizing

$$\left| \frac{n^{-1} \sum_{i=1}^{n} (|X|_{(i)})^q}{(n^{-1} \sum_{i=1}^{n} (|X|_{(i)})^p)^{q/p}} - \frac{\int |x|^q\, dF_r(x)}{\left(\int |x|^p\, dF_r(x)\right)^{q/p}} \right|,$$

with F_r being the distribution function corresponding to φ_r, $0 < p < q$. It is also indicated how p and q should be chosen for given J.

3. Nonrestrictive procedures

Here will be presented estimators of the functions φ_f, some of their asymptotic properties and also the resulting tests or estimators in the two-sample model.

Hájek (1962) proposed the following estimator of φ_f based on ordered sample $X_{(1)} \leqq \cdots \leqq X_{(n)}$:

$$\tilde{\varphi}_n(u) = 2\, \frac{q_n t_n}{n+1} \{(X_{(h_{nj}+q_n)} - X_{(h_{nj}-q_n)})^{-1} - \tag{3.1}$$

$$- X_{(h_{n,j+1}+q_n)} - X_{(h_{n,j+1}-q_n)})^{-1}\} \quad \text{for} \quad h_{nj} n^{-1} \leqq u < h_{n,j+1}\,, \quad 1 \leqq j \leqq t_n\,,$$

where $q_n = [n^{3/4} \varepsilon_n^{-2}]$, $t_n = [n^{1/4} \varepsilon_n^3]$, $\varepsilon_n \to 0$, $n^{1/4} \varepsilon_n^3 \to \infty$, as $n \to \infty$, $h_{nj} = [jn/(t_n + 1)]$, $1 \leqq j \leqq t_n$, with $[a]$ denoting the largest integer part of a.

The motivation of this estimator comes from the following two facts:

1) $X_{(nu)} \to F^{-1}(u)$ in probability as $n \to \infty$, $u \in (0, 1)$,

2)

$$\lim_{r \searrow 0\, s \searrow 0} 2rs \left\{ \frac{1}{F^{-1}(u+r) - F^{-1}(u-r)} - \frac{1}{F^{-1}(u+s+r) - F^{-1}(u+s-r)} \right\} =$$

$$= \varphi_f(u), \; u \in (0, 1).$$

Beran (1974) suggested another estimator (say $\hat{\varphi}_n$) of φ_f through the estimators $\hat{c}_k$ of the Fourier coefficient c_k belonging to $\varphi_f(u)$. Namely,

$$\hat{\varphi}_n(u) = \sum_{|k|=1}^{M_n} \hat{c}_k \exp\{2\pi iku\}, \tag{3.2}$$

$$\hat{c}_k = T_n(\underset{\sim}{X}, \exp\{-2\pi ik\cdot\})$$

where $\underset{\sim}{X} = (X_1, \ldots, X_n)'$, $T_n(\underset{\sim}{X}, g)$ is the functional defined on $L_2(0, 1)$ given by

$$T_n(\underset{\sim}{X}, g) = (2n\theta_n)^{-1} \sum_{v=1}^{n} \{g((n-1)^{-1} \sum_{j=1}^{n} u(X_v - X_j + \theta_n)) -$$

$$- g((n-1)^{-1} \sum_{\substack{j=1 \\ j \neq v}}^{n} u(X_v - X_j - \theta_n))\},$$

$u(x) = 1$ if $x \geqq 0$, $u(x) = 0$ if $x < 0$, $\theta_n = bn^{-1/2}$ for some $b \neq 0$. In fact, $T_n(\underset{\sim}{X}, g)$ is an estimator of the functional

$$T(g) = \int_0^1 \varphi_f(u)\, g(u)\, du = \int \frac{dg(F(x))}{dx}\, dF(x),$$

obtained by replacing the theoretical distribution by the empirical one and replacing the derivative by difference.

The estimator $\tilde{\varphi}_n(u)$ is consistent in the sense that

$$\int_0^1 (\tilde{\varphi}_n(u) - \varphi_f(u))^2\, du \to 0 \quad \text{in probability as} \quad n \to \infty \tag{3.3}$$

and

$$\int_0^1 \tilde{\varphi}_n^2(u)\, du \to I(f) \quad \text{in probability as} \quad n \to \infty. \tag{3.4}$$

If, moreover, $M_n \to \infty$ and $M_n^{7/2} n^{-1} \to 0$ as $n \to \infty$ then in both relations $\tilde{\varphi}_n(u)$ can be replaced by $\hat{\varphi}_n(u)$.

The respective adaptive test statistics in the two-sample model are $S_N(\tilde{\varphi}_N^*)$ and $S_N(\hat{\varphi}_N^*)$, where $S_N(\varphi)$ is given by (1.1) and

$$\tilde{\varphi}_N^*(u) = N^{-1}(n\, \tilde{\varphi}_n(u, \underset{\sim}{X}) + m\, \tilde{\varphi}_m(u, \underset{\sim}{Y})) \quad u \in (0, 1), \tag{3.5}$$

writing $\tilde{\varphi}_n(u, \underset{\sim}{X})$ and $\tilde{\varphi}_m(u, \underset{\sim}{Y})$ for $\tilde{\varphi}_n$ obtained from the sample $\underset{\sim}{X} = (X_1, \ldots, X_n)'$ and $\underset{\sim}{Y} = (Y_1, \ldots, Y_m)'$, resp. $\hat{\varphi}_N^*$ is defined similarly. Under H

$$\mathscr{L}((S_N(\tilde{\varphi}_N^*) - \tilde{E}_N)\, \tilde{D}_N^{-1}) \to_w N(0, 1) \quad \text{as} \quad \min(n, m) \to \infty, \tag{3.6}$$

where $\tilde{E}_N = N^{-1} \sum_{i=1}^{N} \tilde{\varphi}_N^*(i(N+1)^{-1})$, $\tilde{D}_N^2 = (N-1) \sum_{i=1}^{N} (\tilde{\varphi}_N^*(i(n+1)^{-1}) - \tilde{E}_N)^2$. If, moreover, $M_N \to \infty$, $M_N^4(\min(n, m))^{-1} \to 0$, then $\tilde{\varphi}_N^*$ can be replaced by $\hat{\varphi}_N^*$ in (3.6). The tests based on either $S_N(\tilde{\varphi}_N^*)$ or $S_N(\hat{\varphi}_N^*)$ provide the asymptotically most powerful tests for H versus contiguous alternatives.

As for estimators, Beran (1974) proved asymptotic optimality of $\hat{\theta}_L(\hat{\varphi}_N^*)$ given by (1.4). Van Eeden (1970) and Kraft and van Eeden (1970) got the same properties (under somewhat stronger conditions) of $\hat{\theta}_L(\tilde{\varphi}_N^{**})$ and $\hat{\theta}(\tilde{\varphi}_N^{**})$, resp., where $\tilde{\varphi}_N^{**}$ is a modified form of the estimator $\tilde{\varphi}_N^*$ obtained from vanishingly small fraction of the data.

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