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On Nonexistence of Oriented Cycles in Auslander-Reiten Quivers

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Let A be a representation-finite algebra over an algebraically closed field and let $\Gamma(A)$ be the Auslander-Reiten quiver of A. In this note we give a simple necessary and sufficient condition for nonexistence of oriented cycles in $\Gamma(A)$ in the class of representation-finite algebras A with Auslander-Reiten invariant $\beta(A) \leqq 2$.

Kategorii levých modulů nad konečně dimenzionální algebrou nad algebraicky uzavřeným tělesem, která má pouze konečný počet tříd izomorfismů nerozložitelných modulů, lze popsat pomocí Auslanderova-Reiterova grafu. V článku jsmu uvedeny nutné a postačující podmínky pro neexistenci orientovaných cyklů v tomto grafu.

Пусть K — алгебраически замкнутое поле и A — конечно размерная K-алгебра имеющая только конечное число неразложимых A-модулей. Строение категории A-модулей описывается при помощи графа Ауслендера-Рейтена. В статье найдены необходимые и достаточные условия для несуществования ориентированных циклов в этом графе.

Throughout we fix an algebraically closed field K. We use the term algebra to denote a finite-dimensional, connected, basic associative K-algebra with unity. Similarly the term module is used for a left module of finite K-dimension.

Let A be an algebra and let M, N be two indecomposable A-modules. Then a homomorphism $f: M \to N$ is siad to be *irreducible* if for every factorization $f = gh$, g is split epi or h is split mono [3]. An algebra A is said to be *representation-finite* if it has only a finite number of isomorphism classes of indecomposable A-modules. It is well-known that any homomorphism between two indecomposable modules over a representation-finite algebra is a sum of compositions of irreducible homomorphisms [5, 11]. Thus, for a representation-finite algebra A, the full information on the category of A-modules, mod A, is contained in a finite connected graph $\Gamma(A)$ called the *Auslander-Reiten quiver* of A. The vertices of $\Gamma(A)$ are the isomorphism classes of indecomposable A-modules and there is an arrow from a class of M to a class of N if there is an irreducible homomorphism from M to N (see [1]).

One of the open question in the representation theory of algebras is that when,

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for a representation-finite A, $\Gamma(A)$ has no oriented cycles. It is an interesting question since, as it was proved by Happel and Ringel [13] (see also [12]), if A is representation-finite and $\Gamma(A)$ has no oriented cycles, then the indecomposable A-modules are uniquely determined, up to isomorphism, by their simple composition factors. This problem has been considered partially by Bautista and Smalo [8].

In this note we give a complete solution of the raised above problem for representation-finite algebras A with $\beta(A) \leqq 2$. Recall that, for an indecomposable A-module M, the invariant $\beta(M)$ records the largest possible number of nonisomorphic indecomposable A-modules $N_1, \ldots, N_r$ which are not projective-injective and there are irreducible maps $M \to N_i$, $i = 1, \ldots, r$. Then $\beta(A)$ is defined to be the supremum of $\beta(M)$ for M indecomposable A-modules (see [2–6]). Bautista and Brenner proved in [7] that, for any representation-finite algebra A, $\beta(A) \leqq 3$. There is no still classification of all representation-finite algebras. For the case $\beta(A) \leqq$ $\leqq 2$, this problem has been solved by the author and Waschbüsch in [16] (see also [6, 15]), where it was shown that, for a representation-finite algebra A, $\beta(A) \leqq 2$ if and only if A is biserial. Recall that an algebra A is called *biserial* (cf. [9]) if the radical of any indecomposable nonuniserial projective left or right A-module is a sum of two uniserial submodules whose intersection is simple or zero. Consequently we are concerned with the problem when, for a representation-finite biserial algebra A, $\Gamma(A)$ has no oriented cycles. Morever, it is well-known (see [11, 14]) that if $\Gamma(A)$ has no oriented cycles then A is a factor of an hereditary algebra.

In order to formulate the main result of this note we need the notion of a *special family of local modules.* A sequence of nonisomorphic uniserial A-modules $U_1, \ldots$ $\ldots, U_m$, and local nonuniserial A-modules $L_{i_1}, \ldots, L_{i_p}$, $1 \leqq i_1 < \ldots < i_p = m$, is called special if, for $i_t < j < i_{t+1}$, $\operatorname{top}(U_j) = \operatorname{soc}(U_{j+1})$, and for $1 \leqq t \leqq p$, $\operatorname{soc}(L_{i_t}) = \operatorname{top}(U_{i_t}) \oplus \operatorname{soc}(U_{i_t+1})$, where $U_{i_p+1} = U_1$. Recall that a module M is called *uniserial* if the submodule lattice of M is a chain. Moreover, M is *local* if it has unique maximal proper submodule or, equivalently, it is a factor of an indecomposable projective module. Finally, we denote by $\operatorname{top}(M)$ the top of M, by $\operatorname{soc}(M)$ the socle of M, and by $P(M)$ the projective cover of M.

Then we have the following theorem which is the main result of this paper.

Theorem. *For an algebra A being a factor of an hereditary algebra the following conditions are equivalent.*

(i) *A is representation-finite, $\beta(A) \leqq 2$, and $\Gamma(A)$ has no oriented cycles.*

(ii) *A is biserial and there is no special family of local A-modules.*

We know from [15, Theorem 1] that any algebra A satisfying the condition (ii) of Theorem is representation-finite and $\beta(A) \leqq 2$. Thus, for our aim, it is enough to prove that, for a representation finite biserial algebra A, $\Gamma(A)$ has no oriented cycles if and only if there is no special family of local A-modules. In order to prove this fact we use a good presentation of any representation-finite biserial algebra as a bounden quiver algebra.

Let A be a basic algebra and let Q be the ordinary quiver of A [11]: The set Q_0 of vertices is a fixed complete set of primitive orthogonal idempotents $\{e_1, \dots, e_q\}$ of A and for all i, j, $1 \leqq i, j \leqq q$ the set of arrows from e_i to e_j is in one-one correspondence with a K-basis of the space $e_j \operatorname{rad} Ae_i$ modulo the subspace $e_j \operatorname{rad}^2 Ae_i$, where rad A is the Jacobson radical of A. Then the map $\varepsilon : Q \to A$, defined by $\varepsilon(e_i) = e_i$ and $\varepsilon(\alpha) = a$ where a corresponds to the arrow α of Q, extends uniquely to an algebra epimorphism $\varepsilon : K[Q] \to A$, $K[Q]$ is the quiver algebra of Q [11], with kernel I contained in $\operatorname{rad}^2 K[Q]$. In this case we say that A is isomorphic to the *bounden quiver algebra* $K[Q, I]$ of the *bounden quiver* (Q, I). But the ideal I depends on the chosen representatives of the arrows and, in order to prove Theorem, we need a good choice of them. A path (oriented) of Q which does not belong to I is called a *nonzero path* of (Q, I).

We have the following consequence of the facts proved in [16].

Proposition. *Any representation-finite biserial algebra A being a factor of an hereditary algebra is isomorphic to a bounden quiver algebra $K[Q, I]$ where (Q, I) satisfies the following conditions*:

(i) *Q has no oriented cycles.*

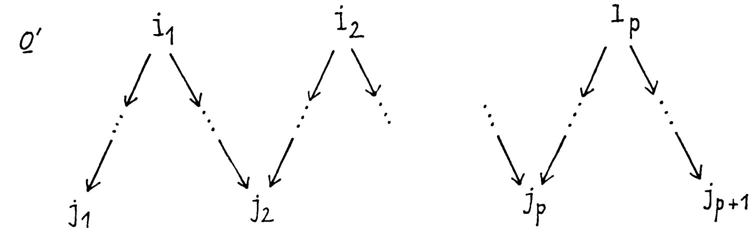
(ii) *The numbers of arrows starting respectively ending in any vertex of Q are bounded by* 2.

(iii) *For any arrow α of Q there is at most one arrow σ and at most one arrow γ such that $\alpha\sigma$ and $\gamma\alpha$ are not in I.*

(iv) *There is an upper bound for the length of paths in Q which are not in I.*

(v) *I is generated by paths and differences of two parallel paths in Q.*

Let A be a representation-finite biserial algebra and assume that $A = K[Q, I]$ where (Q, I) satisfies the conditions $(i)-(v)$ of Proposition. Then mod A is equivalent to the category $\operatorname{mod}_K(Q, I)$ of finite-dimensional K-representations of Q satisfying the relations given by I, from [15] the support of any indecomposable object X in $\operatorname{mod}_K(Q, I)$ is described by a picture



where the first or the last path may be of length 0, the paths

$$i_t \to \dots \to j_t \quad \text{and} \quad i_t \to \dots \to j_{t+1}\,, \quad t = 1, \dots, p\,,$$

are not in I, and $\dim_K X_i = 1$ for any point i from the support of X. Moreover, it is easy to see that there is a special family of local objects in $\operatorname{mod}_K(Q, I)$ if and only if there is a subquiver Q' of Q of the above form satisfying the following conditions:

(1) the paths $i_1 \to \ldots \to j_1$ and $i_p \to \ldots \to j_{p+1}$ are of length $\geqq 1$,

(2) $j_1 = j_{p+1}$ and $i \neq j$ for any pair $\{i, j\}$ of vertices of Q' different from $\{j_1, j_{p+1}\}$,

(3) the paths $i_t \to \ldots \to j_{t+1}$, $t = 1, \ldots, p$, are not in I.

Indeed, assume that Q' is a subquiver of Q, I satisfying the conditions (1)–(3). For $t = 1, \ldots, p$, let

$$v_{t,k} : m_{t,k} \to \ldots \to m_{t,k+1}$$

$k = 0, \ldots, q_t$, be the family of nonzero subpaths of

$$v_t : i_t \to \ldots \to j_t$$

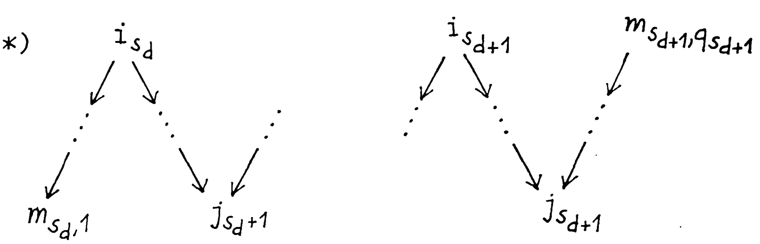
such that $m_{t,0} = i_t$, $m_{t,q_t+1} = j_t$, and inductively $v_{t,k}$ is the maximal nonzero (oriented) subpath of v_t starting in $m_{t,k}$. Since A is representation-finite, from [16, Theorem 1] follows that for at least one t, $q_t \geqq 1$. Let $s_1, \ldots, s_h$, where $s_d < s_{d+1}$, be all integers from $\{1, \ldots, p\}$ with $q_{s_d} \geqq 1$. Without loss of generality we may assume that $s_1 = 1$, (Q' is a cycle). For each $t = 1, \ldots, p$, let L_{i_t} be the local nonuniserial representation of Q, I with support

$$m_{t,1} \leftarrow \ldots \leftarrow i_t \to \ldots \to j_{t+1} .$$

Observe that $\operatorname{soc}(L_{i_t})$ is a direct sum of two simple representations given by the vertices $m_{t,1}$ and j_{t+1}. Further, for each, $d = 1, \ldots, h$, let $U_{s_d,j}$, $0 \leqq j \leqq q_{s_d} - 1$ be the representation of (Q, I) whose support equals to $v_{s_d, q_{s_d} - j}$. Then $U_{s_d,j}$, $0 \leqq j \leqq q_{s_d} - 1$, $1 \leqq d \leqq h$, are uniserial representations of (Q, I), for $j \leqq q_{s_d} - 2$, $\operatorname{top}(U_{s_d,j}) = \operatorname{soc}(U_{s_d,j+1})$, and $\operatorname{soc}(L_{i_{s_d}}) = \operatorname{top}(U_{s_d,q_{s_d}-1}) \oplus S_{j_{s_d+1}}$, where, for $t = 1, \ldots, p$, S_{j_t} denotes the simple representation of (Q, I) given by the vertex j_t. Now it clear that we have a special family of local representations formed by the local representations $L_{i_1}, \ldots, L_{i_p}$, uniserial representations $U_{s_d,j}$, $0 \leqq j \leqq q_{s_d} - 1$, $1 \leqq d \leqq h$, and the simple representations S_{j_t}, for all $t \neq s_1, \ldots, s_h$. The fact that any special family of local representations of (Q, I) is given by a subquiver Q' of Q, satisfying the conditions (1)–(3) follows from the description of supports of indecomposable representations of (Q, I).

Now suppose that is a special family of local A-modules. Then there is a subquiver Q' of (Q, I) satisfying the conditions (1)–(3) and, in our notations, let X_{s_d}, $1 \leqq d \leqq h$, be the indecomposable representation of (Q, I) with support

(*)



where $s_{h+1} = s_1$. Let us observe that there are nonzero maps $U_{s_d, q_{s_d}-1} \to \operatorname{soc}(X_{s_d})$ and $X_{s_d} \to \operatorname{top}(U_{s_{d+1},0})$. Then, since $\operatorname{top}(U_{s_d,j}) = \operatorname{soc}(U_{s_d,j+1})$ for $j \leqq q_{s_d-2}$, and $\operatorname{soc} X_{s_d}$ contains $\operatorname{top}(U_{s_d,q_{s_d}-1})$ as a direct summand, there are chains of nonzero maps

$$X_{s_d} \to \operatorname{top}(U_{s_{d+1},0}) \to X_{s_{d+1}} \quad \text{if} \quad q_{s_d} = 1\,,$$

and

$$X_{s_d} \to U_{s_{d+1},1} \to \ldots \to U_{s_{d+1},q_{s_{d+1}}-1} \to X_{s_{d+1}}\,, \quad \text{if} \quad q_{s_d} \geqq 2$$

$d = 1, \ldots, h$. Consequently there is a chain of nonzero maps

$$X_{s_1} = Y_1 \to \ldots \to Y_m \to Y_{m+1} = X_{s_1}$$

with Y_j's indecomposables in $\operatorname{mod}_K(Q, I)$, and from [1, 5], $\Gamma(A)$ contains an oriented cycle. Thus (i) implies (ii).

Conversely assume that $\Gamma(A)$ contains an oriented cycle. Then there is a chain of nonzero maps

$$X_1 \xrightarrow{f_1} X_2 \to \ldots \to X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} = X_1$$

where all X_i are indecomposable objects in $\operatorname{mod}_K(Q, I)$. Let n be the minimal number with this property. The $f_{i+1} f_i = 0$ for $i = 1, \ldots, n$, and without loss of generality we may assume that any X_i cannot be exchanged by any its proper submodule or factormodule. Then all representations X_i are nonsimple and from the description of supports of indecomposable objects in $\operatorname{mod}_K(Q, I)$ follows that the image of any f_i is simple. We claim that at least one of the modules X_i is nonuniserial. Indeed, if all X_i's are uniserials, the projective covers $P(X_i)$ of X_i are indecomposable modules and the maps f_i induce nonzero maps $q_i : P(X_i) \to P(X_{i+1})$. Hence we get a cycle

$$P(X_1) \xrightarrow{g_1} P(X_2) \to \ldots \to P(X_n) \xrightarrow{g_n} P(X_1)$$

of nonzero maps between indecomposable projective A-modules and this is a contradiction with our assumption that A is a factor of an hereditary algebra (see [14]). Consequently, one of the modules X_i, say X_n, is nonuniserial. Then there is a sequence of integers $1 \leqq s_1 < \ldots < s_h = n$ such that X_{s_j}, $j = 1, \ldots, h$, are all nonuniserial modules in the family $X_1, \ldots, X_n$. Since the image of any f_i is simple, using our description of supports of indecomposable objects in $\operatorname{mod}_K(Q, I)$, it is not hard to check that for any $i = 1, \ldots, h$

$$X_{s_i} = V_i + \sum_{j=1}^{r_i} L_{i,j}$$

where V_i is a nonsimple uniserial subrepresentation of X_{s_i}, $L_{i,j}$ are nonisomorphic nonuniserial local subrepresentations of X_{s_i},

$$\begin{aligned}
&\operatorname{soc}(L_{i,j}) = S_{i,j} \oplus S_{i,j+1} \quad \text{for} \quad 1 \leqq j < r_i\,,\\
&\operatorname{soc}(L_{i,r_i}) = S_{i,r_i} \oplus \operatorname{soc}(V_i)\\
&\ker(f_{s_i}) = \operatorname{rad}(V_i) + \sum_{j=1}^{r_i} L_{i,j}\\
&\operatorname{im}(f_{s_i-1}) = S_{i,1}\,,
\end{aligned}$$

where all $S_{i,j}$ are simple representations. Thus the support of any X_{s_i} is of the form (*). For each $1 \leqq j \leqq h$, let $p_j = \sum_{i=1}^{j} r_i$, and put $p = p_h$ and $m = p + n - h$. Define a sequence $i_1, \ldots, i_p$ of integers with $1 \leqq i_1 < \ldots < i_p = m$ by the equalities

$$i_t = s_{j+1} + t - j - 1 \quad \text{for} \quad p_j < t \leqq p_{j+1}\,, \quad j = 0, \ldots,$$

$h - 1$, where $p_0 = 0$, and put

$$L_{i_t} = L_{j+1,t-p_j}\,.$$

Finally, for any $1 \leqq r \leqq m$, let U_r be the uniserial module defined in the following way:

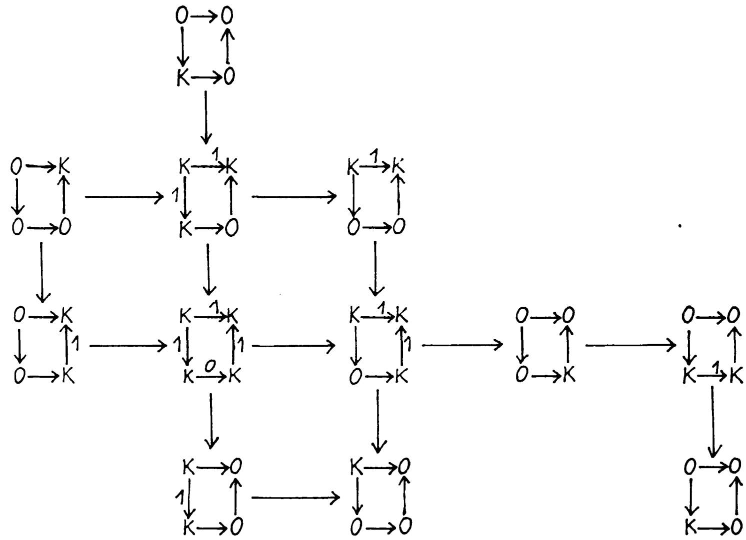
$$\begin{aligned}
U_1 &= V_h\,,\\
U_r &= V_k\,, \quad \text{for} \quad r = s_k + p_k - k + 1\,, \quad 1 \leqq k \leqq h - 1\,,\\
U_r &= S_{k,r-s_k-p_{k-1}+k}\,, \quad \text{for } s_k + p_{k-1} - k + 1 < r \leqq s_k + p_k - k\,,\\
&\quad\ 1 \leqq k \leqq h\,,\\
U_r &= X_{r-p_k+k-1} \quad \text{for} \quad s_k + p_k - k + 1 < r \leqq s_{k+1} + p_k - k\,, \quad 0 \leqq k < h
\end{aligned}$$

where $s_0 = 0$. From our definitions simply follows that the uniserial modules $U, \ldots$ $\ldots, U_m$ together with the nonuniserial local modules $L_{i_1}, \ldots, L_{i_p}$, form a special family of local A-modules. Consequently (ii) implies (i) and this finishes the proof of Theorem.

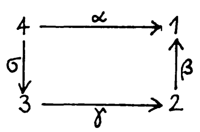
Remark. It is not hard to modify our proof of the implication $(i) \Rightarrow (ii)$ for the case of arbitrary representation-finite algebra A and prove that if $\Gamma(A)$ has no oriented cycles then there is no special family of local A-modules. It would be interesting to know if the converse implication is true.

We end the paper with some example showing that in general there are representation-finite algebras A for which any indecomposable module is uniquely determined by the composition factors but $\Gamma(A)$ contains an oriented cycle.

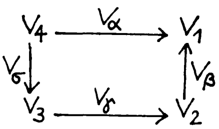
Let us consider the bounden quiver algebra $A = K[Q, I]$ where Q is the quiver



and I is generated by two paths $\beta\gamma$ and $\gamma\sigma$. Then $\mathrm{mod}_K(Q, I)$ is the category of finite-dimensional K-vector spaces and K-linear maps



satisfying the relations $V_\gamma V_\sigma = 0$ and $V_\beta V_\gamma = 0$, and from [10, 11] $\Gamma(A)$ has the form



We see that indecomposable A-modules are uniquely determined by their composition factors and there is an oriented cycle in $\Gamma(A)$.

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