Jindřich Bečvář
N-pure-high subgroups of abelian groups

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 26 (1985), No. 1, 21–33

Persistent URL: http://dml.cz/dmlcz/142544

N-pure-high Subgroups of Abelian Groups

J. BEČVÁŘ
Department of Mathematics, Charles University, Prague*)

Received 13 November 1984

The paper is concerned with N-pure-high subgroups of abelian groups, the study of which is proposed by L. Fuchs in his book Infinite Abelian Groups (Problem 14).

Článek se zabývá N-servantně-vysokými podgrupami Abelových grup, jejichž studium navrhuje L. Fuchs ve své monografii Infinite Abelian Groups (Problem 14).

В статье изучаются N-сервантно-высокие подгруппы абелевых групп, исследование которых предлагается проблемой Но 14 в книге Бесконечные абелевы группы Л. Фукса.

1. Introduction, history and some basic information

The concept of N-high subgroup was introduced into the theory of abelian groups by J. M. Irwin and E. A. Walker [6, 9] in 1961. Since then many papers have been written investigating the various properties of N-high subgroups. One of first questions, namely, for which subgroups N it is true that all N-high subgroups are pure, was posed by Irwin and Walker in [6, 9]. This question has been investigated in several papers (Irwin, Walker, Charles, Khabbaz, Reid), the final result has been done by R. S. Pierce [11]. Some generalizations and related results have been written later (Megibben, Rochlina, Keane, Bečvář).

L. Fuchs, inspired with these relevant questions, proposes the study of N-pure-high subgroups in problem 14 of his book [5]. K. Benabdallah dealt with this problem in [1].

1.1. Definition. Let N be a subgroup of a group G. We say that a subgroup H of G is N-pure-high in G if it is maximal among the pure subgroups disjoint from N.

Zorn's lemma guarantees the existence of N-pure-high subgroups. Moreover, each N-pure-high subgroup of G is contained in an N-high subgroup of G. A natural problem arises to characterize such subgroups N of a group G for which all N-pure-high subgroups are N-high. From this point of view, the mentioned theorem of Pierce describes all subgroups N of a group G for which N-pure-high and N-high subgroups of G coincide. We reformulate the Pierce's result in the following way:

*) 186 00 Praha 8, Sokolovská 83, Czechoslovakia.

1.2. Theorem. ([11]). If N is a subgroup of a group G then the following assertions are equivalent:

(i) A subgroup H of G is N-pure-high in G if and only if H is N-high in G.

(ii) For each prime p either $N[p] \subseteq p^{\omega}G$ or G/N is torsion and there is a natural number n such that $p^{n+2} G[p] \subseteq N[p] \subseteq p^n G$.

The necessary and sufficient condition for a subgroup N under which all N-pure-high subgroups are N-high has not yet been found. In 1974, K. Benabdallah gave the following partial solution (see also theorem 14, [4]):

1.3. Theorem (Theorem 2, [1]). Let N be a subgroup of a group G. If one N-high subgroup is torsion, all N-high subgroups are torsion and N-pure-high subgroups are N-high.

It is easy to see that the assumption and the first assertion of 1.3 are mutually equivalent and that they are also equivalent with the condition that G/N is torsion:

1.4. Remark For a subgroup N of a group G, the following conditions are equivalent:

(i) G/N is torsion.

(ii) There is a torsion N-high subgroup of G.

(iii) Each N-high subgroup of G is torsion.

Hence the theorem of Benabdallah obtains this form: If G/N is torsion then each N-pure-high subgroup of G is N-high in G. The converse is not true; it follows already from Pierce's theorem 1.2.

If G is a torsion group then N-pure-high subgroups of G are exactly pure N-high subgroups of G by 1.3. If G is a torsion free group then N-pure-high and N-high subgroups of G coincide, since N-high subgroups are neat and neat subgroups of a torsion free group are pure. Consequently, the study of N-pure-high subgroups is useful only in the theory of mixed groups. For example, if G does not split then no G_t-high subgroup of G is pure in G and hence G_t-pure-high subgroups of G are not G_t-high in G.

The purpose of this paper is to investigate N-pure-high subgroups (of mixed groups). An important result is theorem 2.5 which asserts that the torsion parts of N-pure-high subgroups of G are pure N_t-high in G_t. A few corollaries of this theorem give a comparison of some elementary properties of N-pure-high and N-high subgroups. If N is a subgroup of a group G and H is an N-high subgroup of G then the following assertions hold:

(i) If $g \in G$ and $pg \in H$ for a prime p then $g \in N \oplus H$ (9.8, [5]).

(ii) H is neat in G.

(iii) $G[p] = N[p] \oplus H[p]$ for each prime p.

(iv) $N \oplus H$ is essential in G.

(v) $G/(N \oplus H)$ is torsion.

The proof of these assertions (in written sequence) can be easily proved. If H is an N-pure-high subgroup of G then the assertions (i)–(iii) hold too (see 2.6 (iii)–(iv)). However, the assertions (iv) and (v) hold if and only if H is N-high in G (see 2.7). Moreover, if M is an N-high subgroup of G containing an N-pure-high subgroup H of G then M/H is torsion free (see 2.6 (v)).

All groups in this paper are assumed to be abelian groups. We follow the terminology and notation of [5]. In addition, a subgroup H of a group G is said to be p-absorbing resp. absorbing in G if $(G/H)_p = 0$, resp. $(G/H)_t = 0$. Obviously, every p-absorbing subgroup of G is p-pure in G and if S is a pure subgroup of G then $S + G_t$ is absorbing in G. The set of all primes is denoted by $\mathbf{P}$.

2. Torsion parts of N-pure-high subgroups of G are N_t-high in G_t

We shall often use the following lemma.

2.1. Lemma. Let N, A and S be subgroups of a group G such that

(i) $A \cap N = 0 = S \cap N$,

(ii) $A \subseteq G_t$,

(iii) if $A_p \neq 0$ then $S_p \subseteq A_p$,

(iv) A and S are pure in G.

Then $A + S$ is pure in G and $(A + S) \cap N = 0$.

Proof. If $a + s = p^i g$, where $a \in A$, $s \in S$ and $g \in G$, then $o(a)\, s = o(a)\, p^i g$ and there is $\bar{s} \in S$ with $o(a)\, s = o(a)\, p^i \bar{s}$ by (iv). Hence $s - p^i \bar{s} \in A$ by (iii). Further $a + s - p^i \bar{s} = p^i(g - \bar{s}) \in A$ and by (iv), there is $\bar{a} \in A$ such that $a + s - p^i \bar{s} = p^i \bar{a}$. Consequently $a + s = p^i(\bar{a} + \bar{s})$.

If $a + s = n$, where $a \in A$, $s \in S$ and $n \in N$, then $o(a)\, s = o(a)\, n \in S \cap N = 0$ and hence $s \in A$ by (iii). Consequently $a + s = n \in A \cap N = 0$ and $(A + S) \cap N = 0$.

2.2. Corollary. Let N be a subgroup of a group G and $\mathbf{R} = \{p \in \mathbf{P};\ N_p = 0\}$. Then the following assertions hold:

(i) Each N-pure-high subgroup of G contains $\bigoplus_{p \in \mathbf{R}} G_p$.

(ii) Each N-pure-high subgroup of G is p-absorbing in G for each $p \in \mathbf{R}$.

Proof. Let H be an N-pure-high subgroup of G and $A = \bigoplus_{p \in \mathbf{R}} G_p$. By lemma 2.1, $H + A$ is a pure subgroup of G and $(H + A) \cap N = 0$. With respect to the maximality of H, we have $A \subseteq H$. If $pg \in H$, where $g \in G$ and $p \in \mathbf{R}$, then $pg = ph$ for some $h \in H$. Hence $g - h \in G_p \subseteq H$ and $g \in H$.

In the following text, we shall often work with a subgroup T which is defined by the equality $(G/N)_t = T/N$.

2.3. Lemma. Let N be a subgroup of a group G and $(G/N)_t = T/N$. Then the following assertions hold:

(i) T is absorbing in G and $N + G_t \subseteq T$.

(ii) $T/(N + G_t) = (G/(N + G_t))_t$.

(iii) T is a maximal essential extension of $N + G_t$ in G.

(iv) T is a pure hull of $N + G_t$ in G.

(v) $T = G \cap D$, where D is a divisible hull of $N + G_t$ contained in a divisible hull E of G.

Proof. (i) $G/N/T/N \cong G/T$ and hence G/T is torsion free. Since $N + G_t/N$ is torsion, $N + G_t \subseteq T$.

(ii) Obviously $T/(N + G_t) \subseteq (G/(N + G_t))_t$. If $g \in G$ and $kg \in N + G_t$ for some integer k then $mg \in N$ for some integer m and hence $g \in T$.

(iii) If $x \in T \setminus (N + G_t)$ then by (ii), $o \neq kx = n + t$ $(n \in N,\ t \in G_t)$ for some integer k and hence $N + G_t$ is essential in T. If $N + G_t$ is essential in a subgroup X of G then $X/(N + G_t)$ is torsion and $X \subseteq T$ by (ii).

(iv) T is pure in G by (i) and hence T/G_t is pure in G/G_t. Let X/G_t be the intersection of all pure subgroups of G/G_t containing $(N + G_t)/G_t$. Then X is the pure hull of $N + G_t$ in G and $N + G_t \subseteq X \subseteq T$. Now, X is pure and essential in T and hence $X = T$.

(v) If E is a divisible hull of G and D a divisible hull of $N + G_t$ which is contained in E then $N + G_t$ is essential in $D \cap G$ and hence $D \cap G \subseteq T$. If $t \in T$ then $kt \in N + G_t \subseteq D$ for an integer k, $kt = kd$ for an element $d \in D$ and hence $t - d \in$ $\in E_t \subseteq D$ and $t \in D$.

2.4. Remark. Let N be a subgroup of a group G and $(G/N)_t = T/N$. Then

(i) if N is pure in G then $T = N + G_t$,

(ii) if $\overline{N}$ is a maximal essential extension of N in G then $\overline{N} + G_t \subseteq T$.

2.5. Theorem. Let N be a subgroup of a group G, $(G/N)_t = T/N$ and Y be a subgroup of G with $G_t \subseteq Y \subseteq T$. Then

(i) If H is an N-pure-high subgroup of G then $H \cap Y = H_t$ and H_t is a pure $N \cap Y$-high subgroup of Y.

(ii) Each pure $N \cap Y$-high subgroup of Y is the torsion part of an N-pure-high subgroup of G.

Proof. Since $Y/(N \cap Y) \cong (Y + N)/N \subseteq T/N$, $Y/(N \cap Y)$ is torsion and $N \cap Y$-high subgroups of Y are torsion (see 1.4).

(i) If H is an N-pure-high subgroup of G then obviously $H \cap Y = H_t$ and H_t is pure in Y. Let A be a pure subgroup of Y such that $H_t \subseteq A$ and $A \cap N \cap Y = 0$. By lemma 2.1, $A + H$ is pure in G and $(A + H) \cap N = 0$. With respect to the maximality of H, we have $A = H_t$. Hence H_t is $N \cap Y$-pure-high in Y and by theorem 1.3, H_t is pure $N \cap Y$-high in Y.

(ii) Let A be a pure $N \cap Y$-high subgroup of Y. Since A is torsion, A is pure in G. If H is a N-pure-high subgroup of G containing A then by (i), $H_t = H \cap Y = A$.

2.6. Corollary. Let N be a subgroup of a group G and $(G/N)_t = T/N$. If H is an N-pure-high subgroup of G then the following assertions hold:

(i) H_t is pure N_t-high in G_t.

(ii) H_t is pure N-high in T and in $N + G_t$.

(iii) $G[p] = N[p] \oplus H[p]$ for each prime p.

(iv) If $g \in G$ and $pg \in H$ for some prime p then $g \in N[p] \oplus H$.

(v) If M is an N-high subgroup of G containing H then M/H is torsion fre.

(vi) Torsion parts of all N-pure-high subgroups of G are exactly all pure N_t-high subgroups of G_t and exactly all pure N-high subgroups of $N + G_t$ (resp. T).

Proof. The assertions (i), (ii), (vi) follow immediately from 2.5. Since H_t is N_t-high in G_t, $G[p] = N[p] \oplus H[p]$ for each prime p.

(iv) If $g \in G$ and $pg \in H$ then $pg = ph$ for some $h \in H$, $g - h \in G[p] = N(p) \oplus \oplus H[p]$ and $g \in N[p] \oplus H$. Note that if $p \in \mathbf{R}$ then $g \in H$ – see 2.2 (ii).

(v) If $g \in M$ and $pg \in H$ then $pg = ph$ for some $h \in H$. Consequently $g - h \in \in M[p] = H[p]$ and $g \in H$.

2.7. Corollary. Let N be a subgroup of a group G and H an N-pure-high subgroup of G. The following assertions are equivalent:

(i) H is N-high in G.

(ii) $N \oplus H$ is essential in G.

(iii) $(N \oplus H)/H$ is essential in G/H.

(iv) $(H + G_t)/G_t$ is $(N + G_t)/G_t$-high in G/G_t.

(v) $G/(N \oplus H)$ is torsion.

Proof. (i) → (ii) Well-known and easy.

(ii) → (iii) Let $g \in G \setminus N \oplus H$. If k is the least natural number such that $kg \in \in N \oplus H$ (see (ii)) then $kg + H$ is a nonzero element of $(N \oplus H)/H$ by 2.6 (iv).

(iii) → (iv) Obviously $(H + G_t)/G_t \cap (N + G_t)/G_t = 0$. Let K/G_t be an $(N + G_t)/G_t$-high subgroup of G/G_t containing $(H + G_t)/G_t$ and $k \in K$. There is an integer r such that $rk = n + h$, where $n \in N$ and $h \in H$ (see (iii)). Hence $n = rk - h \in \in K \cap N = N_t$, $o(n)\, rk = o(n)\, h = o(n)\, r\bar{h}$ for some $\bar{h} \in H$. Consequently $k - \bar{h} \in \in G_t$, i.e. $k \in H + G_t$.

(iv) → (v) For each $g \in G$ there is an integer r such that $r(g + G_t) = (h + G_t) + + (n + G_t)$, where $h \in H$ and $n \in N$ (see (iv)). Hence $rg = h + n + t$, where $t \in G_t$, and $o(t)\, rg \in H \oplus N$.

(v) → (i) If M is an N-high subgroup of G containing H then $M/H \cong \cong (M \oplus N)/(H \oplus N) \subseteq G/(N \oplus H)$ and M/H is torsion by (v). Hence $M = H$ by (2.6) (v).

If H is an N-high subgroup of G then H is $N \cap S$-high in each subgroup S of G which contains H. A similar result holds for N-pure-high subgroups.

2.8. Lemma. Let N be a subgroup of a group G and H be an N-pure-high subgroup of G. If S is a pure subgroup of G containing H then H is $N \cap S$-pure-high in S.

Proof. Easy.

In a sense, the next corollary is dual to the theorem 2.5. Corollary 2.10 afterwards gives a supplementary result.

2.9. Corollary. Let N be a subgroup of a group G and H be an N-pure-high subgroup of G.

(i) If S is a pure subgroup of G such that $H \subseteq S \subseteq H + G_t$ then H is pure $N \cap S$-high in S.

(ii) H is pure N_t-high in $H + G_t$.

Proof. If S is a pure subgroup of G such that $H \subseteq S \subseteq H + G_t$ then H is $N \cap S$-pure-high in S by 2.8. Since $S \subseteq H + G_t$, $S/(H \oplus (N \cap S))$ is torsion and H is pure $N \cap S$-high in S by 2.7. It is easy to see that $H + G_t$ is pure in G and $N \cap (H + G_t) = N_t$. Hence H is pure N_t-high in $H + G_t$ by (i).

2.10. Corollary. Let N be a subgroup of a group G and H an N-pure-high subgroup of G. If K/G_t is an $(N + G_t)/G_t$-high subgroup of G/G_t containing $(H + G_t)/G_t$ then H is N_t-pure-high in K.

Proof. Since G/G_t is torsion free, K/G_t is pure in G/G_t and K is pure in G. Obviously $K \cap H = N_t$. Finally, H is N_t-pure-high in K by 2.8.

3. Splitting pure N-high subgroups

3.1. Theorem. Let N be a subgroup of a group G and $(G/N)_t = T/N$. If $H = H_t \oplus B$ is a splitting N-pure-high subgroup of G then for each subgroup Y of G with $G_t \subseteq Y \subseteq T$ there is a Y-pure-high subgroup X of G such that B is $N \cap X$-high in X. Further B is a T-pure-high subgroup of G.

Proof. Obviously B is pure in G and $B \cap Y = 0$ (see 2.5). Let X be a Y-pure-high subgroup of G containing B and S be an $N \cap X$-high subgroup of X containing B. Since S is pure in X (X is torsion free) and hence in G, $H_t \oplus S$ is pure in G and $(H_t \oplus S) \cap N = 0$ by 2.1. Consequently $S = B$. For the rest put $Y = T$.

Note that $N \cap X$-high subgroups of X are exactly $T \cap X$-high since $N \cap X$ is essential in $T \cap X$.

Conversely, if A is a pure N_t-high subgroup of G_t and B is an $N \cap X$-high subgroup of a Y-pure-high subgroup X of G then $A \oplus B$ is contained in some N-pure-high subgroup H of G by 2.1. Obviously $H_t = A$, $H \cap X = B$ and it is easy to see that $H/A \oplus B$ is torsion free. If $Y = T$ (i.e. B is T-pure-high in G) then B is A-pure-high in H. If $Y = G_t$ then $H \cap (G_t \oplus X) = A \oplus B$. For, if $h = t + x \in H \cap (G_t \oplus X)$ then $o(t)\, h = o(t)\, x \in H \cap X = B$, $o(t)\, h = o(t)\, b$ for some $b \in B$, $h - b \in G_t \cap H = A$ and $h \in A \oplus B$.

3.2. Theorem. Let N be a subgroup of a group G and $(G/N)_t = T/N$, let Y be a subgroup of G such that $G_t \subseteq Y \subseteq T$ and X be a pure Y-high subgroup of G. If A is a pure N_t-high subgroup of G_t and B an $N \cap X$-high subgroup of X then $A \oplus B$ is a splitting pure N-high subgroup of G.

Proof. Since X is torsion free, B is pure in X and hence in G. By 2.1, $A \oplus B$ is pure in G and $(A \oplus B) \cap N = 0$. Let H be an N-high subgroup of G containing $A \oplus B$; obviously $H_t = A$. Let $h \in H$. Since X is Y-high in G, $kh = x + y$ for some $x \in X$, $y \in Y$ and an integer k. Since B is $N \cap X$-high in X and A is $N \cap Y$-high in Y (see 2.5), we have $rx = b + n$ and $my = a + \bar{n}$, where $b \in B$, $a \in A$, $n, \bar{n} \in N$ and m, r are integers. Hence $kmrh = mb + mn + ra + r\bar{n}$, further $kmrh - mb - ra = mn + r\bar{n} \in H \cap N = 0$, i.e. $kmrh \in A \oplus B$. Since $A \oplus B$ is pure in G, $kmrh = kmr(\bar{a} + \bar{b})$, where $\bar{a} \in A$, $\bar{b} \in B$. Consequently, $h - \bar{a} - \bar{b} \in H_t = A$ and $h \in A \oplus B$.

3.3. Corollary. Let N be a subgroup of a splitting group $G = G_t \oplus X$. If A is a pure N_t-high subgroup of G_t and B is $N \cap X$-high subgroup of X then $A \oplus B$ is a splitting pure N-high subgroup of G.

3.4. Corollary. Let G be a group. The following assertions are equivalent:

(i) G is splitting.

(ii) For each subgroup N of G there is a splitting pure N-high subgroup of G.

(iii) For each subgroup N of G there is a pure N-high subgroup of G.

(iv) There is a pure G_t-high subgroup of G.

Proof. (i) → (ii) follows from 3.3, (ii) → (iii) → (iv) is trivial, (iv) → (i) is easy and well-known.

The equivalence (i) ↔ (iii) from 3.4 is proved in [1] (theorem 5). On the other hand, it is proved in [7] (corollary of 3.1) that a reduced group G splits if and only if some N-high subgroup of G splits, where $N \subseteq G^1 \cap G_t$. For the equivalence (i) ↔ (iv) of 3.4 see [12] (proposition 5.1) and [1] (corollary on p. 481).

Note that if one N-pure-high subgroup of a group G splits then all N-pure-high subgroups need not split (even if G itself splits) – see [8] (example on p. 190).

3.5. Theorem. Let N be a subgroup of a group G and $(G/N)_t = T/N$. All splitting pure N-high subgroups of G are exactly all direct sums of a pure N_t-high subgroup of G_t and a pure T-high subgroup of G.

Proof. If $H = H_t \oplus B$ is a pure N-high subgroup of G then H_t is a pure N_t-high subgroup of G_t by 2.5 and B is a T-pure-high subgroup of G by 3.1. If $g \in G$ then $kg = n + h$, where $n \in N$, $h \in H$ and k is a nonzero integer. Hence $kg \in T \oplus B$, $G/T \oplus B$ is a torsion group and B is T-high in G by 2.7. Conversely, if A is a pure N_t-high subgroup of G_t and B a pure T-high subgroup of G then $A \oplus B$ is a pure N-high subgroup of G by 3.2.

Remark that T-high subgroups of G are exactly $N + G_t$-high (see 2.3 (iii)).

3.6. Theorem. Let N be a subgroup of a group G. A subgroup $H = H_t \oplus B$ is pure N-high in G if and only if H_t is a pure N_t-high subgroup of G_t and $(G_t \oplus B)/G_t$ is an $(N + G_t)/G_t$-high subgroup of G/G_t.

Proof. Let $H = H_t \oplus B$ be a pure N-high subgroup of G. Thus $(B \oplus G_t)/G_t \cap \cap (N + G_t)/G_t = 0$; let K/G_t be an $(N + G_t)/G_t$-high subgroup of G/G_t containing $(B \oplus G_t)/G_t$. If $k \in K$ then $rk = n + h$, where $n \in N$, $h \in H$ and r is a nonzero integer, since H is N-high in G. Hence $rk - h = n \in N \cap K \subseteq G_t$ and $rk = n + h \in \in G_t \oplus B$. Since $G_t \oplus B$ is absorbing in G, $k \in G_t \oplus B$. The rest follows from 2.5.

Conversely, let A be a pure N_t-high subgroup of G ane $(B \oplus G_t)/G_t$ be an $(N + G_t)/G_t$-high subgroup of G/G_t. Since B is pure in G, $A \oplus B$ is pure in G and $(A \oplus B) \cap N = 0$ by 2.1. If H is an N-high subgroup of G containing $A \oplus B$ then $G_t \oplus B \subseteq G_t + H$ and $(H + G_t)/G_t \cap (N + G_t)/G_t = 0$. Hence $G_t \oplus B = = G_t + H$. If $h \in H$ then $h = t + b$, where $t \in G_t$ and $b \in B$, Now $t \in H_t = A$ and $h \in A \oplus B$. Consequently $H = A \oplus B$.

4. Intersection of N-pure-high subgroups

The well-known theorem of Grätzer and Schmidt (9.6, [5]) describes the intersection of all complements to a direct summand N of a group G. The intersection of all N-high subgroups has been described by F. V. Krivonos in 1975:

4.1. Theorem (Proposition 9, [10]). If N is a nonzero subgroup of a group G and $\mathbf{R} = \{p \in \mathbf{P}; N_p = 0\}$ then $\bigoplus_{p \in \mathbf{R}} G_p$ is the intersection of all N-high subgroups of G.

Proof. Let H be an N-high subgroup of G and $A = \bigoplus_{p \in \mathbf{R}} G_p$.

If $h + a = n$ $(h \in H, a \in A, n \in N)$ then $o(a)\, n = o(a)\, h \in H \cap N = 0$ and hence $n = 0$. Consequently $(H + A) \cap N = 0$ and $A \subseteq H$.

If $g \in G$ is an element of infinite order such that $\langle g \rangle \cap N = 0$ and $n \in N$ is a nonzero element then $\langle g + n \rangle \cap N = 0$. If $g \in G \setminus N$, $n \in N$ and $o(g) = o(n) =$

$= p \in \mathbf{P} \setminus \mathbf{R}$ then $\langle g + n \rangle \cap N = 0$. In the both cases, an N-high subgroup of G containing $\langle g + n \rangle$ does not contain the element g. Hence A is the intersection of all N-high subgroups of G.

Remark that K. Benabdallah and J. M. Irwin proved that the intersection of all N-high subgroups of a primary group G is trivial whenever N is a nontrivial subgroup of G (Lemma 1.2, [2]). For the original proof of 4.1 see [10]. The first step of our proof corresponds with our assertion 2.2 (i), the second step partially corresponds with the proof of the following theorem.

4.2. Theorem. If N is a subgroup of a group G and $\mathbf{R} = \{p \in \mathbf{P};\ N_p = 0\}$ then $\bigoplus_{p \in \mathbf{R}} G_p$ is the torsion part of the intersection of all N-pure-high subgroups of G.

Proof. With respect to 2.2 it is sufficient to prove that for each prime $p \in \mathbf{P} \setminus \mathbf{R}$ and each element $g \in G[p] \setminus N$ there is an N-pure-high subgroup of G which does not contain the element g. We consider three cases:

Case 1: There are elements at least of two different p-heights in $N[p]$.
In this case there is an element $n \in N[p]$ such that the element $g + n$ is of finite p-height. Hence the element $g + n$ can be embedded in a finite cyclic direct summand Y of G that is disjoint from N (27.2, [5]). Finally, Y can be embedded in an N-pure-high subgroup X of G and obviously $g \notin X$.

Case 2: $N[p] \subseteq p^\omega G_p$.
If $n \in N[p]$ is a nonzero element then there is an N_p-high subgroup Y of G_p containing $\langle g + n \rangle$. Now, Y is pure in G_p by theorem 1.2 and hence Y can be embedded in an N-pure-high subgroup X of G. It is easy to see that $g \notin X$.

Case 3: $N[p] \subseteq p^k G_p \setminus p^{k+1} G_p$.
If there is a nonzero element $n \in N[p]$ such that $g + n$ is of finite p-height then we proceed as in the case 1. Suppose $g + n \in p^\omega G_p$ for each nonzero $n \in N[p]$. If $p^{k+1} G[p] \neq p^\omega G[p]$ then there is a direct summand Y of G_p such that $Y[p] = p^{k+1} G[p]$ by theorem 4.4, [7]; if X is an N-pure-high subgroup of G containing Y then $g \notin X$. If $p^{k+1} G[p] = p^\omega G[p]$ then $G_p = B \oplus D$, where B is bounded and D is divisible, and D can be embedded in an N-pure-high subgroup X of G; obviously $g \notin X$.

4.3. Corollary. Let N be a subgroup of a group G and $\mathbf{R} = \{p \in \mathbf{P};\ N_p = 0\}$. If G/N is a torsion group then $\bigoplus_{p \in \mathbf{R}} G_p$ is the intersection of all pure N-high subgroups of G.

Proof. The N-pure-high subgroups of G are exactly the pure N-high subgroups of G and all these subgroups are torsion, since G/N is torsion (see 1.3 and 1.4). Our corollary follows now from 4.2.

Note that K. Benabdallah and J. M. Irwin proved that the intersection of all pure N-high subgroups of a primary group G is trivial whenever N is a nontrivial subgroup of G (lemma 1.2, [3]).

4.4. Proposition. Let N be a subgroup of a group G, $(G/N)_t = T/N$. If Y is a subgroup of G such that $G_t \subseteq Y \subseteq T$, X is a Y-pure-high subgroup of G such that $N \cap X \neq 0$ and $(G/(X \oplus Y))_t = K/(X \oplus Y)$ then the intersection of all N-pure-high subgroups of G contains no element of infinite order from K.

Proof. With respect to the theorem 2.5 it is sufficient to consider an element $g \in G \setminus T$ such that $kg \in X \oplus Y$ for a nonzero integer k. Hence $kg = x + y$, where $x \in X$ and $y \in Y$. Since $g \notin T$, we have $x \notin T$.

Case 1: $y \notin \bigoplus_{p \in \mathbf{R}} G_p$ $(\mathbf{R} = \{p \in \mathbf{P};\ N_p = 0\})$.

Let A be a pure N_t-high subgroup of G_t such that $y \notin A$ (see 4.3), let B be an $N \cap X$-high subgroup of X containing x. By 2.1, $A \oplus B$ is contained in an N-pure-high subgroup H of G. If $x + y \in H$ then $y \in H \cap Y = H_t = A$ – a contradiction. Hence $g \notin H$.

Case 2: $y \in \bigoplus_{p \in \mathbf{R}} G_p$.

Let A be a pure N_t-high subgroup of G_t and B be an $N \cap X$-high subgroup of X which does not contain x (see 4.1). By 2.1, $A \oplus B$ is contained in an N-pure-high subgroup H of G. If $x + y \in H$ then $x \in H$ by 2.2 and $x \in H \cap X = B$ – a contradiction. Hence $g \notin H$.

4.5. Corollary. Let N be a subgroup of a group G, $(G/N)_t = T/N$. If Y is a subgroup of G such that $G_t \subseteq Y \subseteq T$ and X is a pure Y-high subgroup of G such that $N \cap X \neq 0$ then $\bigoplus_{p \in \mathbf{R}} G_p$ is the intersection of all splitting pure N-high subgroups of G.

Proof. According to proof of 4.4 (in both cases we have $A \oplus B = H$ by 3.2).

4.6. Corollary. Let N be a subgroup of a splitting group G. If N is not torsion then $\bigoplus_{p \in \mathbf{R}} G_p$ is the intersection of all splitting pure N-high subgroups of G.

Corollary 4.6 can be easily proved also by means of theorem 3.6.

If N is a torsion subgroup of a splitting group G then the intersection of all pure N-high subgroups of G can contain also elements of infinite order as the following example shows.

4.7. Example. Let $G = \langle a \rangle \oplus \langle b \rangle$, where $o(a) = 2$ and $o(b) = \infty$. The subgroups $\langle b \rangle$ and $\langle a + b \rangle$ are obviously pure G_t-high in G. It is easy to see that the subgroup $\langle kb \rangle$, where $k \neq \pm 1$, and $\langle a + kb \rangle$, where $k \neq \pm 1$ is an odd integer, are not G_t-pure-high in G. Further

$$2(a + 2kb) = 4kb \in \langle a + 2kb \rangle \cap 4G .$$

If $\langle a + 2kb \rangle$ is pure in G, $4kb \in 4\langle a + 2kb \rangle$, i.e. $4kb = 4ra + 8rkb$, $4k(1 - 2r)\, b = o$ and $k = 0$. Hence the subgroups $\langle a + 2kb \rangle$, where $k \neq 0$ is an integer, are not pure in G. Consequently, there are only two G_t-pure-high subgroups of G. They are G_t-high in G and moreover, they are complements of G_t. Finally, $\langle 2b \rangle$ is the intersection of all pure G_t-high subgroups of G.

5. An example

In the following theorem we shall investigate the well-known group from example 2, § 100 [5].

5.1. Theorem. Let $p_1, p_2, \ldots$ be different primes, and $A = \prod_{i=1}^{\infty} \langle a_i \rangle$, where $o(a_i) = p_i$. Let $G = \langle A_t, b_0, b_1, b_2, \ldots \rangle$, where $b_0 = (a_1, a_2, \ldots) \in A$ and for each $j = 1, 2, \ldots$, b_j has 0 for its j-th coordinate and satisfies

$$p_j b_j = (a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots) = b_0 - a_j \, .$$

Then

(i) If S is a pure subgroup of G then either S is torsion or S is a direct complement of a finite subgroup in G.

(ii) If N is a finite subgroup of G then each N-pure-high subgroup of G is a direct complement of N in G.

(iii) If N is an infinite subgroup of G then the unique N-pure-high subgroup of G is $\bigoplus_{p \in \mathbf{R}} G_p$, where $\mathbf{R} = \{p \in \mathbf{P}; N_p = 0\}$.

(iv) 0 is N-pure-high in G if and only if $G_t \subseteq N$.

The proof of this result is based on the next lemmas. It is easy to see that $G_t = A_t = \bigoplus_{i=1}^{\infty} \langle a_i \rangle$.

5.2. Lemma. Let X be a subgroup of G. If X is not torsion then there is a natural number m such that $mb_0 \in X$.

Proof. Let $x = t + k_0 b_0 + k_1 b_1 + \ldots + k_n b_n$ be an element of infinite order ($x \in X$, $t \in G_t$, k_i are integers). Then

$$(p_1 p_2 \ldots p_n)\, x = t' + k b_0 \, ,$$

where k is an integer and $t' \in G_t$. Hence there is a natural number m such that $mb_0 \in X$.

5.3. Lemma. Let S be a pure subgroup of G and $g \in G$.

(i) If $kp^2 g \in S$ for an integer k and a prime p then $kpg \in S$.

(ii) If m is the least natural number such that $mg \in S$ then m is square-free.

Proof. Let $kp^2 g \in S$. Then $kp^2 g = kp^2 s$ for some $s \in S$ and hence $kp^2(g - s) = o$ and $g - s \in G_t$. With respect to the form of G_t, $kp(g - s) = o$ and $kpg \in S$. Obviously, (i) implies (ii).

5.4. Lemma. Let S be a pure subgroup of G and m be the least natural number such that $mb_0 \in S$. Then

(i) m is square-free.

(ii) If $(p_j, m) = 1$ then $a_j \in S$, S is p_j-absorbing in G and m is the least natural number with $mb_j \in S$.

(iii) If $m = p_j m_j$ then $a_j \notin S$ and m_j is the least natural number with $m_j b_j \in S$.

(iv) $S_t = \bigoplus_{\substack{i=1 \\ p_i \nmid m}}^{\infty} \langle a_i \rangle$.

(v) $mG \subseteq S$.

(vi) $G = S \oplus \bigoplus_{\substack{i=1 \\ p_i | m}}^{\infty} \langle a_i \rangle$.

Proof. By lemma 5.3, m is square-free.

Suppose $(p_j, m) = 1$. We have

$$mp_j b_0 = mp_j(p_j b_j + a_j) = mp_j^2 b_j \in S .$$

Since b_j is divisible by p_j, it is $mb_j \in S$ by lemma 5.3. Further, $ma_j = mb_0 - mp_j b_j \in \in S$ and hence $a_j \in S$. If $g \in G$ and $p_j g \in S$ then $p_j g = p_j s$ for some $s \in S$ and hence $p_j(g - s) = o$, i.e. $g - s \in \langle a_j \rangle \subseteq S$ and $g \in S$. Consequently, S is p_j-absorbing in G. Finally, if $\bar{m} b_j \in S$ and $\bar{m} < m$ then $\bar{m} p_j b_j = \bar{m}(b_0 - a_j) = \bar{m} b_0 - \bar{m} a_j$ and $\bar{m} b_0 \in S$ – a contradiction with the definition of m.

Suppose $m = p_j m_j$. We have

$$mb_0 = m(p_j b_j + a_j) = mp_j b_j = m_j p_j^2 b_j \in S .$$

Since b_j is divisible by p_j, it is $m_j b_j \in S$ by lemma 5.3. Further $m_j a_j = = m_j(b_0 - p_j b_j) = m_j b_0 - mb_j \notin S$ and hence $a_j \notin S$. If $\bar{m} b_j \in S$ and $\bar{m} < m_j$ then $p_j \bar{m} b_0 = p_j \bar{m}(p_j b_j + a_j) = p_j^2 \bar{m} b_j \in S$ – a contradiction with the definition of m.

The assertions (iv), (v) follow from (ii), (iii). Write $T = \bigoplus_{p_i | m} \langle a_i \rangle$. If $g \in G$ then $mg \in S$ and $mg = ms$ for some $s \in S$. Hence $g - s = t \in T$, $g \in S + T$, i.e. $G = = S + T$. By (iv), $S \cap T = 0$ and consequently $G = S \oplus T$.

Proof of theorem 5.1. If S is a pure subgroup of G and S is not torsion then by lemma 5.2 there is a natural number m such that $mb_0 \in S$; let m be the least natural number with this property. By lemma 5.4, S is a complement of a finite subgroup of G.

Let N be a finite subgroup of G and S be an N-pure-high subgroup of G. By 2.6 (iii), $G_t = N \oplus S_t$. Since N is a direct summand of G and each complement of N in G contains S_t, S is not torsion. By lemma 5.4, $G = N \oplus S$.

Let N be an infinite subgroup of G. If S is a pure subgroup of G and S is not torsion then $N \cap S \neq 0$. For, if N is not torsion then there is a natural number k such that $kb_0 \in N \cap S$ by lemma 5.2 and if N is torsion then $N \cap S \neq 0$ by lemma 5.4.

Consequently, each N-pure-high subgroup of G is torsion. With respect to the form of G_t, $H = \bigoplus_{p_i \in \mathbf{R}} \langle a_i \rangle$ is the unique N-pure-high subgroup of G by 2.2 ($\mathbf{R} = \{p \in \mathbf{P};\ N_p = 0\}$) and hence $G_t = N_t \oplus H$.

If N contains G_t then 0 is N-pure-high in G by (iii). If 0 is N-pure-high in G then N is infinite by (ii) and $N \supseteq G_t$ by (iii).

5.5. Remark. The group G is obviously of torsion-free rank 1, G does not split (see 5.1 (i)).

If N is a finite subgroup of G then the intersection of all N-pure-high subgroups of G contains elements of infinite order by lemma 5.4 (compare with 4.3—4.6).

If N is an infinite torsion subgroup of G then G/N is not torsion, there is a unique N-pure-high subgroup H of G, H is torsion and H is not N-high in G.

If N is a subgroup of G which is not torsion then G/N is torsion, there is a unique N-pure-high subgroup H of G, H is torsion and N-high in G.

References

[1] BENABDALLAH, K.: On pure-high subgroups of abelian groups, Canad. Math. Bull. 17 (1974) 479—482.

[2] BENABDALLAH, K. and IRWIN, J.: N-high subgroups and p-adic topology, Comment. Math. Univ. St. Pauli 21 (1972), 43—45.

[3] BENABDALLAH, K. and IRWIN, J.: Pure N-high subgroups, p-adic topology and direct sums of cyclic groups, Canad. J. Math. 26 (1974), 322—327.

[4] BENABDALLAH, K., IRWIN, J. M. and RAFIQ, M.: N-high subgroups of abelian p-groups, Arch. Math. 25 (1974), 29—34.

[5] FUCHS, L.: Infinite abelian groups I, II, Acad. Press, 1970, 1973.

[6] IRWIN, J. M.: High subgroups of abelian torsion groups, Pacific J. Math. 11 (1961), 1375 to 1384.

[7] IRWIN, J. and BENABDALLAH, K.: On N-high subgroups of abelian groups, Bull. Soc. Math. France 96 (1968), 337—346.

[8] IRWIN, J. PEERCY, C. and WALKER, E.: Splitting propeties of high subgroups, Bull. Soc. Math. France 90 (1962), 185—192.

[9] IRWIN, J. M. and WALKER, E. A.: On N-high subgroups of abelian groups, Pacific J. Math. 11 (1961), 1363—1374.

[10] KRIVONOS, F. V.: Ob N-vysokich podgruppach abelovoj grupy, Vest. Moskov. Univ. (1975), 58—64.

[11] PIERCE, R. S.: Centers of purity in abelian groups, Pacific J. Math. 13 (1963), 215—219.

[12] REID, J. D.: On subgroups of an abelian group maximal disjoint from a given subgroup, Pacific J. Math. 13 (1963), 657—664.