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On Bol Loops of Order $4k$

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A general construction for Bol loops is presented and is used to show the existence of at least two non-isomorphic, non-Moufang Bol loops of order $4k$ for every integer k greater than two.

Je dána obecná konstrukce Bolových lup. Ta je pak užita k důkazu existence alespoň dvou neizomorfních nemoufangovských Bolových lup řádu $4k$, pro každé celé číslo $k > 2$.

Представлена общая конструкция луп Бола. Она использованна к доказательству существования не менее чем двух неизоморфных луп Бола порядка $4k$, которые не являются лупами Муфанг, для всякого целого $k > 2$.

Introduction. A set L with a binary operation $(\cdot)$ is called a quasigroup if the specification of any two of the elements a, b, c in the equation $a \cdot b = c$ uniquely determines the third. If a quasigroup $(L, \cdot)$ has a two-sided identity then it is called a loop. A Bol loop (resp. Moufang loop) is a loop in which the identity $((xy)\, z)\, y = x((yz)\, y)$ (resp. $((xy)\, z)\, y = x(y(zy)))$ holds for all x, y, z in L. The question of existence and. classification of nonassociative Bol loops of prescribed orders has been the topic of investigation by several authors recently (see, for example, [1], [2], [3], [4]). In [4], Karl Robinson showed that there exists at least one non-Moufang Bol loop of order $4k$ for each integer k greater than 2. In the present note, we show that, by generalizing the construction of Robinson, one can prove that there exists at least two non-isomorphic, non-Moufang Bol loops of order $4k$ for any integer k greater than 2.

Robinson's Construction. We briefly recall the construction given by Robinson in [4]. Let G, H be two groups with identity elements 1, e respectively and let $f : G \to \mathrm{Aut}(H)$ be a mapping. Let $B = G \times H$ and multiplication in B be defined according to (y, b) . . $(x, a) = (yx, b^{f(x)}a)$ for all x, y in G and a, b in H. (Here $b^{f(x)}$ denotes the image of b under the automorphism $f(x)$). Then B is a Bol loop with identity $(1, e)$ if and only if (i) $f(xyx) = f(x)f(y)f(x)$ for all x, y in G and (ii) $f(1) = 1_H$, the identity map on H. Furthermore, B is Moufang if and only if B is associative if and only if f is a homomorphism of G into $\mathrm{Aut}(H)$. This is Lemma 1 of [4]. Now let G be a group

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of order 4 generated by two elements a, b such that $a^2 = b^2 = 1$, $ab = ba$. Let H be a cyclic group of order $k \geqq 3$. Define the map $f : G \to \mathrm{Aut}(H)$ by $f(1) = 1_H$, $f(a) = f(b) = f(ab) = v$ where v is the automorphism $h \to h^{-1}$ of H. Since G and H are abelian and $(f(x))^2 = 1_H$, $x^2 = 1$ for all x in G, $f(xyx) = f(x)f(y)f(x)$ is satisfied for all x, y in G. Next, $f(1) = 1_H$ by definition. Finally, since $f(ab) \neq f(a)f(b)$, f is not a homomorphism. Thus B is a non-Moufang Bol loop of order $4k$. (This is the corollary to Theorem 1 in [4].)

Generalization of Robinson's construction. We take two maps f, g from G into $\mathrm{Aut}(H)$ (instead of just one map f as has been done by Robinson). Define multiplication in $B = G \times H$ by $(y, b)(x, a) = (yx, b^{f(x)}a^{g(y)})$. If f, g satisfy $f(1) = g(1) = 1_H$, then B is a loop under the above multiplication, with $(1, e)$ as two-sided identity. Let us further assume that $f(x)\,g(y) = g(y)f(x)$ for all x, y in G. Then, a routine checking of the associative identity, the Moufang identity and the Bol identity reveals the following:

(I) The following are equivalent: (1) B is associative (2) B is Moufang (3) $f(xy) = f(x)f(y)$ and $g(xy) = g(y)\,g(x)$ for all x, y in G.

(II) B is a Bol loop if and only if $f(xyx) = f(x)f(y)f(x)$ and $g(xy) = g(y)\,g(x)$ for all x, y in G.

Application. We use the above generalization to prove the following fact:

Proposition. There exist at least two non-isomorphic, non-Moufang Bol loops of order $4k$ for each integer $k > 2$.

Proof: Let G be a group of order 4 generated by two elements a, b with $a^2 = b^2 = 1$ and $ab = ba$. Let H be a cyclic group of order k. Define $f, g : G \to \mathrm{Aut}(H)$ by $f(1) = 1_H$, $f(a) = f(b) = f(ab) = v$ and $g(x) = 1_H$ for all x in G. Then, the loop B obtained with f, g is the same as the one obtained by Robinson. We now define another pair of maps f', g' as follows: $f'(1) = 1_H$, $f'(a) = f'(b) = f'(ab) = v$ and $g'(1) = g'(a) = 1_H$, $g'(b) = g'(ab) = v$. Let us call the loop B given by these two maps as B'. Since $\mathrm{Aut}(H)$ is abelian, the condition $f'(x)\,g'(y) = g'(y)f'(x)$ is satisfied by all x, y in G. Now to check the conditions in (I) and (II): Since f' is the same as the f in Robinson's construction, it satisfies $f'(xyx) = f'(x)f'(y)f'(x)$ for all x, y in G, and $f'(ab) \neq f'(a)f'(b)$ as checked already. Next, g' is a homomorphism from the group $G = \{1, a, b, ab\}$ onto the group $\{1_H, v\}$. So $g'(xy) = g'(x)\,g'(y)$ for all x, y in G. But the group $\{1_H, v\}$ is abelian so that $g'(xy) = g'(y)\,g'(x)$ for all x, y in G. Thus B' is a non-Moufang Bol loop of order $4k$. Now we show that B and B' are non-isomorphic, by counting the number of elements in each, whose square equals the identity. A simple calculation reveals the following: In B, this number equals $3k + 1$ if k is odd and equals $3k + 2$ if k is even. In B', it is $k + 3$ if k is odd and $k + 6$ if k is even. Now $3k + 1 \neq k + 3$ and $3k + 2 \neq k + 6$ since $k > 2$. This shows that B and B' are not isomorphic.

Remark. In [2], Burn proves that, for any odd prime p, there exists exactly two non-Moufang Bol loops of order $4p$. Thus the two loops constructed above, account for all non-Moufang Bol loop of order $4p$, p an odd prime.

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