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Strong decomposability of ultrafilters on cardinals with countable cofinality

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We shall show that for each cardinal κ with countable cofinality and for each ultrafilter $\mathscr{U} \in \mathrm{U}(\kappa)$, $\mathscr{U}$ is a κ^+-point in $\mathrm{U}(\kappa)$.

Dokážeme, že pro každý kardinál κ se spočetnou kofinalitou a každý ultrafiltr $\mathscr{U} \in \mathrm{U}(\kappa)$, $\mathscr{U}$ je κ^+-bodem v $\mathrm{U}(\kappa)$.

Покажем, что для всех кардиналов κ таких что $cf(\kappa) = \omega$ и для всех ультрафильтров $\mathscr{U} \in \mathrm{U}(\kappa)$, $\mathscr{U}$ является κ^+-точкой в $\mathrm{U}(\kappa)$.

0. Introduction

0.1. Definition. Let X be a topological space, τ a cardinal number, $x \in X$. The point x is called a τ-point provided there is a family $\{V_\gamma : \gamma < \tau\}$ satisfying: For each $\gamma < \tau$, the set V_γ is open in X and $x \in \overline{V}_\gamma$, further, if $\gamma < \delta < \tau$, then $V_\gamma \cap V_\delta = \emptyset$.

The aim of the present paper is to prove the following.

0.2. Theorem. Let $\kappa > \omega$ be a cardinal number, $\mathrm{cf}\,\kappa = \omega$. Then every point in $\mathrm{U}(\kappa)$ is a κ^+-point.

This theorem may be regarded as a further step towards the solution of a problem posed by W. W. Comfort and N. B. Hindman in [CH]: Given an arbitrary cardinal κ, is each point in $\mathrm{U}(\kappa)$ a κ^+-point? Let us mention that the answer is affirmative for all regular cardinals, as proved in [BV] and [BS]. We omit remarks concerning the history of this problem; an interested reader can find them in those two papers.

Since the whole paper is devoted to the proof of Theorem 0.2, let us mention few words on its organization. In § 1, the elementary facts are summarized and a notion of strongly decomposable family is introduced. In the case of an ultrafilter, it is a combinatorial restatement of being a τ-point. Our strategy is to give enough examples of strongly κ^+-decomposable families. This is the contents of §§ 2 and 3. Then in the

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last paragraph we shall show that each ultrafilter in $U(\kappa)$ happens to be a special case of at least one family discussed earlier.

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1. Notation and basics

The notation used throughout the paper is the common one, used e.g. in [CN]. Small Greek letters $\kappa, \lambda, \mu, \tau$ (sometimes with subscripts) will stand for cardinal numbers, ϕ, ψ, denote mappings, the other small Greek letters denote ordinals, m, n, i, j are natural numbers. $U(\kappa)$ is a topological space of all uniform ultrafilters on κ, i.e. $U(\kappa)$ is the Stone space of the Boolean algebra $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$. If $\mathscr{F} \subseteq [\kappa]^\kappa$, then $\langle\langle \mathscr{F} \rangle\rangle$ denotes the (possibly improper) filter generated by $\mathscr{F}$.

1.1. Definition. A family $\mathscr{A} \subseteq [\mu]^\tau$ is almost-disjoint, if $|A \cap B| < \tau$ for any two distinct members A, B of $\mathscr{A}$.

1.2. Definition. Let $\mathscr{M} \subseteq [\kappa]^\kappa$. A family $\mathscr{M}$ is called strongly τ-decomposable, provided that there is an almost-disjoint family $\mathscr{A} \subseteq [\kappa]^\kappa$ which can be partitioned to $\mathscr{A} = \bigcup\{\mathscr{A}_\gamma : \gamma < \tau\}$ in such a way that for each $M \in \mathscr{M}$ and for each $\gamma < \tau$ there is some $A \in \mathscr{A}_\gamma$ with $|A \cap M| = \kappa$.

1.3. Observation. Let $\mathscr{U} \in U(\kappa)$. Then $\mathscr{U}$ is a τ-point in $U(\kappa)$ if and only if $\mathscr{U}$ is strongly τ-decomposable.

□ Obvious. □

For the sake of brevity, the phrases like "$\mathscr{A} = \bigcup\{\mathscr{A}_\gamma : \gamma < \tau\}$ witnesses the strong τ-decomposability of $\mathscr{M}$" or "$\mathscr{M}$ has a strong τ-decomposition $\mathscr{A} = \bigcup\{\mathscr{A}_\gamma : \gamma < \tau\}$", etc., will always denote that $\mathscr{A}$ satisfies 1.2.

1.4. Observation. Let $\mathscr{M} \subseteq [\kappa]^\kappa$, $\mathscr{M} = \bigcup\{\mathscr{M}_\xi : \xi < \mu\}$ and suppose that each $\mathscr{M}_\xi$ is strongly τ-decomposable by $\mathscr{A}(\xi) = \bigcup\{\mathscr{A}_\gamma(\xi) : \gamma < \tau\}$. If for each $\xi < \zeta < \mu$ and for each $A \in \mathscr{A}(\xi)$, $B \in \mathscr{A}(\zeta)$, $|A \cap B| < \kappa$, then $\mathscr{M}$ is strongly τ-decomposable, too. In particular, if $\{\bigcup\mathscr{A}(\xi) : \xi < \mu\}$ is an almost-disjoint family on κ, then $\mathscr{M}$ is strongly τ-decomposable.

□ Indeed, denote $\mathscr{A}_\gamma = \bigcup\{\mathscr{A}_\gamma(\xi) : \xi < \mu\}$ for $\gamma < \tau$, $\mathscr{A} = \bigcup\{\mathscr{A}_\gamma : \gamma < \tau\}$. It is clear that this works. □

1.5. Definition. Let $\mathscr{F} \subseteq [\kappa]^\kappa$. The family $\mathscr{F}$ is uniformly centered, if $|F| = \kappa$ for each $F \in \langle\!\langle \mathscr{F} \rangle\!\rangle$. The family $\mathscr{F}$ is countably generated, if there is a subfamily (so-called family of generators) $\{G_n : n < \omega\} \subseteq \langle\!\langle \mathscr{F} \rangle\!\rangle$ such that for each $F \in \langle\!\langle \mathscr{F} \rangle\!\rangle$ there is an $n < \omega$ with $|G_n - F| < \kappa$.

1.6. Proposition. Suppose $\omega = \operatorname{cf} \kappa$. Let $\mathscr{F} \subseteq [\kappa]^\kappa$ be a countably generated uniformly centered family. Then there is a family $\mathscr{A} = \mathscr{A}(\mathscr{F}) \subseteq [\kappa]^\kappa$ with the following properties:

(i) $\mathscr{A}$ is almost-disjoint on κ;

(ii) for each $A \in \mathscr{A}$ and for each $F \in \langle\!\langle \mathscr{F} \rangle\!\rangle$, $|A - F| < \kappa$;

(iii) for each $M \in [\kappa]^\kappa$ such that $\mathscr{F} \cup \{M\}$ is uniformly centered there is an $A \in \mathscr{A}$ with $|A \cap M| = \kappa$.

□ Let $\{F_n : n < \omega\} \subseteq \langle\!\langle \mathscr{F} \rangle\!\rangle$ be a family of generators of $\mathscr{F}$, let $G_n = \bigcap\{F_i : 0 \leqq i \leqq n\}$. Choose any $\mathscr{A} \subseteq [\kappa]^\kappa$ which is maximal with respect to: $\mathscr{A}$ is almost-disjoint and $|A - G_n| < \kappa$ for each $A \in \mathscr{A}$ and $n < \omega$. Then $\mathscr{A}$ clearly satisfies (i). If $F \in \langle\!\langle \mathscr{F} \rangle\!\rangle$, let G_n be such that $|G_n - F| < \kappa$. Then for $A \in \mathscr{A}$, $A - F \subseteq (A - G_n) \cup (G_n - F)$, which shows (ii). If $\mathscr{F} \cup \{M\}$ is uniformly centered, then fix a sequence of cardinals $\kappa_n \uparrow \kappa$ and choose inductively $K_n \subseteq M \cap G_n$ with $|K_n| = \kappa_n$. Then for $K = \bigcup\{K_n : n < \omega\}$ we have $K \subseteq M$, $|K| = \kappa$, $|K - G_n| < \kappa$ for each G_n, thus for some $A \in \mathscr{A}$, $|K \cap A| = \kappa$ according to the maximality of $\mathscr{A}$. For this particular A, $|M \cap A| = \kappa$, too, hence (iii) is proved. □

1.7. Proposition. Suppose $\omega = \operatorname{cf} \kappa$. Let Φ be a collection of countably generated uniformly centered families on κ and suppose that for any two distinct $\mathscr{F}, \mathscr{G} \in \Phi$ there exist $F \in \langle\!\langle \mathscr{F} \rangle\!\rangle$ and $G \in \langle\!\langle \mathscr{G} \rangle\!\rangle$ with $|F \cap G| < \kappa$. If for $\mathscr{F} \in \Phi$ the family $\mathscr{A}(\mathscr{F}) \subseteq [\kappa]^\kappa$ is as in 1.6, then $\bigcup\{\mathscr{A}(\mathscr{F}) : \mathscr{F} \in \Phi\}$ is almost-disjoint.

□ Indeed, for $A, B \in \mathscr{A}(\mathscr{F})$ we have $|A \cap B| < \kappa$ by 1.6. (i), for $A \in \mathscr{A}(\mathscr{F})$ and $B \in \mathscr{A}(\mathscr{G})$ we have $A \cap B \subseteq (A - F) \cup (B - G) \cup (F \cap G)$, where $|A - F| < \kappa$, $|B - G| < \kappa$, by 1.6 (ii) and $|F \cap G| < \kappa$ by the assumption. □

We shall need one combinatorial fact concerning uniform filters on ω. The Balcar-Vojtáš Theorem states that for every uniform filter $\mathscr{F}$ on ω there is an almost-disjoint family $\mathscr{B} \subseteq [\omega]^\omega$ such that for each $F \in \mathscr{F}$ there is a $B \in \mathscr{B}$ with $B \subseteq F$. The full proof can be found in [BV]. Though the next lemma is a bit stronger, its proof coincides (modulo some slight modifications) with the one given in [BV], so we omit it.

1.8. Lemma. For each uniform filter $\mathscr{F}$ on ω there is an almost-disjoint family $\mathscr{B} \subseteq [\omega]^\omega$ such that for each sub-family $\{F_n : n < \omega\} \subseteq \mathscr{F}$ there is a $B = \{b_0 < b_1 < b_2 < \ldots\} \in \mathscr{B}$ with $B - F_n \subseteq \{b_i : i < n\}$, for each $n < \omega$. □

2. The cardinal λ and strongly λ-decomposable families

In this paragraph we shall permanently assume that $\omega = \operatorname{cf} \kappa$ and $2^\omega < \kappa$.

2.1. Definition. Let $Q \in [\omega]^\omega$, let $\{\beta_n : n \in Q\}$ be a sequence of ordinals satisfying

$$(*)\quad \begin{cases} (\forall n \in Q)(\beta_n < \kappa), \quad \text{and} \\ (\forall n < m,\ n, m \in Q)(\beta_n < \beta_m \,\&\, \operatorname{cf} \beta_n < \operatorname{cf} \beta_m), \quad \text{and} \\ \sup \{\operatorname{cf} \beta_n : n \in Q\} = \kappa . \end{cases}$$

Consider $\Pi\{\beta_n : n \in Q\}$ ordered in a usual manner mod fin, i.e. $f \leqq g$ iff $|\{n \in Q : f(n) > g(n)\}| < \omega$.

Define

$\lambda\{\beta_n, Q\} = \min \{|H| : H \subseteq \Pi\{\beta_n : n \in Q\} \,\&\, H$ has no upper bound$\}$,

$\lambda = \min \{\lambda\{\beta_n, Q\} : Q \in [\omega]^\omega, \{\beta_n : n \in Q\}$ satisfies $(*)\}$.

For the rest of the paper, the letter λ will have the meaning just defined. Let us mention several trivial observations without giving proofs.

2.2. Observation. Suppose $\{\beta_n : n < \omega\}$ to satisfy 2.1. $(*)$, let $Q, Q' \in [\omega]^\omega$. Then

(a) $\kappa < \lambda \leqq \lambda\{\beta_n, Q\} \leqq \kappa^\omega$.

(b) λ as well as $\lambda\{\beta_n, Q\}$ are regular cardinals.

(c) If $Q' \subseteq Q$, if $\{f_\xi : \xi < \lambda\{\beta_n, Q'\}\} \subseteq \Pi\{\beta_n : n \in Q\}$ and if the family $\{f_\xi \upharpoonright Q' : \xi < \lambda\{\beta_n, Q'\}\}$ has no upper bound in $\Pi\{\beta_n : n \in Q'\}$, then $\{f_\xi : \xi < \lambda\{\beta_n, Q'\}\}$ has no upper bound in $\Pi\{\beta_n : n \in Q\}$.

(d) If $|Q' - Q| < \omega$, then $\lambda\{\beta_n, Q\} \leqq \lambda\{\beta_n, Q'\}$.

(e) If $f \in \Pi\{\beta_n : n \in Q\}$ is given, then there is a family $\{g_\xi : \xi < \lambda\{\beta_n, Q\}\} \subseteq \Pi\{\beta_n : n \in Q\}$ satisfying the following:

(i) For each $n \in Q$ and each $\xi < \lambda\{\beta_n, Q\}$, we have $f(n) \leqq g_\xi(n)$;

(ii) for each $\xi < \eta < \lambda\{\beta_n, Q\} : g_\xi \leqq g_\eta$;

(iii) there is no upper bound for $\{g_\xi : \xi < \lambda\{\beta_n, Q\}\}$ in $\Pi\{\beta_n : n \in Q\}$.

2.3. Notation. Let $Q \in [\omega]^\omega$, let $\beta_n < \kappa$ for each $n \in Q$, let $f, g \in \Pi\{\beta_n : n \in Q\}$. We shall denote

$$\kappa(f, g) = \{\alpha < \kappa : (\exists n \in Q)(f(n) \leqq \alpha < g(n))\},$$
$$\kappa(f, \rightarrow) = \{\alpha < \kappa : (\exists n \in Q)(f(n) \leqq \alpha < \beta_n)\}.$$

Similarly, if $M \in [\kappa]^\kappa$, we shall denote

$$M(f, g) = \{\alpha \in M : (\exists n \in Q)(f(n) \leqq \alpha < g(n))\},$$
$$M(f, \rightarrow) = \{\alpha \in M : (\exists n \in Q)(f(n) \leqq \alpha < \beta_n)\}.$$

2.4. Definition. Suppose $\{\beta_n : n < \omega\}$ to satisfy 2.1. (*), let $Q \in [\omega]^\omega$ and let $\{g_\xi : \xi < \lambda\{\beta_n, Q\}\} \subseteq \Pi\{\beta_n : n \in Q\}$ satisfy 2.2. (e) with f defined by $f(0) = 0$, $f(n+1) = \beta_n$ for $n + 1 \in Q$.

For $M \in [\kappa]^\kappa$, let us denote $T(M) = \{n < \omega : \sup(M \cap \beta_n) = \beta_n\}$. We shall define

$\mathscr{I}^+\{\beta_n, g_\xi, Q\} = \{M \in [\kappa]^\kappa : |T(M) \cap Q| = \omega \,\&\, \{g_\xi \restriction T(M) \cap Q : \xi < \lambda\{\beta_n, Q\}\}$ has no upper bound in $\Pi\{\beta_n : n \in T(M) \cap Q\}\}$,

$\text{Big}\{\beta_n, Q\} = \{M \in [\kappa]^\kappa : |T(M) \cap Q| = \omega\}$,

$\text{Big}\{\beta_n\} = \text{Big}\{\beta_n, \omega\}$.

2.5. Lemma. Under the notation from 2.4 the family $\mathscr{I}^+\{\beta_n, g_\xi, Q\}$ is strongly λ-decomposable by $\mathscr{A} = \bigcup\{\mathscr{A}_\gamma : \gamma < \lambda\}$ with the following property: For each $A \in \mathscr{A}$ there is a $\xi < \lambda\{\beta_n, Q\}$ such that for each $\eta < \xi$, $|A - \kappa(g_\eta, g_\xi)| < \kappa$.

□ Put $\mathscr{I}^+ = \mathscr{I}^+\{\beta_n, g_\xi, Q\}$. Denote $S = \{\xi < \lambda\{\beta_n, Q\} : \operatorname{cf} \xi = \omega\}$. For $\xi \in S$, fix a strictly increasing sequence $\{\xi(n) : n < \omega\}$ of ordinals converging to ξ. Denote $\mathscr{F}(\xi) = \{\kappa(g_{\xi(n)}, g_\xi) : n < \omega\}$. For each $\mathscr{F}(\xi)$, if $\mathscr{F}(\xi)$ is uniformly centered, choose $\mathscr{A}(\xi) = \mathscr{A}(\mathscr{F}(\xi))$ having the properties 1.6, otherwise let $\mathscr{A}(\xi) = \emptyset$.

The set S is stationary in $\lambda\{\beta_n, Q\}$; applying Fodor's theorem, find a pairwise disjoint family $\{S_\gamma : \gamma < \lambda\}$ of subsets of S with each member stationary. Define $\mathscr{A}_\gamma = \bigcup\{\mathscr{A}(\xi) : \xi \in S_\gamma\}$, $\mathscr{A} = \bigcup\{\mathscr{A}_\gamma : \gamma < \lambda\}$.

We have to show that $\mathscr{A} = \bigcup\{\mathscr{A}_\gamma : \gamma < \lambda\}$ witnesses the strong λ-decomposability of $\mathscr{I}^+$.

First, notice that if $\eta < \xi$, $\xi, \eta \in S$, then there is some $i < \omega$ with $g_{\xi(i)} \geqq g_\eta$. Let $n_0 < \omega$ be such that $g_{\xi(i)}(n) \geqq g_\eta(n)$ whenever $n \geqq n_0$. Then $\kappa(g_{\xi(i)}, g_\xi) \in \mathscr{F}(\xi)$ and $\kappa(g_{\eta(0)}, g_\eta) \in \mathscr{F}(\eta)$, and

$$\begin{aligned}
&\kappa(g_{\xi(i)}, g_\xi) \cap \kappa(g_{\eta(0)}, g_\eta) \subseteq \\
&\subseteq \{\alpha < \kappa : (\exists n < \omega)(g_{\xi(i)}(n) \leqq \alpha < g_\xi(n) \,\&\, g_{\eta(0)}(n) \leqq \alpha < g_\eta(n))\} \subseteq \\
&\subseteq \{\alpha < \kappa : (\exists n < \omega)(g_{\xi(i)}(n) \leqq \alpha < g_\xi(n) \,\&\, \beta_{n-1} \leqq \alpha < g_\eta(n))\} \subseteq \\
&\subseteq \{\alpha < \kappa : (\exists n < \omega)(g_{\xi(i)}(n) \leqq \alpha < g_\eta(n))\} \subseteq \bigcup\{\beta_n : n < n_0\} \subseteq \beta_{n_0},
\end{aligned}$$

thus by 1.7, the family $\mathscr{A}$ is almost-disjoint. The disjointness of $\mathscr{A}_\gamma$'s follows from the disjointness of S_γ's immediately.

Fix $M \in \mathscr{I}^+$, $\gamma < \lambda$. We have to show that there is an $A \in \mathscr{A}_\gamma$ with $|M \cap A| = \kappa$. To abbreviate the notation, denote for a moment $M(\xi, \eta) = M(g_\xi, g_\eta)$ and $T = T(M) \cap Q$.

Claim 1. For each $\xi < \lambda\{\beta_n, Q\}$, the set $M(\xi, \rightarrow)$ belongs to $\mathscr{I}^+$. Indeed, if $M \cap \beta_n$ is cofinal in β_n, then $M \cap \beta_n - g_\xi(n)$ is cofinal in β_n, too. Thus $Q \cap T(M(\xi, \rightarrow)) = T$.

Claim 2. For each $\xi < \lambda\{\beta_n, Q\}$ there is an $\eta < \lambda\{\beta_n, Q\}$ with $|M(\xi, \eta)| = \kappa$.

Denote $\tau_n = \operatorname{cf} \beta_n$. Then for each $n \in T$, $M(\xi, \to) \cap \beta_n$ is cofinal in β_n by claim 1, so let us define a function $h \in \Pi\{\beta_n : n \in T\}$ as follows: $h(n) = \min \{\alpha < \beta_n : : |M(\xi, \to) \cap \alpha| \geqq \tau_{n-1}\}$.

Since h cannot be an upper bound for the family $\{g_\xi \restriction T : \xi < \lambda\{\beta_n, Q\}\}$, there is some $\eta < \lambda\{\beta_n, Q\}$ such that the set $P = \{n \in T : h(n) \leqq g_\eta(n)\}$ is infinite. Let $\tau < \kappa$ be arbitrary. Then there is some $i < \omega$ with $\tau_i > \tau$. Choose $n \in P$ greater than $i + 1$. Then $|M \cap g_\eta(n) - g_\xi(n)| = |M(\xi, \to) \cap g_\eta(n)| \geqq |M(\xi, \to) \cap h(n)| > \tau_i > \tau$. Since τ was arbitrary, $|M(\xi, \eta)| \geqq \sup \{\tau : \tau < \kappa\} = \kappa$.

Claim 3. For each $\xi_0 < \lambda\{\beta_n, Q\}$ there is an $\eta < \lambda\{\beta_n, Q\}$ such that $\xi_0 < \eta$ and $|M(\xi, \eta)| = \kappa$ whenever $\xi < \eta$.

Indeed, applying inductively claim 2, we obtain a sequence $\xi_0 < \eta_0 < \eta_1 < \ldots \ldots < \eta_n < \ldots < \lambda\{\beta_n, Q\}$ with $|M(\eta_i, \eta_{i+1})| = \kappa$ for each $i < \omega$. Then it suffices to put $\eta = \sup \{\eta_i : i < \omega\}$.

Denote

$$C(M) = \{\eta < \lambda\{\beta_n, Q\} : (\forall \xi < \eta)\, |M(\xi, \eta)| = \kappa\}\,.$$

Claim 4. The set $C(M)$ is closed unbounded in $\lambda\{\beta_n, Q\}$.

By virtue of claim 3, $C(M)$ is unbounded. The closedness of $C(M)$ is an immediate consequence of an obvious fact that if $\xi < \zeta < \eta < \lambda\{\beta_n, Q\}$, then $M(\xi, \zeta) \cup \cup M(\zeta, \eta) \subseteq M(\xi, \eta) \cup Z$, for some suitable $Z \in [\kappa]^{<\kappa}$.

Now we are ready to finish the proof. If $\gamma < \lambda$ is arbitrary, then there is a $\xi \in S_\gamma \cap \cap C(M)$, since S_γ is stationary and $C(M)$ closed unbounded. Then for $F \in \langle\langle \mathscr{F}(\xi) \rangle\rangle$, $|M \cap F| = \kappa$. Hence by 1.6 (iii), $|M \cap A| = \kappa$ for some $A \in \mathscr{A}(\xi) \subseteq \mathscr{A}_\gamma$. □

Let us remark that the preceding lemma was the essential part of this paragraph. All what follows are just more complicated variations on this example. They culminate in Lemma 2.8, which is a substantial tool for proving 0.2.

Notice that up to now we did not need the assumption $2^\omega < \kappa$. But the cardinal 2^ω has to be small in our next example.

2.6. Lemma. Under the notation from 2.4, the family $\mathrm{Big}\,\{\beta_n\}$ is strongly λ-decomposable.

□ Denote $\mathfrak{c} = 2^\omega$. Enumerate $[\omega]^\omega = \{Q_\iota : \iota < \mathfrak{c}\}$. We wish to apply Lemma 2.5, but for doing this, we need also families $\{g_{\xi,\iota} : \xi < \lambda\{\beta_n, Q_\iota\}\}$ each without an upper bound in $\Pi\{\beta_n : n \in Q_\iota\}$, hence some preparatory steps are necessary.

Using a transfinite induction to $\mathfrak{c}$, we shall define for several $\iota < \mathfrak{c}$ the function f_ι, set of functions $\{g_{\xi,\iota} : \xi < \lambda\{\beta_n, Q_\iota\}\}$ and a set of indices $I(\iota)$ as follows:

Let $\nu < \mathfrak{c}$ and suppose all $\iota < \nu$ have been considered.

CASE 1. $\nu = 0$ or if $\iota < \nu$ and $\iota \in I(\iota)$, then $|Q_\iota \cap Q_\nu| < \omega$.

Define $f_\nu(0) = 0$, $f_\nu(n + 1) = \beta_n$ for $n + 1 \in Q_\nu$ and choose any family

$\{g_{\xi,\nu} : \xi < \lambda\{\beta_n, Q_\nu\}\}$ satisfying 2.2(e) with f_ν considered as f. Define $I(\nu) = \bigcup\{I(\iota) : \iota < \nu\} \cup \{\nu\}$.

CASE II. There is a $\iota < \nu$ with $\iota \in I(\iota)$ such that the family $\{g_{\xi,\iota} \upharpoonright Q_\iota \cap Q_\nu : \xi < \lambda\{\beta_n, Q_\iota\}\}$ has no upper bound in $\Pi\{\beta_n : n \in Q_\iota \cap Q_\nu\}$, and $|Q_\iota \cap Q_\nu| = \omega$.

Define $I(\nu) = \bigcup\{I(\iota) : \iota < \nu\}$, the other notions will remain undefined.

CASE III. The remaining. Since not case I, there is a $\iota < \nu$ with $|Q_\iota \cap Q_\nu| = \omega$ and $\iota \in I(\iota)$. But since not case II, too, for each such ι there is a function $h_\iota \in \Pi\{\beta_n : n \in Q_\iota \cap Q_\nu\}$, which is an upper bound for $\{g_{\xi,\iota} \upharpoonright Q_\iota \cap Q_\nu : \xi < \lambda\{\beta_n, Q\}\}$. Let $\tilde{h}_\iota \supseteq h_\iota$, $\tilde{h}_\iota \in \Pi\{\beta_n : n \in Q\}$ be an arbitrary extension of h_ι. Since $\nu < \mathfrak{c} < \kappa < \lambda\{\beta_n, Q_\nu\}$, the family $\{\tilde{h}_\iota : \iota < \nu$ and $\tilde{h}_\iota$ is defined$\}$ has an upper bound $f_\nu \in \Pi\{\beta_n : n \in Q_\nu\}$, we may and shall assume that $f_\nu(n + 1) \geqq \beta_n$ for each $n + 1 \in Q_\nu$.

Choose a family $\{g_{\xi,\nu} : \xi < \lambda\{\beta_n, Q\}\}$ satisfying 2.2(e) with f_ν in the rôle of f. Finally, put $I(\nu) = \bigcup\{I(\iota) : \iota < \nu\} \cup \{\nu\}$.

Having completed the inductive definitions, let us remark that $I = \bigcup\{I(\iota) : \iota < \mathfrak{c}\}$ is just the list of those indices $\iota < \mathfrak{c}$ for which f_ι and $g_{\xi,\iota}$ were defined.

As might be expected, Big $\{\beta_n\} = \bigcup\{\mathscr{I}^+\{\beta_n, g_{\xi,\iota}, Q_\iota\} : \iota \in I\}$.

To see this, let $M \in$ Big $\{\beta_n\}$, let $\nu < \mathfrak{c}$ be the one with $Q_\nu = T(M)$. If $\nu \in I$, then $M \in \mathscr{I}^+\{\beta_n, g_{\xi,\nu}, Q_\nu\}$ automatically. But if $\nu \notin I$, then according to case II, there is some $\iota < \nu$, $\iota \in I$ with $M \in \mathscr{I}^+\{\beta_n, g_{\xi,\iota}, Q_\iota\}$ – compare simply the condition in case II with the definition of $\mathscr{I}^+\{\beta_n, g_\xi, Q\}$.

For $\iota \in I$, apply Lemma 2.5, let $\mathscr{A}(\iota) = \bigcup\{\mathscr{A}_\gamma(\iota) : \gamma < \lambda\}$ be the result. Then $\mathscr{A} = \bigcup\{\mathscr{A}_\gamma : \gamma < \lambda\}$, where $\mathscr{A}_\gamma = \bigcup\{\mathscr{A}_\gamma(\iota) : \iota \in I\}$, witnesses the strong λ-decomposability of Big $\{\beta_n\}$. In view of the preceding, we need only to check that $\mathscr{A}$ is almost-disjoint.

Choose $\iota < \nu$, $\iota, \nu \in I$ and let $A \in \mathscr{A}(\iota)$, $B \in \mathscr{A}(\nu)$. Then, by 2.5, there is some $\sigma < \lambda\{\beta_n, Q_\iota\}$ such that $|A - \kappa(g_{\zeta,\iota}, g_{\sigma,\iota})| < \kappa$ whenever $\zeta < \sigma$, and there is some $\xi < \lambda\{\beta_n, Q_\nu\}$ such that $|B - \kappa(g_{\eta,\nu}, g_{\xi,\nu})| < \kappa$ whenever $\eta < \xi$. In particular, $|A - \kappa(f_\iota, g_{\sigma,\iota})| < \kappa$ and $|B - \kappa(f_\nu, g_{\xi,\nu})| < \kappa$.

Since $A \cap B \subseteq (A - \kappa(f_\iota, g_{\sigma,\iota})) \cup (B - \kappa(f_\nu, g_{\xi,\nu})) \cup (\kappa(f_\iota, g_{\sigma,\iota}) \cap \kappa(f_\nu, g_{\xi,\nu}))$, we need to check that $|\kappa(f_\iota, g_{\sigma,\iota}) \cap \kappa(f_\nu, g_{\xi,\nu})| < \kappa$ only.

If $|Q_\iota \cap Q_\nu| < \omega$, then there is some $\beta < \kappa$ with $\beta > \max\{\beta_n : n \in Q_\iota \cap Q_\nu\}$. Clearly, $\kappa(f_\iota, g_{\sigma,\iota}) \cap \kappa(f_\nu, g_{\xi,\nu}) \subseteq \bigcup\{\beta_n : n \in Q_\iota \cap Q_\nu\} \subseteq \beta$.

If $|Q_\iota \cap Q_\nu| = \omega$, then case II is ruled out by $\nu \in I$, hence according to case III, $f_\nu \upharpoonright Q_\iota \cap Q_\nu \geqq g_{\sigma,\iota} \upharpoonright Q_\iota \cap Q_\nu$. Hence for some $n_0 < \omega$, $f_\nu(n) \geqq g_{\sigma,\iota}(n)$ whenever $n \in Q_\iota \cap Q_\nu$, $n > n_0$. Hence $\kappa(f_\iota, g_{\sigma,\iota}) \cap \kappa(f_\nu, g_{\xi,\nu}) \subseteq \bigcup\{\beta_n : n \leqq n_0\} \subseteq \beta_{n_0} < \kappa$.

Thus $\mathscr{A}$ is almost-disjoint, which completes the proof. □

2.7. Definition. Let $\mathscr{D} \subseteq [\kappa]^\kappa$. The family $\mathscr{D}$ will be called helpful if for each $D \in \mathscr{D}$ there is a sequence of ordinals $\beta_n(D)$ $(n < \omega)$, which satisfies 2.1(*) and a mapping $f_D \in \Pi\{\beta_n(D) : n < \omega\}$ such that $f_D(n) \geqq \beta_{n-1}$ for each n (β_{-1} is understood to equal 0) and $D = \bigcup\{\beta_n(D) - f_D(n) : n < \omega\}$.

2.8. Lemma. Let $\mathscr{D} \subseteq [\kappa]^\kappa$ be helpful and almost-disjoint. Then $\bigcup\{\text{Big}\{\beta_n(D)\} : D \in \mathscr{D}\}$ is strongly λ-decomposable.

□ According to the definition of Big $\{\beta_n(D)\}$, if $M \in$ Big $\{\beta_n(D)\}$, then $M \cap D \in$ Big $\{\beta_n(D)\}$, too. Thus we may assume that if $\mathscr{A}(D) = \bigcup\{\mathscr{A}_\gamma(D) : \gamma < \lambda\}$ witnesses the strong λ-decomposability of Big $\{\beta_n(D)\}$, then $\bigcup\mathscr{A}(D) \subseteq D$. It remains to apply 1.4. □

3. Strongly κ^ω-decomposable families

The present paragraph contains only one statement. Here we have no assumptions concerning 2^ω, only $\omega = \text{cf}\,\kappa < \kappa$ is assumed. But let us start with an explicit formulation of what we understand by $\mathscr{F}^+$ and $\mathscr{F}^c$ for a uniform filter $\mathscr{F}$ on κ, for it may perhaps slightly differ from the usually adopted meaning.

3.1. Notation. Let $\mathscr{F} \subseteq [\kappa]^\kappa$ be a uniform filter. Then

$$\mathscr{F}^+ = \{X \in \mathscr{P}(\kappa) : (\forall F \in \mathscr{F})(|X \cap F| = \kappa)\},$$
$$\mathscr{F}^c = \{X \in \mathscr{P}(\kappa) : (\exists F \in \mathscr{F})(|X \cap F| < \kappa)\}.$$

3.2. Lemma. Let $\{\kappa_n : n < \omega\}$ be a strictly increasing sequence of regular uncountable cardinals with $\sup\{\kappa_n : n < \omega\} = \kappa$. For each $n < \omega$, let $\mathscr{R}_n = \{R^n_\xi : \xi < \kappa_n\}$ be a partition of κ, and suppose that the family $\mathscr{F} = \langle\langle\{\bigcup\{R^n_\xi : \eta \leqq \xi < \kappa_n\} : n < \omega, \eta < \kappa_n\}\rangle\rangle$ is a uniform filter on κ. Then $\mathscr{F}^+$ is strongly κ^ω-decomposable.

□ Denote for $\xi < \eta \leqq \kappa_n$, $R^n[\xi, \eta) = \bigcup\{R^n_\zeta : \xi \leqq \zeta < \eta\}$. For $\xi < \kappa_n$, let $\mathscr{G}_{n,\xi} = \{R^n[\zeta, \xi) : \zeta < \xi\}$, for $f \in \Pi\{\kappa_n : n < \omega\}$ let $\mathscr{G}_f = \bigcup\{\mathscr{G}_{n,f(n)} : n < \omega\}$. If $\text{cf}\,f(n) = \omega$ for each $n < \omega$ and if $\mathscr{G}_f$ is uniformly centered, then obviously it is countably generated. Let $\mathscr{A}(f) = \mathscr{A}(\mathscr{G}_f)$ be as in 1.6 in this case, otherwise let $\mathscr{A}(f) = \emptyset$.

Denote $S_n = \{\xi < \kappa_n : \text{cf}\,\xi = \omega\}$. Using Fodor's theorem, choose $\{S_{n,\eta} : \eta < \kappa_n\}$ a pairwise disjoint family of stationary subsets of S_n. For $\phi \in \Pi\{\kappa_n : n < \omega\}$ let $\mathscr{A}_\phi = \bigcup\{\mathscr{A}(f) : f \in \Pi\{S_{n,\phi(n)} : n < \omega\}\}$. Let $\mathscr{A} = \bigcup\{\mathscr{A}_\phi : \phi \in \Pi\{\kappa_n : n < \omega\}\}$. We need to show that $\mathscr{A}$ is a required.

The family $\mathscr{A}$ is almost-disjoint, indeed, for if $f \neq g$, then for some $n < \omega$, $f(n) \neq g(n)$, say $f(n) < g(n)$. Then $R^n[0, f(n)) \in \mathscr{G}_f$, $R^n[f(n), g(n)) \in \mathscr{G}_g$ and $R^n[0, f(n)) \cap R^n[f(n), g(n)) = \emptyset$. Now by the fact that if $A \in \mathscr{A}(f) \subseteq \mathscr{A}_\phi$, $B \in \mathscr{A}(g) \subseteq \mathscr{A}_\psi$ and $\phi \neq \psi$, then $f \neq g$, and by 1.7, $\mathscr{A}$ is almost-disjoint.

Clearly $|\{\mathscr{A}_\phi : \phi \in \Pi\{\kappa_n : n < \omega\}\}| = \kappa^\omega$, provided that for each ϕ, $\mathscr{A}_\phi$ is non-empty; this will be shown together with the property that $\mathscr{A} = \bigcup\{\mathscr{A}_\phi : \phi \ldots\}$ is a strong κ^ω-decomposition of $\mathscr{F}^+$.

For $p < \omega$, denote by $\mathscr{F}(p)$ the uniform filter generated by $R^n[\xi, \kappa_n) : p \leqq n < \omega, \xi < \kappa_n\}$. Clearly $\mathscr{F} = \mathscr{F}(0) \supseteq \mathscr{F}(1) \supseteq \mathscr{F}(2) \supseteq \ldots$, hence $\mathscr{F}^+ = \mathscr{F}(0)^+ \subseteq \mathscr{F}(1)^+ \subseteq \mathscr{F}(2)^+ \subseteq \ldots$.

Claim 1. For each $p < \omega$ and for each $F \in \mathscr{F}(p)^+$ there is a $\xi < \kappa_p$ such that $F \cap R^p[0, \xi) \in \mathscr{F}(p+1)^+$. (Notice that $F \cap R^p[0, \xi) \in \mathscr{F}(p)^c$, thus the above inclusions are proper.)

Suppose not, denote $F_\xi = F \cap R^p[0, \xi)$, and find a set $H_\xi \in [\omega - (p+1)]^{<\omega}$ and for each $n \in H_\xi$ an ordinal $\eta(n, \xi) < \kappa_n$ such that $|F_\xi \cap \bigcap\{R^n[\eta(n, \xi), \kappa_n) : n \in H_\xi\}| < \kappa$. Since κ_p is uncountable and regular, there is a set $H \in [\omega - (p+1)]^{<\omega}$ such that $W = \{\xi < \kappa_p : H = H_\xi\}$ is cofinal in κ_p. For $n \in H$, let $\eta_n = \sup\{\eta(n, \xi) : \xi \in W\}$. Then $\eta_n < \kappa_n$ for κ_n is regular, $\kappa_n > \kappa_p$. Let $G = \bigcap\{R^n[\eta_n, \kappa_n) : n \in H\}$. Then $G \in \mathscr{F}(p+1)$, hence $G \in \mathscr{F}(p)$, thus $|F \cap G| = \kappa$ for $F \in \mathscr{F}(p)^+$. Suppose there is some $n < \omega$ such that $|F_\xi \cap G| < \kappa_n$ for each $\xi \in W$. Since W is cofinal in κ_p and since $\bigcup\{F_\xi : \xi < \kappa_p\} = F$, we have $|F \cap G| \leqq \kappa_p \cdot \kappa_n < \kappa$, which is impossible. Thus for each $n < \omega$ there is a $\xi_n < \kappa_p$ such that $|F_{\xi_n} \cap G| \geqq \kappa_n$. Since W is cofinal in κ_p and since κ_p is regular uncountable, there is some $\xi \in W$, $\xi > \sup\{\xi_n : n < \omega\}$. According to the definition of F_ξ, we have $F_\xi \supseteq F_{\xi_n}$ for all $n < \omega$, hence $|F_\xi \cap G| = \kappa$. But this is a contradiction, because $G \subseteq \bigcap\{R^n[\eta(n, \xi), \kappa_n) : n \in H\}$ and $|F_\xi \cap \bigcap\{R^n[\eta(n, \xi), \kappa_n) : n \in H\}| < \kappa$. Therefore claim 1 is proved.

Claim 2. For each $p < \omega$ and for each $F \in \mathscr{F}(p)^+$ there is a $\xi < \kappa_p$ such that $F \cap R^p[\eta, \xi) \in \mathscr{F}(p+1)^+$ whenever $\eta < \xi$.

Indeed, according to claim 1, the set $\{\xi < \kappa_p : F \cap R^p[0, \xi) \in \mathscr{F}(p+1)^+\}$ is non-void, let ξ be its first element. Then for $\eta < \xi$, $F \cap R^p[0, \eta) \in \mathscr{F}(p+1)^c$, but $F \cap R^p[0, \xi) \in \mathscr{F}(p+1)^+$. Thus $F \cap R^p[\eta, \xi) \in \mathscr{F}(p+1)^+$, which was to be proved.

Denote $C(p, F) = \{\xi < \kappa_p : F \cap R^p[\eta, \xi) \in \mathscr{F}(p+1)^+ \text{ for each } \eta < \xi\}$. According to claim 2, $C(p, F)$ is non-void for $F \in \mathscr{F}(p)^+$, but more is true:

Claim 3. Let $p < \omega$, $F \in \mathscr{F}(p)^+$. Then $C(p, F)$ is closed unbounded in κ_p.

$C(p, F)$ is obviously closed: If ξ is a limit point of $C(p, F)$ and if $\eta < \xi$, then there is a $\zeta \in C(p, F)$, $\eta < \zeta \leqq \xi$. But then $F \cap R^p[\eta, \xi) \supseteq F \cap R^p[\eta, \zeta)$ and $F \cap R^p[\eta, \zeta) \in \mathscr{F}(p+1)^+$, hence $\xi \in C(p, F)$, too.

$C(p, F)$ is unbounded: choose $\eta < \kappa_p$ arbitrary, we want to find a $\xi \in C(p, F)$ with $\eta \leqq \xi$. To this end, let $F' = F \cap R^p[\eta, \kappa_p)$. Then $F' \in \mathscr{F}(p)^+$, hence claim 2 applies: there is some $\xi \in C(p, F')$. Obviously $\xi \geqq \eta$.

Claim 4. Let $F \in \mathscr{F}^+$. Then there is a family $\{C_n : n < \omega\}$ such that for each $n < \omega$, C_n is closed unbounded in κ_n, and if $f \in \Pi\{C_n : n < \omega\}$, then $F \in \mathscr{G}_f^+$.

Put $C_0 = C(0, F)$, where $C(0, F)$ is closed unbounded in κ_0 by claim 3.

For each $\xi \in C_0$, for each $\eta < \xi$, the set $F \cap R^0[\eta, \xi)$ belongs to $\mathscr{F}(1)^+$, hence by Claim 3, the set $C(1, F \cap R^0[\eta, \xi))$ is closed unbounded in κ_1. Denote $C_1 = \bigcap\{C(1, F \cap R^0[\eta, \xi)) : \xi \in C_0, \eta < \xi\}$. Then C_1 is closed unbounded in κ_1, too, being an intersection of less than κ_1 closed unbounded subsets of κ_1.

Following by induction,

$C_2 = \bigcap\{C(2, F \cap R^0[\eta_0, \xi_0) \cap R^1[\eta_1, \xi_1)) : \xi_0 \in C_0,\ \xi_1 \in C_1,\ \eta_0 < \xi_0,\ \eta_1 < \xi_1\}$,

$C_3 = \bigcap\{C(3, F \cap R^0[\eta_0, \xi_0) \cap R^1[\eta_1, \xi_1) \cap R^2[\eta_2, \xi_2)) :$

$: \xi_0 \in C_0,\ \xi_1 \in C_1,\ \xi_2 \in C_2,\ \eta_0 < \xi_0,\ \eta_1 < \xi_1,\ \eta_2 < \xi_2\}$,

and so on.

Now, if $\xi_n \in C_n$ are chosen and if $\eta_n < \xi_n$ for each $n < \omega$, then $F \cap R^0[\eta_0, \xi_0) \in \mathscr{F}(1)^+$, in particular $|F \cap R^0[\eta_0, \xi_0)| = \kappa$.

Further, $F \cap R^0[\eta_0, \xi_0) \cap R^1[\eta_1, \xi_1) \cap \ldots \cap R^n[\eta_n, \xi_n) \in \mathscr{F}(n + 1)^+$, in particular its cardinality is κ. Thus if $f(n) = \xi_n$, $F \in \mathscr{G}_f^+$, for η_n's were chosen arbitrarily.

To finish the proof of the lemma, let $F \in \mathscr{F}^+$, $\phi \in \Pi\{\kappa_n : n < \omega\}$. Choose closed unbounded $C_n \subseteq \kappa_n$ using claim 4, choose $f(n) \in C_n \cap S_{n,\phi(n)}$. This is always possible, since $S_{n,\phi(n)}$ is stationary in κ_n and C_n is closed unbounded. Then $f \in \Pi\{S_{n,\phi(n)} : : n < \omega\}$ and $\operatorname{cf} f(n) = \omega$ for each $n < \omega$, because $S_{n,\phi(n)} \subseteq S_n = \{\xi < \kappa_n : : \operatorname{cf} \xi = \omega\}$.

According to claim 4, $F \in \mathscr{G}_f^+$. This in particular means, that $\mathscr{G}_f$ is uniformly centered, hence $\mathscr{A}(f)$ is non-void. By 1.6, there is an $A \in \mathscr{A}(f)$ with $|F \cap A| = \kappa$.

Hence for arbitrary $\phi \in \Pi\{\kappa_n : n < \omega\}$ we were lucky enough to find an f and an $A \in \mathscr{A}(f) \subseteq \mathscr{A}(\phi)$ with $|F \cap A| = \kappa$, which completes the proof. □

4. Strong decomposability of uniform ultrafilters

We have promised to prove Theorem 0.2 here. But in fact we prove a result a bit stronger, namely:

4.1. Theorem. Let κ be a cardinal number, $\omega = \operatorname{cf} \kappa < \kappa$, let $\mathscr{U} \in \mathrm{U}(\kappa)$. Then

(i) if $\mathscr{U}$ is $(\kappa, \omega, \emptyset)$-regular, then $\mathscr{U}$ is strongly κ^ω-decomposable,

(ii) if $\mathscr{U}$ is arbitrary, then $\mathscr{U}$ is strongly λ-decomposable.

(Recall that $\mathscr{U}$ is $(\kappa, \omega, \emptyset)$-regular provided that there is a $\mathscr{V} \in [\mathscr{U}]^\kappa$ such that $\bigcap\mathscr{W} = \emptyset$ for each $\mathscr{W} \in [\mathscr{V}]^\omega$.)

□ (i): Let $\mathscr{U}$ be $(\kappa, \omega, \emptyset)$-regular, let $\mathscr{V} = \{V_\xi : \xi < \kappa\} \subseteq \mathscr{U}$ be such that $\bigcap\mathscr{W} = \emptyset$ for each $\mathscr{W} \in [\mathscr{V}]^\omega$. We may and shall assume $V_0 = \kappa$.

Choose a strictly increasing sequence $\{\kappa_n : n < \omega\}$ of regular uncountable cardinals converging to κ. Define for $n < \omega$ and $\xi < \kappa_n$, $R^n_\xi = \{\sigma < \kappa : \xi = \max\{\eta < \kappa_n : \sigma \in V_\eta\}\}$, denote $\mathscr{R}_n = \{R^n_\xi : \xi < \kappa_n\}$.

Fix $n < \omega$. Since for each $\sigma < \kappa$ there is some $\eta < \kappa_n$ with $\sigma \in V_\eta$, namely $\eta = 0$, and since the family of such η's must be finite according to $(\kappa, \omega, \emptyset)$-regularity, there is some $\xi < \kappa_n$ with $\sigma \in R^n_\xi$. Clearly $R^n_\xi \cap R^n_\zeta = \emptyset$ whenever $\xi < \zeta < \mu_n$. Thus $\mathscr{R}_n$ is a partition of κ.

Further, if $\sigma \in V_\xi$, then for some $\zeta \geqq \xi$, $\sigma \in R^n_\zeta$. Thus for each $\xi < \kappa_n$, we have $V_\xi \subseteq \bigcup\{R^n_\zeta : \xi \leqq \zeta < \kappa_n\}$, therefore $\bigcup\{R^n_\zeta : \xi \leqq \zeta < \kappa_n\} \in \mathscr{U}$.

The assumptions of Lemma 3.2 being satisfied, $\mathscr{U}$ is strongly κ^ω-decomposable.

(ii): Case $2^\omega > \kappa$.

Choose an arbitrary *partition* $\mathscr{R} = \{R_n : n < \omega\}$ of κ such that for each $n < \omega$, $|R_n| < |R_{n+1}| < \kappa$. Define a mapping $f : \kappa \to \omega$ by $f(\alpha) = n$ iff $\alpha \in R_n$. Then $\mathscr{F} = f[\mathscr{U}] = \{F \subseteq \omega : f^{-1}[F] \in \mathscr{U}\}$ is a uniform ultrafilter on ω. Let $\mathscr{B} \subseteq [\omega]^\omega$ be a family with the properties from 1.8.

For $B \in \mathscr{B}$, choose $\{B_\gamma : \gamma < 2^\omega\} \subseteq [B]^\omega$ an arbitrary almost-disjoint family of subsets of B. Let $\mathscr{A}_\gamma = \{f^{-1}[B_\gamma] : B \in \mathscr{B}\}$, $\mathscr{A} = \bigcup\{\mathscr{A}_\gamma : \gamma < 2^\omega\}$.

It is clear that $\mathscr{A}$ is almost-disjoint, $\mathscr{A} \subseteq [\kappa]^\kappa$.

Let $U \in \mathscr{U}$ and $\gamma < 2^\omega$ be given. For $i < \omega$, let $F_i = \{n < \omega : |U \cap R_n| > |R_i|\}$. Then $F_i \in \mathscr{F}$ because $f^{-1}[\omega - F_i] \cap U$ is of cardinality $\leqq \omega \, . \, |R_i| < \kappa$, thus does not belong to $\mathscr{U}$. Using 1.8, find $B \in \mathscr{B}$ corresponding to $\{F_i : i < \omega\}$. If $C \in [B]^\omega$, then $|f^{-1}[C] \cap U| = \kappa$, because if $b_n \in C$, then $|f^{-1}\{b_n\} \cap U| = |R_{b_n} \cap U| > |R_n|$. Since this applies for B_γ, too, we conclude that $|A \cap U| = \kappa$ for $A = f^{-1}[B_\gamma] \in \mathscr{A}_\gamma$.

It remains to note that $2^\omega = \kappa^\omega$ by the assumption $2^\omega > \kappa$.

Case $2^\omega \leqq \kappa$.

Because of $\omega = \text{cf}\,\kappa$, actually we have $2^\omega < \kappa$ in this case, so we can use all the results from § 2. This case is the most complicated, but the idea is rather simple: We want to find a helpful almost-disjoint family $\mathscr{D}$ and to verify, that 2.8 can be used then, or to prove that $\mathscr{U}$ is $(\kappa, \omega, \emptyset)$-regular, hence to reduce the question to the (i) part of the theorem, which we know to hold. To do this, we shall introduce some not unnatural transfinite procedure and then observe, what happens.

Call a partition $\mathscr{R}$ of κ to be admissible if each $R \in \mathscr{R}$ is a bounded interval of ordinals in κ, i.e. there are $\alpha < \beta < \kappa$ such that $R = \beta - \alpha$.

Let us define for $R = \beta - \alpha$ with $\alpha < \beta < \kappa$ and for $X \subseteq \kappa$, $\text{ct}\,(X, R) = \text{cf}\,\beta$ if β is limit and $X \cap R$ is cofinal in β, $\text{ct}\,(X, R) = 1$ otherwise.

For an admissible partition $\mathscr{R}$ of κ and for $U \in \mathscr{U}$, define

$$\text{ct}\,(U, \mathscr{R}) = \sup\,\{\text{ct}\,(U, R) : R \in \mathscr{R}\}\,,$$

finally, let

$$\text{ct}\,(\mathscr{U}, \mathscr{R}) = \min\,\{\text{ct}\,(U, \mathscr{R}) : U \in \mathscr{U}\}\,.$$

Observation 1. If $\mathscr{R}$ is an admissible partition of κ, U, $V \in \mathscr{U}$ and $U \subseteq V$, then $\text{ct}\,(U, \mathscr{R}) \leqq \text{ct}\,(V, \mathscr{R})$.

Indeed, there is nothing to prove if $\text{ct}\,(U, \mathscr{R}) = 1$ and almost nothing otherwise, for if $U \cap R$ is cofinal in R, then $V \cap R$ is cofinal in R, too.

Observation 2. Let $\mathscr{R}$ be an admissible partition of κ, let $1 < \tau = \text{ct}\,(\mathscr{U}, \mathscr{R})$, let $\mu < \tau$. Then the set $V = \kappa - \bigcup\{R \in \mathscr{R} : R = \beta - \alpha \;\&\; \mu < \text{cf}\,\beta \leqq \tau\}$ does not belong to $\mathscr{U}$.

Choose $U \in \mathscr{U}$ such that $\mathrm{ct}(U, \mathscr{R}) = \mathrm{ct}(\mathscr{U}, \mathscr{R}) = \tau$, suppose on the contrary that $V \in \mathscr{U}$. Let us try to check $\mathrm{ct}(V \cap U, \mathscr{R})$: If $R = \beta - \alpha$ and β is isolated, then $\mathrm{ct}(V \cap U, R) = 1$. If β is a limit and $\mathrm{cf}\,\beta \leqq \mu$, then $\mathrm{ct}(V \cap U, R) \leqq \mathrm{cf}\,\beta \leqq \mu$. If $\mu < \mathrm{cf}\,\beta \leqq \tau$, then $R \cap V = \emptyset$, thus $\mathrm{ct}(V \cap U, R) = 1$. If $\tau < \mathrm{cf}\,\beta$, then $U \cap R$ is not cofinal in β, because $\sup\{\mathrm{ct}(U, R) : R \in \mathscr{R}\} \leqq \tau < \mathrm{cf}\,\beta$. Thus $\mathrm{ct}(U, R) = 1$ and by the previous observation, $\mathrm{ct}(V \cap U, R) = 1$, too. So $\mathrm{ct}(V \cap U, \mathscr{R}) \leqq \mu < \tau = \min\{\mathrm{ct}(W, \mathscr{R}) : W \in \mathscr{U}\}$, which is a contradiction.

The procedure

Let $\alpha < \beta < \kappa$, $R = \beta - \alpha$, $X \subseteq \kappa$. Let us define a partition $\mathscr{P}(R, X)$ fo R as follows:

(a) there is some $0 < n < \omega$ and some limit $\gamma > \alpha$ such that $\beta = \gamma + n$: let $\mathscr{P}(R, X) = \{\beta - \gamma, \gamma - \alpha\}$,

(b) β is a limit and $\alpha \leqq \sup(X \cap \beta) < \beta$: let $\mathscr{P}(R, X) = \{\beta - (\sup(X \cap \beta) + 1), (\sup(X \cap \beta) + 1) - \alpha\}$,

(c) β is a limit and $\beta = \sup(X \cap \beta)$: Let $\mathscr{P}(R, X)$ be a partition of R of cardinality $\mathrm{cf}\,\beta$ and such that for each $P \in \mathscr{P}(R, X)$ there are $\alpha(P)$, $\beta(P)$, $\alpha \leqq \alpha(P) < \beta(P) < \beta$ with $P = \beta(P) - \alpha(P)$,

(d) otherwise let $\mathscr{P}(R, X) = \{R\}$.

Observation 3. $|\mathscr{P}(R, X)| \leqq \omega \,.\, \mathrm{ct}(X, R)$.

Let $\mathscr{R}$ be an admissible partition of κ, let $U \in \mathscr{U}$. Let us define $\mathscr{P}(\mathscr{R}, U) = \bigcup\{\mathscr{P}(R, U) : R \in \mathscr{R}\}$.

Let us mention one straightforward consequence of definitions and of Observation 3.

Observation 4. The partition $\mathscr{P}(\mathscr{R}, U)$ is admissible and $|\mathscr{P}(\mathscr{R}, U)| \leqq \omega \,.\, \mathrm{ct}(U, \mathscr{R}) \cdot |\mathscr{R}|$.

Fix some increasing sequence of cardinals $\{\kappa_n : n < \omega\}$ converging to κ, $\kappa_0 = 0$. Define $\mathscr{R}_0 = \{\kappa_{n+1} - \kappa_n : n < \omega\}$. Clearly $\mathscr{R}_0$ is an admissible partition of κ. We shall proceed via transfinite induction as follows:

Let η be a limit ordinal and suppose that for each $\xi < \eta$, $\mathscr{R}_\xi$ has been defined. Then $\mathscr{R}_\eta$ is the coarsest partition of κ refining all $\mathscr{R}_\xi$, $\xi < \eta$, i.e. $R \in \mathscr{R}_\eta$ iff for each $\xi < \eta$ there is a $T \in \mathscr{R}_\xi$ with $T \supseteq R$, but no R' properly containing R satisfies the same.

Let ξ be an ordinal and suppose that $\mathscr{R}_\xi$ is defined. If the set $\bigcup\{R \in \mathscr{R} : |R| < \omega\} \in \mathscr{U}$, then $\mathscr{R}_{\xi+1}$ is not defined and the induction stops here. Otherwise we can fix some $U_\xi \in \mathscr{U}$ such that $\mathrm{ct}(U_\xi, \mathscr{R}_\xi) = \mathrm{ct}(\mathscr{U}, \mathscr{R}_\xi)$ and $U_\xi \cap \bigcup\{R \in \mathscr{R}_\xi : |R| < \omega\} = \emptyset$, then $\mathscr{R}_{\xi+1}$ is defined by $\mathscr{R}_{\xi+1} = \mathscr{P}(\mathscr{R}_\xi, U_\xi)$.

We have defined the procedure and it remains to exploit it. Notice that the procedure guarantees that the following holds.

Observation 5. If $\xi < \eta$ and if $\mathscr{R}_\xi$, $\mathscr{R}_\eta$ are defined, then $\mathscr{R}_\eta$ refines $\mathscr{R}_\xi$, i.e. for each $R \in \mathscr{R}_\eta$ there is some $T \in \mathscr{R}_\xi$ with $R \subseteq T$.

Subcase I. The procedure does not stop before κ.

We claim that in this subcase $\mathscr{U}$ is $(\kappa, \omega, \emptyset)$-regular. Since the induction already defined a family $\{U_\xi : \xi < \kappa\} \subseteq \mathscr{U}$, we need only to verify that $\bigcap \{U_{\xi_i} : i < \omega\} = \emptyset$ whenever $\xi_0 < \xi_1 < \ldots < \kappa$. Suppose not, let $\sigma \in \bigcap U_{\xi_i}$. Let $R_n \in \mathscr{R}_{\xi_n}$ be the member of the partition $\mathscr{R}_{\xi_n}$ with $\sigma \in R_n$, denote $\beta_n = \sup R_n$. By observation 5, $R_0 \supseteq \supseteq R_1 \supseteq \ldots \supseteq R_n \supseteq \ldots$, hence $\beta_0 \geqq \beta_1 \geqq \ldots \geqq \beta_n \geqq \ldots$. According to our choice of U_{ξ_n}, each R_n must be infinite and $U_{\xi_n} \cap R_n \neq \emptyset$ since $\sigma \in U_{\xi_n} \cap R_n$. Thus, when constructing $\mathscr{R}_{\xi_n+1} = \mathscr{P}(\mathscr{R}_{\xi_n}, U_{\xi_n})$, the case (d) never took place, unfortunately, either (a) or (b) or (c) imply that $\beta_0 > \beta_1 > \ldots > \beta_n > \ldots$, a contradiction.

$\mathscr{U}$ being $(\kappa, \omega, \emptyset)$-regular, 4.2(i) applies.

Subcase II. There is some $\xi < \kappa$ such that $\mathscr{R}_{\xi+1}$ cannot be defined.

Observation 6. If $\eta \leqq \xi$ is a limit ordinal, then $|\mathscr{R}_\eta| \leqq \Sigma\{|\mathscr{R}_\zeta| : \zeta < \eta\}$.

Indeed, for $\zeta \leqq \eta$, consider the set $K(\zeta) = \{\sup R : R \in \mathscr{R}_\zeta\} \subseteq \kappa$.

Since each $\mathscr{R}_\zeta (\zeta \leqq \eta)$ is admissible, the correspondence between $\mathscr{R}_\zeta$ and $K(\zeta)$ given by $R \mapsto \sup R$ is one-to-one and onto, therefore $|K(\zeta)| = |\mathscr{R}_\zeta|$. Choose $\beta \in K(\eta)$, let $R \in \mathscr{R}_\eta$ be the member with $\beta = \sup R$. Then for each $\zeta < \eta$ there is precisely one $R_\zeta \supseteq R$, $R_\zeta \in \mathscr{R}_\zeta$, denote $\beta(\zeta) = \sup R_\zeta$. Clearly $\beta(\zeta) \geqq \beta(\zeta') \geqq \beta$ whenever $\zeta < < \zeta' < \eta$, since $R_\zeta \supseteq R_{\zeta'} \supseteq R$. Thus there is some $\bar{\beta}$ with $\bar{\beta} = \beta(\zeta)$ for eventually many ζ's less than η. Now $\bar{\beta} = \beta$, for otherwise the set $(\bar{\beta} - \beta) \cup R$ would properly contain R and yet be contained in each R_ζ. Thus $\beta \in \bigcup\{K(\zeta) : \zeta < \eta\}$, consequently $K(\eta) \subseteq \bigcup\{K(\zeta) : \zeta < \eta\}$ and the observation follows.

Observation 7. For each $\tau < \kappa$ there is an $\eta \leqq \xi$ with $\operatorname{ct}(\mathscr{U}, \mathscr{R}_\eta) > \tau$.

Suppose not, let $\tau < \kappa$ be such that $\operatorname{ct}(\mathscr{U}, \mathscr{R}_\eta) \leqq \tau$ whenever $\eta \leqq \xi$. We have $|\mathscr{R}_0| = \omega$ and an easy induction using Observation 4 on successor stages and Observation 6 on limits gives immediately that for each $\eta \leqq \xi$, $|\mathscr{R}_\eta| \leqq |\eta| \cdot \omega \cdot \tau < \kappa$. In particular, $|\mathscr{R}_\xi| \leqq |\xi| \cdot \omega \cdot \tau < \kappa$, thus $|\bigcup\{R \in \mathscr{R}_\xi : |R| < \omega\}| \leqq \omega \cdot |\mathscr{R}_\xi| < \kappa$. But then, $\mathscr{U}$ being uniform, $\bigcup\{R \in \mathscr{R}_\xi : |R| < \omega\} \notin \mathscr{U}$. According to the rules of procedure, $\mathscr{R}_{\xi+1}$ is defined then, but this contradicts our assumption and proves the observation.

By virtue of the last observation, there are only two possibilities we have to consider,

IIa: There is an $\eta \leqq \xi$ with $\operatorname{ct}(\mathscr{U}, \mathscr{R}_\eta) = \kappa$, or

IIb: for each $\tau < \kappa$ there is an $\eta \leqq \xi$ with $\tau < \operatorname{ct}(\mathscr{U}, \mathscr{R}_\eta) < \kappa$.

We shall show that both the possibilities guarantee that 2.8 may be applied.

To see this, assume IIa first. Denote by $\mathscr{R}$ the admissible partition of κ with $\operatorname{ct}(\mathscr{U}, \mathscr{R}) = \kappa$.

Let $\Psi \subseteq {}^{\omega}\kappa$ satisfy the following:

(i)$_{\mathscr{R}}$ for each $g \in \Psi$ and for each $n < \omega$ there is some $R \in \mathscr{R}$ with $\sup R = g(n)$;

(ii)$_{\mathscr{R}}$ for each $g \in \Psi$, the family $\{\beta_n = g(n) : n < \omega\}$ satisfies 2.1(*);

(iii)$_{\mathscr{R}}$ for any two distinct $f, g \in \Psi$, the intersection $\{f(n) : n < \omega\} \cap \{g(n) : : n < \omega\}$ is finite;

(iv)$_{\mathscr{R}}$ Ψ is the maximal one having (i), (ii), (iii).

For $g \in \Psi$ and $n < \omega$, denote $R(g(n))$ the (unique, by the admissibility of $\mathscr{R}$) member $R \in \mathscr{R}$ with $g(n) = \sup R$, let $D(g) = \bigcup\{R(g(n)) : n < \omega\}$, $\mathscr{D} = \{D(g) : : g \in \Psi\}$. The existence of a set $D(g)$ is guaranteed by (i), (ii) implies that $\mathscr{D} \subseteq [\kappa]^{\kappa}$ and, if for $D = D(g)$ we denote $\beta_n(D) = g(n), f_D(n) = \min R(g(n))$, that $\mathscr{D}$ is helpful. Moreover, $\mathscr{D}$ is almost-disjoint by (iii) and by the admissibility of $\mathscr{R}$.

If $U \in \mathscr{U}$, then $\mathrm{ct}(U, \mathscr{R}) = \kappa$, which means that $\sup\{\mathrm{cf}\,\beta\colon (\exists\alpha < \beta < \kappa) (\beta - \alpha \in \mathscr{R}\ \&\ \sup(U \cap \beta) = \beta)\} = \kappa$. Therefore there is a subset $\{R_n : n < \omega\} \subseteq \mathscr{R}$ such that $U \cap R_n$ is cofinal in R_n and the set $\{\beta_n = \sup R_n : n < \omega\}$ satisfies 2.1(*). Using the maximality of Ψ, there is some $g \in \Psi$ with $\{g(n) : n < \omega\} \cap \{\beta_n = \sup R_n : : n < \omega\}$ infinite. Now clearly for $D = D(g)$, $U \in \mathrm{Big}\,\{\beta_n(D)\}$.

We have found a helpful almost-disjoint family $\mathscr{D} \subseteq [\kappa]^{\kappa}$ such that $\mathscr{U} \subseteq \subseteq \bigcup\{\mathrm{Big}\,\{\beta_n(D)\} : D \in \mathscr{D}\}$.

Next, assume IIb. A trivial induction coupled with Observation 5 gives us immediately the following.

Observation 8. There is an increasing sequence $\{\tau_n : n < \omega\}$ of cardinals, $\sup\{\tau_n : n < \omega\} = \kappa$ and an increasing sequence $\{\eta(n) : n < \omega\}$ of ordinals such that $\mathrm{ct}(\mathscr{U}, \mathscr{R}_{\eta(n)}) = \tau_n$ and for each $n < \omega$, $\mathscr{R}_{\eta(n)}$ refines $\mathscr{R}_0$.

Therefore, making if necessary appropriate choices between the κ_n's, τ_n's, and $\eta(n)$'s, we need to work under the assumption that the situation is like this:

There are increasing sequences $\{\kappa_n : n < \omega\}$, $\{\tau_n : n < \omega\}$ of cardinals converging to κ and such that $\kappa_n < \tau_n < \kappa_{n+1}$ for all n, and a family $\{\mathscr{R}_n : n < \omega\}$ of admissible partitions of κ such that for each $n < \omega$ and for each $R \in \mathscr{R}_n$ there is some $i < \omega$ with $R \subseteq \kappa_{i+1} - \kappa_i$, and $\mathrm{ct}(\mathscr{U}, \mathscr{R}_n) = \tau_n$ for each $n < \omega$.

Let us fix this notation for the rest of the proof and forget about the procedure at all.

For $n < \omega$, $U \in \mathscr{U}$, define a set $F_n(Q)$ as follows:

$$F_n(Q) = \{i < \omega : (\exists R \in \mathscr{R}_n)(\kappa_n < \mathrm{ct}(U, R) \leqq \tau_n\ \&\ R \subseteq \kappa_{i+1} - \kappa_i)\}\,.$$

Observation 9. The family $\{F_n(U) : n < \omega,\ U \in \mathscr{U}\}$ is uniformly centered on ω.

To see this, denote $H_n(U) = \bigcup\{\kappa_{i+1} - \kappa_i : i \in F_n(U)\}$. It is clear that $\{F_n(U) : : n < \omega, U \in \mathscr{U}\}$ is uniformly centered on ω if and only if the family $\{H_n(Q) : n < \omega, U \in \mathscr{U}\}$ is uniformly centered on κ. But according to Observation 2. the set $V_n = = \bigcup\{R \in \mathscr{R}_n : \kappa_n < \mathrm{cf}\,R \leqq \tau_n\}$ belongs to $\mathscr{U}$ and $V_n \cap U \subseteq H_n(U)$ obviously from the definitions. Thus each $H_n(U) \in \mathscr{U}$, which shows the observation.

Let $\mathscr{B} \subseteq [\omega]^\omega$ be the almost disjoint family given by 1.8 for $\mathscr{F} = \langle\!\langle \{F_n(U) : n < \omega,\ U \in \mathscr{U}\} \rangle\!\rangle$. For $B \in \mathscr{B}$, $B = \{b_0 < b_1 < \ldots < b_n < \ldots\}$, let $\mathscr{T}_n(b_n) = \{R \in \mathscr{R}_n : \kappa_n < \operatorname{cf} R \leqq \tau_n \ \&\ R \subseteq \kappa_{b_n+1} - \kappa_{b_n}\}$,

$$\mathscr{T}(B) = \begin{cases} \bigcup\{\mathscr{T}_n(b_n) : n < \omega\} \text{ provided that each } \mathscr{T}_n(b_n) \neq \emptyset \\ \emptyset \text{ otherwise.} \end{cases}$$

$T(B) = \bigcup \mathscr{T}(B)$.

Observation 10. If $B, C \in \mathscr{B}$ and $T(B)$, $T(C)$ are non-empty, then $|T(B) \cap T(C)| < \kappa$.

Indeed, choose $j < \omega$ such that $B \cap C \subseteq j$, then $T(B) \cap T(C) \subseteq \kappa_{j+1}$.

Observation 11. For $B \in \mathscr{B}$, $\mathscr{T}(B)$ is an admissible partition of $T(B)$. This is, of course, obvious.

Fix $B \in \mathscr{B}$ with $\mathscr{T}(B) \neq \emptyset$, and denote $\mathscr{T}_n = \mathscr{T}_n(b_n)$. Similarly as in the previous, let $\Psi(B) \subseteq {}^\omega\kappa$ be the family satisfying:

(i)$_B$ for each $g \in \Psi$ and for each $n < \omega$ there is some $R \in \mathscr{T}_n$ with $\sup R = g(n)$;

(ii)$_B$ for any two distinct $f, g \in \Psi$, the set $\{n < \omega : f(n) = g(n)\}$ is finite;

(iii)$_B$ $\Psi(B)$ is the maximal one having (i)$_B$, (ii)$_B$.

Exactly in the same way as in the possibility IIa, a family $\Psi(B)$ determines an almost-disjoint collection $\mathscr{D}(B) = \{D(g) : g \in \Psi(B)\} \subseteq [\kappa]^\kappa$ and $\mathscr{D}(B)$ is helpful. We don't need to repeat the proof here.

Finally, let $\mathscr{D} = \bigcup\{\mathscr{D}(B) : B \in \mathscr{B} \ \&\ \mathscr{T}(B) \neq \emptyset\}$. By Observation 10 and by the fact that $\bigcup\mathscr{D}(B) \subseteq T(B)$, the family $\mathscr{D}$ is almost-disjoint, too.

Pick now an arbitrary $U \in \mathscr{U}$. By 1.8, there is a $B = \{b_0 < b_1 < \ldots\} \in \mathscr{B}$ such that $B - F_n(U) \subseteq \{b_i : i < n\}$, in particular, $b_n \in F_n(U)$.

But then there is a set $R_n \in \mathscr{R}_n$ such that $\kappa_n < \operatorname{ct}(U, R_n) \leqq \tau_n$ and $R_n \subseteq \kappa_{b_n+1} - \kappa_{b_n}$. In particular, $R_n \in \mathscr{T}_n(b_n)$. Since this holds for each $n < \omega$, $\mathscr{T}(B)$ is non-empty. Define $h \in {}^\omega\kappa$ by $h(n) = \sup R_n$. Then according to the maximality of $\Psi(B)$, there is some $D = D(g) \in \mathscr{D}(B)$ with $U \in \operatorname{Big}\{\beta_n(D)\}$, for $\{n < \omega : h(n) = g(n)\}$ has to be infinite for some $g \in \Psi(B)$. Hence $\mathscr{U} \subseteq \{\operatorname{Big}\{\beta_n(D)\} : D \in \mathscr{D}\}$.

Since we succeeded to find a helpful almost-disjoint family $\mathscr{D}$ such that $\mathscr{U} \subseteq \bigcup\{\operatorname{Big}\{\beta_n(D)\} : D \in \mathscr{D}\}$ in IIa as well as in IIb, we infer from 2.8 that $\mathscr{U}$ is strongly λ-decomposable. Having considered all possible cases, the proof of the theorem is complete. □

4.2 Concluding remarks

The author wishes the reader to know that Lemmas 2.5, 2.6 are due to Peter Vojtáš, who used them in his Ph. D. Thesis (unpublished). With a quite different way of reasoning, he proved 4.1(ii) for a special case $\kappa = \omega_\omega$ there.

Further, the basic trick used in the proof of 4.1(i), i.e. finding the decompositions $\mathscr{R}_n$ with help of the regularity of $\mathscr{U}$, is widely known and may be found e.g. in [CN]. We gave here all the details and proved both lemmas to make the paper self-contained.

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