Acta Universitatis Carolinae. Mathematica et Physica

Alberto Facchini

Algebraically compact modules

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 26 (1985), No. 2, 27--37

Persistent URL: http://dml.cz/dmlcz/142552

Algebraically Compact Modules

ALBERTO FACCHINI
Mathematical Institute of Udine University*)

Received 26 February 1985

The paper is devoted to a study of algebraically compact modules.

Článek je věnován studiu algebraicky kompaktních modulů.

Статья посвещена изучению алгебраически компактных модулей.

This paper is a reasonably self-contained report on some techniques employed in the study of algebraically compact (pure-injective) modules. In particular we are concerned with the direct sum decompositions of algebraically compact modules. We do not provide the proofs but give all the necessary references to them.

We begin with a quick review of the definition and main properties of algebraically compact modules in § 1, following Warfield's paper [35]. In § 2 we present spectral categories as they are introduced in Gabriel's and Oberst's note [14]. In §§ 3, 4 and 5 we examine various possibilities of applying spectral categories to the study of algebraically compact modules, presenting a paper of the autor [7], Gruson's and Jensen's category D(R) [19], and Gabriel's functor ring of the finitely presented modules [13] respectively. Finally in § 6 we show how it is possible to apply the results of the previous sections to the study of the structure of algebraically compact modules over valuation rings.

1. Algebraically compact modules: definition and main properties

The study of algebraically compact Abelian groups was begun by Kaplansky [22], Łoś [24], Balcerzyk [1] and Maranda [25]. Then the theory was extended to modules by Stenström [33], Fuchs [10] and Warfield [35] and to general algebraic systems by Mycielski [26]. Here we present the notion of algebraic compactness following Warfield's paper [35].

Fix an associative ring R with 1.

Proposition 1. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of left R-modules. The following properties are equivalent:

*) Istituto di Matematica, Informatica e Sistemistica, Università di Udine, Udine, Italy.

(a) For any finitely presented left R-module M, the sequence $0 \to \mathrm{Hom}(M, A) \to$ $\to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ is exact.

(b) For any right R-module F, the sequence $0 \to F \otimes A \to F \otimes B \to F \otimes C \to$ $\to 0$ is exact.

(c) Any finite set of linear equations $\sum_{j=1}^{n} r_{ij}x_j = a_i$, $1 \leqq i \leqq k$, with $r_{ij} \in R$ and $a_j \in A$, which is soluble in B is soluble in A.

A short exact sequence satisfying the conditions of Proposition 1 is called *pure*. In this case A is said to be a *pure submodule* of B. A module D is *pure-injective* if for any pure short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to$ $\to \mathrm{Hom}(C, D) \to \mathrm{Hom}(B, D) \to \mathrm{Hom}(A, D) \to 0$ is exact, or, equivalently, if any morphism $A \to D$ can be extended to a morphism $B \to D$ whenever A is a pure submodule of B.

The interest of the notion of pure-injectivity mainly lies in its connection to the notion of (topological) compactness and its equivalence to the concept of algebraic compactness. A left R-module M is *compact* if there exists a compact Hausdorff topology on M making it a topological module over the ring R, where R is endowed with the discrete topology (i.e., M is a topological group and the multiplication $R \times M \to M$ is continuous.) A left R-module M is *algebraically compact* if every family of linear equations $\sum_{j \in J} r_{ij}x_j = m_i$ $(i \in I)$ that is finitely soluble is soluble. Here the r_{ij}'s are in R, the m_i's in M and for any $i \in I$ the elements r_{ij}'s are zero for all but a finite number of $j \in J$.

Theorem 2. The following conditions on a left R-module M are equivalent:

(a) M is pure-injective.
(b) M is isomorphic to a summand of a compact R-module.
(c) M is algebraically compact.

It is also possible to construct a theory similar to the theory of injective envelopes and essential submodules in the following way. If A is a pure submodule of B, then B is a *pure-essential extension* of A if there are no nonzero submodules $S \subseteq B$ with $A \cap S = 0$ and $A + S/S$ pure in B/S. A module B is a *pure-injective envelope* of a submodule A if B is a pure-essential extension of A and a pure-injective module.

Theorem 3. Pure-injective envelopes exist and are unique up to isomorphism.

We conclude this introductory section with a theorem due to B. Zimmermann-Huisgen and W. Zimmermann [38] on the endomorphism rings of algebraically compact modules.

Theorem 4. Let M be an algebraically compact left R-module, $S = \mathrm{End}_R(M)$ and $J(S)$ the Jacobson radical of S. Then $S/J(S)$ is a right self-injective von Neumann regular ring, and idempotents lift modulo $J(S)$.

2. Spectral categories

Theorems 3 and 4 seem to suggest that pure-injectivity might be regarded as a sort of injectivity (or quasi-injectivity, see [8]) in a suitable ambient category. The two main ideas are the following :1) modify the category R-Mod of all left R-modules so that pure-injective modules are exactly the injective objects in the modified category; 2) give the algebraically compact modules a structure of spectral category. The various developments of these two ideas, which are closely linked together, will be the main theme of this paper. Here we shall not report on the approach to the study of algebraically compact modules via model theory. For this approach we refer the reader to the papers of Fisher [9], Garavaglia [15], Ziegler [37] and Prest [28]. We tackle the problem only with module theory and its immediate extension, that is, abelian category theory.

Since the injective objects of a Grothendieck category can be given the structure of a spectral category and the decompositions of injective modules are studied with spectral categories, we will briefly describe these categories, following Gabriel's and Oberst's note [14]. We also refer the reader to [32] and [18].

Let C be a *Grothendieck* category, that is, an abelian category with a generator and exact direct limits. It is known that for every object A of C there exists an essential monomorphism into a unique (up to isomorphism) injective object, the *injective envelope* of A. Moreover, if A is an object of C the set of essential subobjects of A is directed under reverse inclusion. Therefore we can define a category Spec C, the *spectral category of* C, in the following way: Spec C has the same objects as C; if A and B are objects of Spec C, then

$$\mathrm{Hom}_{\mathrm{Spec}\,C}(A, B) = \varinjlim_{A'} \mathrm{Hom}_C(A', B)$$

where the direct limit is taken over all essential subobjects A' of A. There is a canonical functor $P: C \to \mathrm{Spec}\, C$ which is the identity on objects and induces the canonical homomorphism of $\mathrm{Hom}_C(A, B)$ into $\varinjlim \mathrm{Hom}_C(A', B)$ on morphisms. The P-images of an object of C and its injective envelope in C are isomorphic objects in Spec C.

Proposition 5. If C is a Grothendieck category, then Spec C is a Grothendieck category in which every exact sequence splits.

(If C is a generator in C, a generator in Spec C is $U = P(\bigoplus_{C'} C/C')$ where C' ranges in the set of all subobjects of C.)

If we call *spectral* category any Grothendieck category in which every exact sequence splits, the proposition says that Spec C is spectral for any Grothendieck category C. (Note that other authors call spectral category any abelian category in which every exact sequence splits.)

Spectral categories can be characterized algebraically.

Proposition 6. Let S be a spectral category with a generator U and set $S =$

$= \mathrm{End}_S(U)$. Then S is a von Neumann regular right self-injective ring and S is equivalent to the full subcategory $N(S)$ of Mod-S whose objects are the nonsingular injective modules. Conversely, for any von Neumann regular right self-injective ring S, the full subcategory $N(S)$ of Mod-S is a spectral category.

It is easy to see that in this case the nonsingular injective modules are exactly the direct summands of direct products of copies of R [18].

There is a considerable advantage in passing to the category Spec C, because it has a much simpler structure than the category C itself. For instance, in Spec C every object is injective, every subobject is a direct summand, simple objects are exactly indecomposable injective objects, two objects A and B of C have isomorphic injective envelopes if and only if $P(A)$ and $P(B)$ are isomorphic in Spec C, two injective objects of C are isomorphic if and only if their P-images are isomorphic in Spec C, and so on. In particular Spec C is the natural tool to study the injective objects of C.

All these considerations apply in particular to $C = R$-Mod for any ring R, and thus the study of injective modules is reduced to the study of nonsingular injective modules over a regular self-injective ring or, equivalently, to the study of spectral categories. Both methods lead to situations that are now well understood.

We refer the reader to Renault [29], Goodearl and Boyle [18] and Goodearl's books [16] and [17] for the study of nonsingular injective modules over self-injective regular rings and to Gabriel and Oberst [14], Roos [31], [32], Warfield [34], Popescu [27] and Facchini [2], [3], [5] for the study of spectral categories. In particular we want to point out the dimension theory for the nonsingular injective modules in [18], the decomposition into types according to von Neumann and Murrays classification in [32] and [18], the decomposition of a spectral category into a discrete and a continuous part in [14], the study of the lattice of subobjects in a spectral category in [32] and the study of spectral categories via enriched category theory and topos theory in [5].

3. The direct approach

In this section we show how it is possible to give a structure of spectral category to algebraically compact modules.

Let C be any additive category (i.e., C is preadditive and has finite coproducts.) A (two-sided) *ideal* I in C is a function which assigns to each ordered pair (A, B) of objects of C a subgroup $I(A, B)$ of $\mathrm{Hom}_C(A, B)$ such that the following property holds: if $f: A \to B$, $g: B \to C$, $h: C \to D$ are morphisms of C and $g \in I(B, C)$, then $hgf \in I(A, D)$.

Let I be an ideal in C. Then the *factor category* of C modulo I is defined in the following way: the objects of C/I are the objects of C (if A is an object of C, then the same object considered as an object of C/I will be denoted by $[A]$); if $[A]$ and $[B]$ are objects of C/I, then $\mathrm{Hom}_{C/I}([A],[B]) = \mathrm{Hom}_C(A, B)/I(A, B)$.

It is easy to verify that C/I is an additive category; moreover there is a natural functor $C \to C/I$ that is the identity on objects and the reduction modulo $I(A, B)$ on morphisms $A \to B$.

Given an additive category C, the *Jacobson radical* $J(C)$ of C is the ideal in C defined as follows: if A and B are objects of C and $p: A \oplus B \to A$ and $i: B \to A \oplus B$ are the canonical projection and the canonical injection, set $J(C)(A, B) = \{f \in \mathrm{Hom}_C(A, B) \mid ifp \in J(\mathrm{End}_C(A \oplus B))\}$, where $J(\mathrm{End}_C(A \oplus B))$ denotes the Jacobson radical of the ring $\mathrm{End}_C(A \oplus B)$. It is possible to verify that $J(C)$ is really an ideal in C and that it is the *unique* ideal I in C such that $I(A, A)$ is the Jacobson radical of the ring $\mathrm{Hom}_C(A, A)$ for every object A of C. For the Jacobson radical of a category we refer the reader to [23] and [21].

Let us go back to algebraically compact modules. Let A be the full subcategory of R-Mod whose objects are all algebraically compact left R-modules. Since the direct sum of two algebraically compact left R-modules is algebraically compact, the category A is additive and has a Jacobson radical $J(A)$.

Theorem 7. Let A be the full subcategory of R-Mod of all algebraically compact modules, $J(A)$ its Jacobson radical and $A/J(A)$ the factor category. Then $A/J(A)$ is a spectral category.

A direct proof of Theorem 7 will appear in [7]. Note that A is not an abelian category in general, but that $A/J(A)$ is a *Grothendieck* category.

We briefly describe some properties of the category $A/J(A)$. The monomorphisms in $A/J(A)$ are all of type $[f]: [A] \to [B]$ where $f: A \to B$ is a splitting monomorphism in R-Mod [here if A, B are objects of A, and $f: A \to B$ is a morphism in A, then $[A], [B], [f]$ are their images in $A/J(A)$, in particular $[f] = f + J(A)(A, B) \in \mathrm{Hom}_{A/J(A)}([A], [B])$.] If $f: A \to B$ is a morphism in A, then the kernel of $[f]: [A] \to [B]$ can be constructed in the following way: since $\mathrm{End}_{A/J(A)}[(A \oplus B)] = \mathrm{End}_R(A \oplus B)/J(\mathrm{End}_R(A \oplus B))$ is a von Neumann regular ring (Theorem 4), there exists $g: B \to A$ such that $[f] = [fgf]$; the idempotent $[gf] \in \mathrm{End}_R(A)/J(\mathrm{End}_R(A))$ can be lifted to an idempotent $e \in \mathrm{End}_R(A)$ (Theorem 4). Then the subobject (direct summand) $[(1-e)A]$ of $[A]$ is the kernel of $[f]: [A] \to [B]$.

Given a family $\{A_\lambda \mid \lambda \in \Lambda\}$ of algebraically compact modules, the coproduct of the $[A_\lambda]$, $\lambda \in \Lambda$, in $A/J(A)$ is the image in $A/J(A)$ of the pure-injective envelope of the direct sum of the A_λ's: $\bigoplus_\lambda [A_\lambda] = [PE(\bigoplus_\lambda A_\lambda)]$. Finally, a generator in $A/J(A)$ is given by the image in $A/J(A)$ of the pure-injective envelope of the direct sum of a family of representatives (up to isomorphism) of all modules of cardinality $\leqq \max\{\aleph_0, |R|\}$.

For a similar use of factor categories (modulo the Jacobson radical) we refer the reader to Harada's book [21].

4. The approach via the functor category

In this section we consider a method of rendering algebraically compact modules injective in a suitable ambit. It is due to Gruson and Jensen [19], [20].

Consider the category $Pf(R)$ of all finitely presented *right* R-modules, regarded as a full subcategory of Mod-R. The additive functors of $Pf(R)$ into the category Ab of all abelian groups are the objects of a category $D(R)$. The morphisms in $D(R)$ are the natural transformations between two functors. The functor category $D(R)$ turns out to be a Grothendieck category.

Two important functors are defined: the first one is the functor $\otimes_R$: R-Mod $\to$ $\to D(R)$ which maps the *left* R-module ${}_RM$ into the functor $-\otimes_R M$ (this is an object of $D(R)$); the second one is the functor "evaluation in R" e_R: $D(R) \to R$-Mod which maps the functor F: $Pf(R) \to Ab$ into $F(R)$. [The abelian group $F(R)$ can be regarded as a left R-module in the following way: if $r \in R$ and $x \in F(R)$, define $rx = F(\mu_r)(x)$, where μ_r: $R \to R$ is the left multiplication by r, μ_r: $t \mapsto rt$, so that $F(\mu_r)$: $F(R) \to F(R)$ is a homomorphism of abelian groups, and $F(\mu_r)(x)$ is an element of $F(R)$.] One obtains that $\otimes_R$ is a left adjoint for e_R. The fundamental result for the study of algebraically compact modules is the following

Theorem 8. Let F be an object of $D(R)$. Then F is injective in $D(R)$ if and only if F is naturally isomorphic to $-\otimes_R M$ for some algebraically compact left R-module M.

Now we may study an algebrically compact module M by passing to the functor $-\otimes_R M$: $Pf(R) \to Ab$. This turns out to be an injective object in $D(R)$, which is a Grothendieck category, so that we can pass to the spectral category Spec $D(R)$, study the object in this category, and then bring back the information obtained in this way to the algebraically compact module M.

It is possible to prove that the category Spec $D(R)$ and the category $A/J(A)$ of § 3 are equivalent, and this yields a second proof of Theorem 7.

5. The functor ring

We now consider a second technique of making the algebraically compact modules injective. It is based on Gruson's and Jensen's results (§ 4) and a result in Gabriel's thesis [13, Chap. II, § 1]. This technique was employed in a different problem by Fuller [11], [12]. I am grateful to Robert Wisbauer, who suggested to me the possibility of applying this idea in our context.

Let R be a ring with identity and $\{U_\lambda \mid \lambda \in \Lambda\}$ a set containing one isomorphic copy of each finitely presented right R-module. Set $U_R = \bigoplus_{\lambda \in \Lambda} U_\lambda$ and define

$$S = \{f\colon U_R \to U_R \mid f(U_\lambda) = 0 \text{ a.e.}\} \subseteq \operatorname{End}(U_R)$$

where a.e. means for all but a finite number of $\lambda \in \Lambda$. It is obvious that S is a subring of End (U_R). In general S has not an identity, but it has enough idempotents [11]: a ring T is a *ring with enough idempotents* if there is an orthogonal set of idempotents $\{e_\lambda \mid \lambda \in \Lambda\}$ in T such that $T = \bigoplus_{\lambda \in \Lambda} Te_\lambda = \bigoplus_{\lambda \in \Lambda} e_\lambda T$ (for our ring S it is sufficient to take the idempotents for the decomposition $\bigoplus_{\lambda \in \Lambda} U_\lambda$.) The ring S si called the *functor ring of the finitely presented right R-modules.*

Now let T be any ring with enough idempotents, let T-Mod be the category of all left T-modules, and let T-Mod^ be the full subcategory of T-Mod whose objects are the T-modules M with the property that for every $x \in M$ there exists an idempotent $e = e^2 \in R$ such that $x = ex$.

The category T-Mod^ contains ${}_TT$ and all submodules, factor modules, and direct sums of its membeis. There is a slight difference for direct products though. The direct product of $\{M_\gamma \mid \gamma \in \Gamma\}$ in T-Mod^ is not the ordinary cartesian product $\prod_{\gamma \in \Gamma} M_\gamma$, but rather its submodule $\prod^\wedge_{\gamma \in \Gamma} M_\gamma = \{x \in \prod_{\gamma \in \Gamma} M_\gamma \mid x = ex$ for some idempotent $e = e^2 \in R\}$. Of course $\bigoplus_{\gamma \in \Gamma} M_\gamma \subseteq \prod^\wedge_{\gamma \in \Gamma} M_\lambda \subseteq \prod_{\gamma \in \Gamma} M_\gamma$, and the injections and projections are the usual ones.

Proposition 9. Let R be a ring with identity, and let S be the functor ring of the finitely presented right R-modules. Then the categories S-Mod^ and $D(R)$ are equivalent.

The functors T: S-Mod^ $\to D(R)$ and V: $D(R) \to S$-Mod^ that give the equivalence are defined as follows on objects:

Given any object ${}_SM$ in S-Mod^, $T({}_SM)$ is the functor $Pf(R) \to Ab$ that maps the finitely presented module U_λ into the abelian group $e_\lambda M$;

Given any object $F\colon Pf(R) \to Ab$ in $D(R)$, then $V(F) = \bigoplus_{\lambda \in \Lambda} F(U_\lambda)$ with the S-module structure defined by $f \,.\, x = (\sum_{\lambda \in \Lambda} (F(f_{\mu\lambda}))(x_\lambda))_\mu$ for $f = (f_{\mu\lambda}) \in S \subseteq$ $\subseteq$ End $(\bigoplus_{\lambda \in \Lambda} U_\lambda)$ and $x = (x_\lambda) \in \bigoplus_\lambda F(U_\lambda)$ [Note that if $f_{\mu\lambda}\colon U_\lambda \to U_\mu$, then $F(f_{\mu\lambda})$: $F(U_\lambda) \to F(U_\mu)$, so that $F(f_{\mu\lambda})(x_\lambda) \in F(U_\mu)$. Since $f_{\mu\lambda} = 0$ for almost all λ, $F(f_{\mu\lambda})(x_\lambda) = 0$ for almost all λ, so that $\sum_{\lambda \in \Lambda} (F(f_{\mu\lambda}))(x_\lambda) \in F(U_\mu)$.]

The next Corollary follows immediately from Theorem 8 and Proposition 9. Note that U is a End (U_R)-R-bimodule, so that if ${}_RM$ is any left R-module, $U \otimes {}_RM$ is a left End (U_R)-module, hence a left S-module by restriction of scalars. Moreover $U \otimes {}_RM$ is in S-Mod^.

Corollary 10. Let N be a module in S-Mod^. Then N is injective in S-Mod^ if and only if it is isomorphic to $U \otimes {}_RM$ for some algebraically compact left R-module M.

6. Algebraically compact modules over valuation rings

In § 3 we saw that it is possible to give a structure of spectral category to the algebraically compact modules by constructing the category $A/J(A)$, the full subcategory A of algebraically compact modules modulo its Jacobson radical. In § 4 we saw that $A/J(A)$ is equivalent to the spectral category of the Grothendieck category $D(R)$ of all additive functors from finitely presented modules to abelian groups. In § 5 we saw that $A/J(A)$ is equivalent to the spectral category of the category S-Mod^ for a suitable ring S with enough idempotents, the functor ring of the finitely presented modules. Now we apply the results of the previous sections to the study of algebraically compact modules over valuation rings. This part of the paper is a survey of a research that is now in progress, and its form is still unfinished.

A *valuation ring* is a commutative ring with identity such that for any two elements r and s, either r divides s or s divides r. Algebraically compact modules over valuation domains already received particular attention in Warfield's paper [35]. In this section we apply Gruson's and Jensen's techniques (§ 4) to the study of the structure of these modules.

Let R be a valuation ring and let R^* be the totally ordered monoid of all the principal ideals of R, i.e., $R^* = \{Rr \mid r \in R\}$ totally ordered by reverse inclusion and with multiplication defined by $r^*s^* = (rs)^*$ for $r^*, s^* \in R^*$; here we set $x^* = Rx$ for all $x \in R$. When R is a valuation domain, R^* is isomorphic to $G^+ \cup \{\infty\}$ where G^+ is the positive cone of the valuation group G of R.

If M is a module over a valuation ring R, let $L(M)$ denote the lattice of all its submodules: $L(M) = \{N \mid N \leqq M\}$. An *$R^*$-filtration for M* is an antihomomorphism of bounded ordered sets $R^* \to L(M)$ compatible with the module structure of M. Hence an R^*-filtration for M is a mapping $\Phi: R^* \to L(M)$ such that:

(i) If $r^* \leqq s^*$ (that is, if $Rr \supseteq Rs$), then $\Phi(r) \supseteq \Phi(s)$;

(ii) $\Phi(0^*) = \{0\}$, $\Phi(1^*) = M$;

(iii) $r\,\Phi(s^*) \subseteq \Phi(r^*s^*)$ for $r, s \in R$.

If Φ is an R^*-filtration for M, we say that the ordered pair (M, Φ) is an *R^*-filtered* module (or simply a *filtered* module).

For any R-module M there is a *natural filtration* υ_M for M given by $\upsilon_M(r^*) = rM$ for all $r \in R$. More generally, for any $M \leqq N$ there is a filtration $\mu_{M,N}$ for M given by $\mu_{M,N}(r^*) = rN \cap M$ for all $r \in R$. All filtrations are of this type:

Lemma 11. Let R be a valuation ring. For any filtered R-module (M, Φ) there exists an algebraically compact module N such that $M \leqq N$ and $\Phi = \mu_{M,N}$. An R-module homomorphism $f: M \to N$, where (M, Φ), (N, ψ) are filtered R-modules, is an *R^*-filtered homomorphism* if $f(\Phi(r^*)) \subseteq \psi(r^*)$ for all $r^* \in R^*$. Then R^* – filtered R-modules and R^*-filtered homomorphisms form a *preabelian* category R-Filt, which is complete and cocomplete (i.e., with limits and colimits).

The reader expert in abelian groups will recognize the resemblance between this theory and the theory of valuated groups due to Richman and Walker [30] and others. Valuated groups arise in a completely different context.

There is a connection between the category R-Filt of filtered R-modules and Gruson's and Jensen's category $D(R)$ (§ 2). Define a functor T: R-Filt $\to D(R)$ in the following way: if (M, Φ) is an object of R-Filt, let $T(M, \Phi)$: $Pf(R) \to Ab$ be the functor such that $T(M, \Phi)(R/rR) = M/\Phi(r^*)$; if $f: (M, \Phi) \to (N, \psi)$ is a morphism in R-Filt, let $T(f)$: $T(M, \Phi) \to T(N, \psi)$ be the natural transformation which assigns to each object R/rR of $Pf(R)$ the group homomorphism $T(f)_{R/rR}: M/\Phi(r^*) \to$ $\to N/\psi(r^*)$ induced by $f: M \to N$. Note that we have defined functors $Pf(R) \to Ab$ not on all of $Pf(R)$, but only on its objects R/rR $(r \in R)$; but this does not cause problems because every finitely presented R-module (when R is a valuation ring) is isomorphic to a direct sum of modules of type R/rR $(r \in R)$ [36].

Lemma 12. The functor T: R-Filt $\to D(R)$ is full and faithful.

By Lemma 12 R-Filt is equivalent to a full subcategory of $D(R)$, and it is possible to prove that R-Filt is equivalent to the full subcategory of $D(R)$ whose objects are the epic-preserving functors, i.e., the functors F: $Pf(R) \to Ab$ such that $F(f)$ is an epimorphism in Ab whenever f is an epimorphism in $Pf(R)$.

The category R-Filt is a preabelian category, but it is not an abelian category (for instance a monomorphism which is also an epimorphism need not be an isomorphism), and it is not difficult to show that the injective objects of R-Filt are exactly the algebraically compact R-modules endowed with their natural filtration. Again we have been able to "modify" the category R-Mod (in this case by putting filtrations on modules), so that the injective objects of the modified category (the category R-Filt now) are exactly the algebraically compact R-modules. Here it is even possible to define a sort of "injective envelope" in the preabelian category R-Filt and to show that every algebraically compact module over a valuation ring R is the injective envelope (in R-Filt) of a direct sum of cyclic modules. These ideas will be developed in a forthcoming paper [6].

References

[1] Balcerzyk S.: On algebraic compact groups of I. Kaplansky, Fund. Math. 44 (1957) 91—93.

[2] Facchini A.: Spectral categories and varieties of preadditive categories, J. Pure Applied Algebra 29 (1983), 219—239.

[3] Fachini A.: Varieties of preadditive categories and their application to the study of the structure of spectral categories, Proc. International Conference of Mathematics in the Gulf Area, Riyadh (Saudi Arabia), 17—21 October 1982.

[4] Facchini A.: Decompositions of algebraically compact modules, Pacific J. Math. 115 (1984).

[5] Facchini A.: Spectral categories and enriched categories, to appear.

[6] FACCHINI A.: Algebraically compact modules over valuation rings and elementary divisor rings, to appear.
[7] FACCHINI A.: On the category of algebraically compact modules, to appear.
[8] FAITH C., UTUMI Y.: Quasi-injective modules and their endomorphism rings, Arch. Math. 15 (1964), 166—174.
[9] FISHER E. R.: Abelian astructures I, Abelian Group Theory, 2nd New Mexico State University Conference, 1976, Lecture Notes in Mathematics 616, Springer Verlag, Berlin, 1977. 270—322.
[10] FUCHS L.: Algebraically compact modules over Noetherian rings, Indian J. Math. 9 (1967), 357—374.
[11] FULLER K. R.: On rings whose left modules are direct sums of finitely generated modules, Proc. Amer. Math. Soc. 54 (1976), 39—44.
[12] FULLER K. R., HULLINGER H.: Rings with finiteness conditions and their categories of functors, J. Algebra 55 (1978), 94—105.
[13] GABRIEL P.: Des catégories abéliennes, Bull. Soc. Math. France 90 (1962), 323—448.
[14] GABRIEL P., OBERST U.: Spektralkategorien und reguläre Ringe im von Neumannschen Sinn, Math. Z. 92 (1966), 389—395.
[15] GARAVAGLIA S.: Decomposition of totally transcendental modules, J. Symbolic Logic 45 (1980), 155—164.
[16] GOODEARL K. R.: Ring theory — Nonsingular rings and modules, Marcel Dekker Inc., New York, 1976.
[17] GOODEARL K. R.: Von Neumann regular rings, Pitman, London, 1979.
[18] GOODEARL K. R., BOYLE A. K.: Dimension theory for nonsingular injective modules, Mem. Amer. Math. Soc. 177 (1976).
[19] GRUSON L., JENSEN C. U.: Modules algébriquement compacts et foncteur $\varinjlim^{(i)}$, C. R. Acad. Sci. Paris 276 (1973), 1651—1653.
[20] GRUSON L., JENSEN C. U.: Dimensions cohomologiques reliées aux foncteurs $\varinjlim^{(i)}$, Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, Lecture Notes in Mathematics 867, Springer, Berlin, 1981, 234—294.
[21] HARADA M.: Factor categories with applications to direct decomposition of modules, Lecture Notes in Pure and Applied Mathematics 88, Marcel Dekker, New York, 1983.
[22] KAPLANSKY I.: Infinite Abelian Groups, Ann Arbor, 1954. Revised Edition 1969.
[23] KELLY G. M.: On the radical of a category, J. Australian Math. Soc. 4 (1964), 299—307.
[24] ŁOŚ J.: Abelian groups that are direct summands of every abelian group which contains them as pure subgroups, Fund. Math. 44 (1957), 84—90.
[25] MARANDA J. M.: On pure subgroups of Abelian groups, Archiv der Math. 11 (1960), 1—13.
[26] MYCIELSKI J.: Some compactifications of general algebras, Coll. Math. 13 (1964), 1—9.
[27] POPESCU N.: Abelian categories with applications to rings and modules, Academic Press, London, 1973.
[28] PREST M.: Rings of finite representation type and modules of finite Morley rank, J. Algebra 88 (1984), 502—533.
[29] RENAULT G.: Anneaux réguliers auto-injectifs à droite, Bull. Soc. Math. France 101 (1973) 237—254.
[30] RICHMAN F., WALKER E. A.: Valuated groups, J. Algebra 56 (1979), 145—167.
[31] ROOS J.-E.: Sur la structure des catégories spectrales et les coordonnées de von Neumann des treillis modulaires complémentés, C. R. Acad. Sci. Paris 265 (1967), 42—45.
[32] ROOS J.-E.: Locally distributive spectral categories and strongly regular rings, Reports of the Midwest Category Seminar, Lecture Notes in Mathematics 47, Berlin, Springer, 1967 156—181.
[33] STENSTRÖM B. T.: Pure submodules, Arkiv för Mat. 7 (1967), 159—171.

[34] Warfield R. B., Jr: Decompositions of injective modules, Pacific J. Math. 31 (1969), 263—276.
[35] Warfield R. B., Jr: Purity and algebraic compactness for modules, Pacific J. Math. 28 (1969), 699—719.
[36] Warfield R. B., Jr: Decomposability of finitely presented modules, Proc. Amer. Math. Soc. 25 (1970), 167—172.
[37] Ziegler M.: Model theory of modules, Ann. Pure Applied Logic 26 (1984), 149—213.
[38] Zimmermann-Huisgen B., Zimmermann W.: Algebraically compact rings and modules, Math. Z. 161 (1978), 81—93.