On the Open Mapping Principle and Convex Multivalued Mappings

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Received 17 September 1984

We generalize the open mapping principle and apply it to study convex multivalued mappings.

Zobecníme princip otevřených zobrazení a využijeme ho k vyšetřování konvexních mnohoznačných zobrazení.

Дается обобщение принципа открытых отображений и применится к исследовании выпуклых многозначных отображений.

We generalize the open mapping principle for mappings defined on closed subsets of a Banach space and some results in [4], [5], [7] derive as its corollaries. In the conclusion we prove that a convex multivalued mapping F whose value at some point x_0 is a closed convex bounded subset and whose domain is the whole space is of the form $F(x) = F(x_0) + T(x)$, where T is a linear mapping.

Let X, Y be metric spaces, $f: X \to Y$ a mapping. For each $r > 0$, $x \in X$, we put:

$$k^r(f, x) = r^{-1} \sup \{s: s \geqq 0,\ B_s(f(x)) \subseteq \overline{f(B_r(x))}\}$$

where $B_r(x)$ denotes the ball with center x and radius r. Of course $k^r(f, x) \geqq 0$ for all $r > 0$, $x \in X$. Put:

$$k(f, x_0) = \lim_{\substack{x \to x_0 \\ r \to 0}} k^r(f, x)$$

Lemma 1. Let X, Y be Banach spaces, A, B convex subsets of X, $T \in L(X, Y)$. Suppose that $\overline{T(A)} \supseteq B_r(y_0)$ and $h(A, B) \leqq \varepsilon/\|T\|$, $0 < \varepsilon < r$, where $h(A, B)$ denotes the Hausdorff distance between sets A, B. Then $\overline{T(B)} \supseteq B_{r-\varepsilon}(y_0)$.

Proof: In contrary we suppose that $B_{r-\varepsilon}(y_0) \nsubseteq \overline{T(B)}$. Let $y_1 \in B_{r-\varepsilon}(y_0)$ and $y_1 \notin \overline{T(B)}$. For $\overline{T(B)}$ is a closed convex subset of Y, there exists a $y_1^* \in Y^*$, $\|y_1^*\| = 1$ and α, β such that $y_1^*(y_1) = \alpha > \beta \geqq y_1^*(y)$ for all $y \in \overline{T(B)}$. $y_1^*(y_1 - y_0) = y_1^*(y_1) - y_1^*(y_0) = \alpha - y_1^*(y_0) \leqq \|y_1 - y_0\| \leqq r - \varepsilon$. Take $y_n \in B_r(y_0)$ such that

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$\lim_n y_1^*(y_n - y_0) = r$. Thus $\|y_n - y\| \geqq y_1^*(y_n - y) \geqq y_1^*(y_n - y_0) - y_1^*(y) + y_1^*(y_0) \geqq y_1^*(y_n - y_0) - \beta + y_1^*(y_0)$ for all $y \in \overline{T(B)}$. Hence $b(T(A), T(B)) \geqq \lim_n y_1^*(y_n - y_0) - \beta + y_1^*(y_0) \geqq r - \beta + y_1^*(y_0) > r - \alpha + y_1^*(y_0) = \varepsilon$.

That contradicts the fact $h(T(A), T(B)) \leqq \|T\| \, h(A, B) \leqq \varepsilon$. This finishes the proof of Lemma 1.

Proposition 1. Let X, Y be Banach spaces, A a closed convex subset of X, $T \in L(X, Y)$. Then

(1) $k^r(T \mid A, x) \leqq k^{r'}(T \mid A, x)$ for all $r > r' > 0$ and $x \in A$,

(2) $k(T \mid A, x) = \lim_{r \to 0} k^r(T \mid A, x)$ for all $x \in A$,

where $T \mid A$ denotes the restriction of T on A.

Proof: It is clear that $B_{\lambda r}(x) \cap A \supseteq \lambda(B_r(x) \cap A - x) + x$ for all $r > 0$, $0 < \lambda \leqq 1$ and $x \in A$. If $B_s(T(x)) \subseteq \overline{T(B_r(x) \cap A)} = \overline{T(B_r(x) \cap A - x)} + T(x)$ then $B_{\lambda s}(T(x)) \subseteq \lambda \overline{T(B_r(x) \cap A - x)} + T(x) \subseteq \overline{T(B_{\lambda r}(x) \cap A)}$. Thus $k^{\lambda r}(T \mid A, x) \geqq k^r(T \mid A, x)$ for $0 < \lambda \leqq 1$ or $k^r(T \mid A, x) \leqq k^{r'}(T \mid A, x)$ for all $r > r' > 0$, $x \in A$. By the Dini theorem we have:

$$\lim_{\substack{x \to x_0 \\ r \to 0}} k^{r'}(T \mid A, x) = \lim_{r \to 0} \lim_{x \to x_0} k^r(T \mid A, x) .$$

On the other hand $h(B_r(x) \cap A, B_r(x_0) \cap A) \leqq \|x - x_0\|$. Then

$$\lim_{x \to x_0} h(B_r(x) \cap A, B_r(x_0) \cap A) = 0 .$$

By Lemma 1 $\lim_{x \to x_0} k^r(T \mid A, x) = k^r(T \mid A, x_0)$. Thus $k(T \mid A, x_0) = \lim_{r \to 0} k^r(T \mid A, x_0)$, for all $x_0 \in A$.

Theorem 1. Let X be a complete metric space, $x_0 \in X$, Y a normed space, $f: X \to Y$ a continuous mapping. Suppose that there is a continuous mapping $g: X \to Y$ and an $r > 0$ such that $g(x_0) = f(x_0)$ and

(1) $k(g, x_0) > 0$,

(2) $\|f(x) - f(x') - g(x) + g(x')\| \leqq K \, d(x, x')$ for all $x, x' \in X$, $d(x, x_0) < r$, $d(x', x_0) < r$,

(3) $K(k(g, x_0))^{-1} < 1$.

Then $f(x_0) \in \operatorname{int}(f(B_s(x_0)))$ for all $s > 0$.

Proof: Choose $\theta \in (0, 1)$, $\varepsilon \in (0, 1)$ such that $(K + \theta)(k(g, x_0) - \varepsilon)^{-1} < 1$. Put $\varkappa = k(g, x_0) - \varepsilon$. Then there exists a $b > 0$ such that $B_{\varkappa s}(g(x)) \subseteq \overline{g(B_s(x))}$ for all $x \in X$, $d(x, x_0) < b < r$ and $0 < s < b$. Put $q = (K + \theta)\varkappa^{-1} < 1$. Let $y \in Y$ and $\|y - f(x_0)\| \leqq \varkappa(1 - q)\, s$, $0 < s < b$. We construct inductively the following sequence $\{x_n\}$ such that: (1) $d(x_{n+1}, x_n) \leqq q^n(1 - q)\, s$, (2) $\|g(x_{n+1}) - g(x_n) +$

$+ f(x_n) - y\| \leqq \theta d(x_{n+1}, x_n)$ for all n. Since $\|y - f(x_0)\| \leqq \varkappa(1 - q)s$ then $y \in B_{\varkappa(1-q)s}(g(x_0)) \subseteq \overline{g(B_{(1-q)}(x_0))}$. If $y = f(x_0) = g(x_0)$ then put $x_1 = x_0$. If $a = \|y - g(x_0)\| > 0$ then by the continuity of g there is a $\delta > 0$ such that $\|g(x) - g(x_0)\| < a/2$ for all x, $d(x, x_0) < \delta$. Choose $x_1 \in B_{(1-q)s}(x_0)$ such that $\|g(x_1) - y\| \leqq \theta \min\{a/2, \delta\}$. Then of course $d(x_1, x_0) \geqq \delta$ and $\|g(x_1) - y\| \leqq \theta\delta \leqq \theta\, d(x_1, x_0)$. Suppose that we have constructed $\{x_k\}$, $0 < k \leqq n$ satisfying the inductive assumptions. Then

$$d(x_n, x_0) \leqq \sum_{k=0}^{n-1} d(x_{k+1}, x_k) \leqq s(1 - q)(1 - q^n)/(1 - q) < s\,.$$

Consider

$$y_n = g(x_n) - g(x_{n-1}) + f(x_{n-1}) - y\,, \quad z_n = g(x_n) - f(x_n) + y\,.$$

By the inductive assumptions we have $\|y_n\| \leqq \theta\, d(x_n, x_{n-1})$, $\|z_n - g(x_n)\| = \|f(x_n) - f(x_{n-1}) - g(x_n) + g(x_{n-1}) + y_n\| \leqq Kd(x_n, x_{n-1}) + \theta\, d(x_n, x_{n-1}) = (K + \theta)\, d(x_n, x_{n-1}) \leqq (K + \theta)\, q^n(1 - q)\, s$. Thus $z_n \in B_{(K+\theta)q^{n-1}(1-q)s}(g(x_n)) \subseteq \overline{g(B_{q^n(1-q)s}(x_n))}$. In the same argument as in the construction of x_1, we choose $x_{n+1} \in B_{q^n(1-q)s}(x_n)$ such that: $\|g(x_{n+1}) - z_n\| \leqq \theta\, d(x_{n+1}, x_n)$. Then

$$\|g(x_{n+1}) - g(x_n) + f(x_n) - y\| \leqq \theta\, d(x_{n+1}, x_n)\,,$$
$$d(x_{n+1}, x_n) \leqq q^n(1 - q)\, s\,.$$

That completes the inductive construction. Of course $\{x_n\}$ is a Cauchy sequence in X, then there exists an $x \in X$, $x = \lim x_n$; it is $d(x, x_0) \leqq s$ and $0 = \lim\,(g(x_{n+1}) - g(x_n) + f(x_n) - y) = f(x) - y$. Thus $y = f(x)$. This proves that $B_{\varkappa(1-q)s}(f(x_0)) \subseteq f(B_s(x_0))$, i.e. $f(x_0) \in \operatorname{int} f(B_s(x_0))$. This ends the proof of Theorem 1.

Corollary 1. Let X, Y be Banach spaces, $A \subseteq X$ a convex closed subset of X, $T \in L(X, Y)$ such that $T(A)$ is a set of the second category. Then $\operatorname{int} \overline{T(A)} \neq \emptyset$ and if $x \in A$, $T(x) \in \operatorname{int} T(A)$, then $k(T \mid A, x) > 0$.

Proof: Let x_0 be any point of A. Without loss of generality we can suppose $x_0 = 0$. Then for $r > 0$ we have

$$A = \bigcup B_{nr}(0) \cap A \subseteq \bigcup_{n=1}^{\infty} n(B_r(0) \cap A)\,,$$

$$T(A) = \bigcup_{n=1}^{\infty} T(B_{nr}(0) \cap A) \subseteq \bigcup_{n=1}^{\infty} nT(B_r(0) \cap A)\,.$$

Since $T(A)$ is of the second category, there exists an n_0 such that $\operatorname{int} \overline{T(B_{n_0r}(0) \cap A)} \neq \emptyset$. Choose $y_1 = T(x_1) \in T(B_{n_0r}(0) \cap A)$ and $s > 0$ such that: $T(x_1) + B_s(0) \subseteq \overline{T(B_{n_0r}(0) \cap A)} \subseteq \overline{T(B_{n_0r + \|x_1\|}(x_1) \cap A)}$. Then by Proposition 1, we have

$k(T \mid A, x_1) \geqq k^{n_0 r + \|x_1\|}(T \mid A, x_1) \geqq s/(n_0 r + \|x_1\|) > 0$. In Theorem 1, put $X = A$, $f = g = T$; we have $T(x_1) \in \operatorname{int} T(B_r(x_1) \cap A) \subseteq \operatorname{int} T(A)$ for all $r > 0$. Thus $\operatorname{int} T(A) \neq \emptyset$. If $0 = T(0) \in \operatorname{int} T(A)$ then there is a $K > 0$ such that $-(y_1/K) \in T(A)$. Let $n_1 \in N$ such that $-(y_1/K) \in T(B_{n_1 r}(0) \cap A) \subseteq n_1 T(B_r(0) \cap A)$; then

$$B_{s/K}(0) \subseteq \frac{1}{K} \overline{T(B_{n_0 r}(0) \cap A)} + n_1 T(B_r(0) \cap A) \subseteq \left(\frac{n_0}{K} + n_1\right) \overline{T(B_r(0) \cap A)}\,.$$

Then

$$k(T \mid A, 0) \geqq k^r(T \mid A, 0) \geqq \frac{s}{Kn_1 + n_0} > 0\,.$$

That finishes the proof of Corollary 1.

Let A be a convex subset of a Banach space X, put $\overset{r}{\operatorname{Cor}} A = \{x \in A\colon$ for each $y \in A$, $y \neq x$ there is a $z \in A$ and a $\lambda \in (0, 1)$ such that $x = (1 - \lambda)\, y + \lambda z\}$.

Corollary 2 (P. C. Duong - H. Tuy [7]). Let X, Y be Banach spaces, $F\colon X \to 2^Y$ a multivalued closed convex mapping such that $F(X)$ is of the second category. Then for each $x_0 \in \overset{r}{\operatorname{Cor}}(\operatorname{dom} F)$ and for each open set $U \ni x_0$, $F(x_0) \cap \operatorname{int} F(U) \neq \emptyset$.

Proof. Put $A = \operatorname{Gr}(F) = \{(x, y)\colon y \in F(x),\ x \in X\}$. By the assumption, A is a closed convex subset of the Banach space $X \times Y$. We define $T\colon X \times Y \to Y$ by $T(x, y) = y$. Then $F(U) = T(U \times Y \cap A)$. By Corollary 1 there is $y_1 \in \operatorname{int} T(A) \neq \emptyset$, for $T(A) = F(X)$ is of the second category. Let $y_1 \in F(x_1)$, $x_1 \in \operatorname{dom} F$. There is an $x_2 \in \operatorname{dom} F$ and a $\lambda \in (0, 1)$ such that $x_0 = \lambda x_1 + (1 - \lambda)\, x_2$. Take a $y_2 \in F(x_2)$. Then $y_0 = \lambda y_1 + (1 - \lambda)\, y_2 \in \operatorname{int} T(A)$, $y_0 \in F(\lambda x_1 + (1 - \lambda)\, x_2) = F(x_0)$. By Corollary 1, $k(T \mid A, (x_0, y_0)) > 0$. Putting $f = g = T$, $X = A$ in Theorem 1, we have $y_0 \in \operatorname{int} T(B_r(x_0, y_0) \cap A)$ for all $r > 0$, hence $y_0 \in \operatorname{int} F(U) = \operatorname{int} T(U \times Y \cap A)$ for all open sets U containing x_0.

Corollary 3. (Robinson [4] - P. C. Duong - H. Tuy [7].) Let X, Y be Banach spaces, $F\colon X \to 2^Y$ a multivalued closed convex mapping such that $F(X)$ is open. Then $F(U)$ is open for each open set U.

Proof. Put $A = \operatorname{Gr}(F)$, $T((x, y)) = y$. Then $T((x, y)) \in \operatorname{int} T(A)$ for each $(x, y) \in A$. Thus $k(T \mid A, (x, y)) > 0$. Then $T(V \cap A)$ is open for each open set V in $X \times Y$. Hence $F(U) = T(U \times Y \cap A)$ is open for each open set U.

Recall that a multivalued mapping $F\colon X \to 2^Y$ is surjective at a point x_0 if it carries every neighbourhood U of x_0 onto a neighborhood $F(U)$ of $F(x_0)$.

Let M be a subset of Y. We say that a singlevalued mapping f is M-surjective at x_0 if the mapping $f(x) - M$ is surjective at x_0.

Let X, Y be Banach spaces, $F: X \to 2^Y$ be a multivalued convex mapping. Put $\tilde{k}^r(F, (x_0, y_0)) = r^{-1} \sup \{\inf\{\|y - y_0\|, y \in F(x)\}, \|x - x_0\| \leqq r, x \in \operatorname{dom} F\}$ for $r > 0$, $y_0 \in F(x_0)$.

It is obvious that $\tilde{k}^r(F, (x, y)) \geqq \tilde{k}^{r'}(F, (x, y))$ if $r > r' > 0$. Put $\tilde{k}(F, (x, y)) = \lim_{r \to 0} \tilde{k}^r(F, (x, y))$, $F^{-1}(y) = \{x \in X: y \in F(x)\}$. It is clear tht if F is convex then F^{-1} is convex, if F is closed then F^{-1} is closed.

Corollary 4. (P. C. Duong - H. Tuy [7]). Let X, Y be Banach spaces, U an open subset of X, $x_0 \in U$, $f: U \to Y$ a continuous mapping, M a closed convex subset of Y. Suppose that there is a continuous mapping $g: X \to Y$ and $r > 0$ such that $g(x_0) = f(x_0)$ and

(1) $G(x) = g(x) - M$ is a closed convex mapping,
(2) $a = \tilde{k}(G^{-1}, (f(x_0), x_0)) > 0$,
(3) $\|f(x) - f(x') - g(x) + g(x')\| \leqq K\|x - x'\|$ for all x, x', $\|x - x_0\| \leqq r$, $\|x' - x_0\| \leqq r$,
(4) $K \,.\, a < 1$,
(5) $G(X) = Y$.

Then f is M-surjective at x_0.

Proof. Put $Z = X \times Y$, $\|(x, y)\| = \max\{\|x\|, a \,.\, \|y\|\}$, $A = \operatorname{Gr}(G)$, $T(x, y) = y$, $h: A \to Y$, $h(x, y) = f(x) - g(x) + y$. Then $k(T \mid A, (x_0, f(x_0)) = a^{-1}$, $\|h(x, y) - h(x', y') - T((x, y) - (x', y'))\| = \|f(x) - f(x') - g(x) + g(x')\| \leqq K\|x - x'\|$. By Theorem 1 $f(x_0) = y_0 = h(x_0, y_0) \in \operatorname{int}(h(B_r(x_0, y_0) \cap A))$, hence $y_0 \in \operatorname{int}(h(U \times Y \cap A))$ for every open set U containing x_0. $h(U \times Y \cap A) = \{f(x) - g(x) + y: x \in U, y \in g(x) - M\} = \{f(x) - M: x \in U\} = F(U)$, where $F(U) = f(x) - M$. That proves that $f(x_0) \in \operatorname{int} F(U)$ for every open set U containing x_0, i.e. f is M-surjective at x_0.

Now let X be a Banach space, X^* denotes the linear space of all linear forms on X. Let $f: X \to \mathbb{R}$ be a convex function. Linear form $x^* \in X^*$ is said to be an algebraic subgradient of f at x_0 if $\langle x^*, x - x_0 \rangle \leqq f(x) - f(x_0)$ for all $x \in X$. Put $\partial^a f(x_0) = \{x^* \in X^*: x^*$ is an algebraic subgradient of f at $x_0\}$. It is obvious that if $x_0 \in \operatorname{int}(\operatorname{dom} f)$, then by Hahn-Banach theorem $\partial^a f(x_0) \neq \emptyset$.

Remark. If F is a multivalued convex mapping, $\operatorname{dom} F = X$ and there exists an $x_0 \in X$ such that $F(x_0)$ is bounded then $F(x)$ is bounded for all $x \in X$. In fact, if there were an $x \in X$ such that $F(x)$ is unbounded, then $F(x_0) = F(\frac{1}{2}x + \frac{1}{2}(2x_0 - x)) \supseteq \frac{1}{2}F(x) + \frac{1}{2}F(2x_0 - x)$ would be unbounded too. It is a contradiction.

Theorem 2. Let X, Y be Banach spaces, $F: X \to 2^Y$ be a convex closed multivalued mapping such that $\operatorname{dom} F = X$ and $F(x_0)$ is bounded for an $x_0 \in X$. Then there exists a unique linear singlevalued mapping $T: X \to Y$ such that $F(x) = F(0) + T(x)$.

Proof. By the remark, $F(x)$ is bounded closed for all $x \in X$.

(1) Let $Y = \mathbb{R}$. Put $-\infty < \varphi(x) = \max\{y: y \in F(x)\} < \infty$, $-\infty < \psi(x) = \min\{y: y \in F(x)\} < \infty$. It is clear that ψ is convex, φ is concave and $\operatorname{dom} \varphi = \operatorname{dom} \psi = X$. Put $h(x) = \psi(x) - \varphi(x) \leqq 0$; h is a convex function and $\partial^a h(x) \neq \emptyset$ for all $x \in X$. Let $\hat{x}$ be any point of X, $x^* \in \partial^a h(\hat{x})$. Then $\langle x^*, x - \hat{x}\rangle \leqq h(x) - h(\hat{x})$ for all $x \in X$, hence $\langle x^*, k\rangle \leqq h(\hat{x} + k) - h(\hat{x}) \leqq -h(\hat{x})$ for all $k \in X$. This shows that linear form x^* is upper bounded, thus $\langle x^*, k\rangle = 0$ for all $k \in X$. That means $\partial^a h(x) = \{0\}$ for all $x \in X$ and thus h is a constant. Let $h(x) = -a$; then $\varphi(x) = a + \psi(x)$. It follows that φ, ψ are simultaneously convex and concave functions. Thus φ, ψ are affine. Put $T(x) = \psi(x) - \psi(0)$, then T is a linear form on X and $F(x) = [\psi(x), \varphi(x)] = [\psi(x), \psi(x) + a] = \psi(x) - \psi(0) + [\psi(0), \psi(0) + a] = T(x) + [\psi(0), \psi(0) + a] = T(x) + F(0)$.

(2) Let Y be any Banach space. For each $y^* \in Y^*$, put $(y_c^* F)(x) = \overline{y^*(F(x))}$; then $y_c^* F$ is a convex multivalued mapping of X into $2^{\mathbf{R}}$. Without loss of generality we can suppose that $0 \in F(0)$. Let $x \in X$, $x \neq 0$, $1 \leqq \lambda_1 < \lambda_2$; then

$$\lambda_1 x = \frac{\lambda_1}{\lambda_2}(\lambda_2 x) + \left(1 - \frac{\lambda_1}{\lambda_2}\right) 0,$$

$$F(\lambda_1 x) \supseteq \frac{\lambda_1}{\lambda_2} F(\lambda_2 x) + \left(1 - \frac{\lambda_1}{\lambda_2}\right) F(0) \supseteq \frac{\lambda_1}{\lambda_2} F(\lambda_2 x)$$

and hence

$$\frac{1}{\lambda_1} F(\lambda_1 x) \supseteq \frac{1}{\lambda_2} F(\lambda_2 x).$$

On the other hand, for each $y^* \in Y^*$ there exists a unique linear form $T_{y^*}: X \to \mathbb{R}$ such that $(y_c^* F)(x) = (y_c^* F)(0) + T_{y^*}(x)$. Then

$$\operatorname{diam} F(x) = \sup_{\|y^*\| = 1} \operatorname{diam} (y_c^* F)(x) = \sup_{\|y^*\| = 1} \operatorname{diam} ((y_c^* F)(0) + T_{y^*}(x)) =$$
$$= \sup_{\|y^*\| = 1} \operatorname{diam} (y_c^* F)(0) \operatorname{diam} F(0).$$

Thus $\lim_{\lambda \to \infty} \operatorname{diam} [(1/\lambda) F(\lambda x)] = 0$. By the Cantor theorem, there is a unique element, which is denoted by $T(x)$, such that $\{T(x)\} = \bigcap_{\lambda \geqq 1} (1/\lambda) F(\lambda x)$. Of course $T(x) \in F(x)$ for all $x \in X$. Now we claim that $T(x)$ is positively homogeneous. Let $x_1, x_2 \in X$, and $x_2 = \lambda_0 x_1$. Without loss of generality we can suppose that $\lambda_0 > 1$. Then $\lambda\lambda_0 > \lambda$ for all $\lambda \geqq 1$. It holds

$$F(\lambda x_2) = F(\lambda \lambda_0 x_1), \quad \frac{1}{\lambda\lambda_0} F(\lambda x_2) \supseteq \frac{1}{\lambda} F(\lambda x_1)$$

and hence

$$\frac{1}{\lambda} F(\lambda x_2) \supseteq \frac{\lambda_0}{\lambda} F(\lambda x_1).$$

Thus

$$\{T(x_2)\} = \bigcap_{\lambda \geqq 1} \frac{1}{\lambda} F(\lambda x_2) \supseteq \lambda_0 \bigcap_{\lambda \geqq 1} \frac{1}{\lambda} F(\lambda x_1) = \lambda_0 \{T(x_1)\},$$

$$\text{i.e.} \quad T(x_2) = \lambda_0 T(x_1).$$

This shows that T is positively homogeneous and of course $(y_c^* T)(x) \in (y_c^* F)(x) = (y_c^* F)(0) + T_{y*}(x)$ for all $x \in X$. Then $\lambda((y_c^* T) - T_{y*})(x) \in (y_c^* F)(0)$ for all $\lambda > 0$, $((y_c^* T) - T_{y*})(x) = \lim (1/\lambda)(y_c^* F)(0) = \{0\}$, hence $y_c^* T = T_{y*}$. Let $\alpha, \beta \in \mathbb{R}$, $u, v \in X$, $y^* \in Y^*$; then $y^*(T(\alpha u + \beta v)) = T_{y*}(\alpha u + \beta v) = \alpha T_{y*}(u) + \beta T_{y*}(v) = \alpha(y_c^* T)(u) + \beta(y_c^* T)(v) = y^*(\alpha T(u) + \beta T(v))$. Thus $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$. Hence T is a linear mapping. On the other hand we have $(y_c^* F)(x) = \overline{y^*(F(x))} = (y_c^* F)(0) + y^*(T(x)) = \overline{y^*(F(0) + T(x))}$. Then $F(x) = F(0) + T(x)$ and the proof of Theorem 2 is completed.

We denote the linear hull of a subset A by $\mathscr{L}(A)$.

Corollary 5. Let X, Y be Banach spaces, $F: X \to 2^Y$ a continuous multivalued closed convex mapping, dom $F = X$, $F(x_0)$ bounded for an $x_0 \in X$. Suppose that: 1) $\mathscr{R}(F) = Y$, 2) $\dim(\mathscr{L}(F(0))) < \infty$, 3) F is 1-1, i.e. $F(x) \neq F(x')$ if $x \neq x'$. Then $X \cong Y$.

Proof. By Theorem 2, $F(x) = F(0) + T(x)$ for a $T \in L(X, Y)$. Of couse T is an injection. It is sufficient to prove that $\mathscr{R}(T) = Y$ (that means that T is open). Suppose that $\mathscr{R}(T) \neq Y$ and $\hat{y} \in Y$, $\hat{y} \notin \mathscr{R}(T)$. By the assumption, for each $n \in \mathbb{N}$ there is an $a_n \in F(0)$, $x_n \in X$ such that $n\hat{y} = a_n + T(x_n)$. Put $H = \mathscr{L}(\{T(x_n)\}) \subseteq \mathscr{L}(F(0)) \oplus \mathscr{L}(\{\hat{y}\})$. Then $H \subseteq \mathscr{R}(T)$ and $\dim(H) \leqq \dim(\mathscr{L}(F(0))) + 1$. Therefore H is a closed subspace of Y. By the Hahn-Banach theorem there is a $y^* \in Y^*$ such that $y^*(\hat{y}) = 1$, $y^*(y) = 0$ for all $y \in H$. Thus $y^*(a_n) = n$ for all $n \in \mathbb{N}$ and $\sup_{y \in F(0)} y^*(y) = \infty$. This contradicts the boundedness of $F(0)$. That shows that $\mathscr{R}(T) = Y$ and the proof is over.

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