Acta Universitatis Carolinae. Mathematica et Physica

N. V. Loi

On the radical theory of involution algebras

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 27 (1986), No. 1, 29--40

Persistent URL: http://dml.cz/dmlcz/142562

On the Radical Theory of Involution Algebras

N. V. LOI
Mathematial Institute of the Hungarian Academy of Sciences, Budapest*)

Received 1 March 1985

In this paper we continue the study of the general radical theory of involution algebras as started in [5] with special emphasis on semisimple classes. We prove that if a class $\mathbb{C}$ is regular, coinductive and closed under extensions, and its semisimple closure is hereditary, then $\mathbb{C}$ is a semisimple class (Theorem 3). The proof of this assertion is based on a method which can be successfully applied in characterizing semisimple classes in other varieties (cf. [2]). As a semisimple class is not always hereditary, it need not be a coradical class. We shall show that also a coradical class is not necessarily a semisimple class (Proposition 14).

In [5] we have given necessary and sufficient conditions on a radical class $\mathbb{R}$ to satisfy the A—D—S property in terms of involution algebras with zero-multiplication. We shall prove that a radical class $\mathbb{R}$ satisfies A—D—S if and only if its semisimple class $\mathbb{F}$ satisfies the following condition: If $A^* \in \mathbb{F}$ and A is nilpotent then any nilpotent involution algebra built on the additive group of A, is in $\mathbb{F}$ (Theorem 7). Here we may write also $\mathbb{R}$ in the place of $\mathbb{F}$ (Theorem 10).

V článku se zabýváme obecnou teorií radikálů v algebrách s involucí. Je dána postačující podmínka k tomu, aby regulární třída $\mathbb{C}$ byla polojednoduchá.

В докалде исследуется общая теория радикалов в алгебрах с инволуцией. Дается достаточное условие для полупростоты регулярного класса $\mathbb{C}$.

0.

First, we shall recall the basic definitions needed in this paper.

A K-algebra A is an *involution algebra*, if in A an unary operation $*$ is defined such that $x^{**} = x$, $(x + y)^* = x^* + y^*$, $(xy) + = y^*x^*$, $(kx)^* = kx^*$ for all x, $y \in A$, $k \in K$. Throughout this paper we shall work with involution algebras over a commutative associative ring K with identity. The universal class we consider will be the variety $\mathbb{V}$ of all K-algebras with involution. A^* will always stand for a K-algebra with involution $*$, whenever there is no ambiguity we write only A. In particular, *id* will denote the operation $x^{id} = x$, and $-*$ the operation $x^{-*} = -x^*$. Let us notice that for an involution algebra A^*, such that $A^2 = 0$, A^{-*} is an involution algebra. An ideal I of an involution algebra A^* will always mean an ideal of A such

*) Reáltanoda u. 13-15, H-1053 Hungary.

that I is an involution subalgebra of A^*. This fact will be indicated by $I \lhd A^*$. By a homomorphism ϕ we mean an algebra-homomorphism, such that $\phi(x)^* = \phi(x^*)$. For any algebra A the zero-algebra built on the additive group of A will be denoted by $A_{\mathbb{V}}$, that is, $A_{\mathbb{V}}^2 = 0$.

For future reference we list the following properties of non-empty classes of involution algebras. Let $\mathbb{M}$ be a class of involution algebras:

A class $\mathbb{M}$ is *closed under extensions* if whenever an involution algebra A contains an ideal I such that I, $A/I \in \mathbb{M}$, then $A \in \mathbb{M}$.

A class $\mathbb{M}$ is *inductive* if whenever an involution algebra A contains an ascending chain of ideals I_i such that $\bigcup I_i = A$ and $I_i \in \mathbb{M}$, for each i, then $A \in \mathbb{M}$.

A class $\mathbb{M}$ is *coinductive* if whenever an involution algebra A contains a descending chain of ideals I_i such that $\bigcap I_i = 0$ and $A/I_i \in \mathbb{M}$, for each i, then $A \in \mathbb{M}$.

A class $\mathbb{M}$ is *closed under homomorphisms* if an involution alegbra A is contained in $\mathbb{M}$, then every image of A is also contained in $\mathbb{M}$.

A class $\mathbb{M}$ is *closed under subdirect sums* if whenever an involution algebra A contains a family of ideals I_i such that $\bigcap I_i = 0$ and $A/I_i \in \mathbb{M}$, for each i, then $A \in \mathbb{M}$.

A class $\mathbb{M}$ is *regular* if $0 \in \mathbb{M}$ and whenever I is a non-zero ideal of an involution algebra in $\mathbb{M}$, then I has a non-zero image in $\mathbb{M}$.

A class $\mathbb{M}$ is *hereditary* if $A \in \mathbb{M}$ then every ideal of A is also in $\mathbb{M}$. A *radical class* $\mathbb{R}$ (in the sense of Kurosh and Amitsur) of involution algebras is a subclass of $\mathbb{V}$, which is closed under extensions, homomorphisms and is inductive. A class $\mathbb{C}$ is called a *coradical class*, if $\mathbb{C}$ is hereditary and closed under extensions and subdirect sums. Further, the class

$$\mathscr{S}\mathbb{R} = \{A^* \in \mathbb{V} : \mathbb{R}(A) = 0\}$$

is called a *semisimple class* of radical class $\mathbb{R}$. If $\mathbb{M}$ is any regular class, then the class

$$\mathscr{U}\mathbb{M} = \{A \in \mathbb{V} : A/I \in \mathbb{M} \Rightarrow I = A\}$$

is a radical class, which is referred to as the *upper radical class* of $\mathbb{M}$. For further details of the basic facts of radical theory we refer to [1], [9] and [10]. Radical of involution algebra has been studied in the recent papers [5], [6].

Given a class $\mathbb{M}$ it may happen that

(ID): $A^{id} \in \mathbb{M}$ if and only if $A^{-id} \in \mathbb{M}$ whenever $A^2 = 0$. In this case we say that $\mathbb{M}$ *satisfies condition* (ID). Let $\mathbb{R}$ be any radical class. We say that $\mathbb{R}$ *satisfies* A$-$D$-$S, if

$$(\mathrm{A-D-S}): \quad I^* \lhd A^* \quad \text{implies} \quad \mathbb{R}(I^*) \lhd A^* .$$

In [5] Theorem 1 it has been proved that a radical class $\mathbb{R}$ satisfies A$-$D$-$S if and only if $\mathbb{R}$ satisfies condition (ID) and that this is equivalent to the fact that $\mathscr{S}\mathbb{R}$ satisfies condition (ID) (cf. also [5] Corollary 2). In this paper following the investigations of [5] we shall deal mainly with the characterization of semisimple classes.

1.

We shall make use of the following two Propositions. For their proof we refer to [2] Propositions 1 and 2.

Proposition 1. *Let* $\mathbb{S}$ *be a class of involution algebras, which is closed under extensions and let* A *be any involution algebra. If* $I_n \lhd \ldots \lhd I_1 \lhd I_0 = A$ *and* $I_n \lhd A$, *and* $I_k/I_{k+1} \in \mathbb{S}$ *for each* $k = 0, 1, \ldots, n-1$, *then* $A/I_n \in \mathbb{S}$.

Proposition 2. *Let* $\mathbb{S}$ *be a regular class of involution algebras, and let* A *be any involution algebra. If* $I_n \lhd I_{n-1} \lhd \ldots \lhd I_1 \lhd I_0 = A$ *and* $I_k/I_{k+1} \in \mathbb{S}$ *for each* $k = 0, 1, \ldots, n$, *then for radical* $\mathbb{R} = \mathcal{U}\mathbb{S} : \mathbb{R}(A) \subseteq \mathbb{R}(I_n)$ *holds.*

Theorem 3. *Let* $\mathbb{S}$ *be a subclass of involution algebras, which is regular, coinductive, and closed under extensions. If the semisimple class* $\mathcal{S}\mathcal{U}\mathbb{S}$ *is hereditary, then* $\mathbb{S} = \mathcal{S}\mathcal{U}\mathbb{S}$, *that is,* $\mathbb{S}$ *is a semisimple class.*

Proof. We have to prove the inclusion $\mathcal{S}\mathcal{U}\mathbb{S} \subseteq \mathbb{S}$ only, as the opposite inclusion is trivially fulfilled. Let us take an algebra $A \in \mathcal{S}\mathcal{U}\mathbb{S}$ and choose an ideal I of A and an ideal M of I such that they are minimal relative to $A/I \in \mathbb{S}$ and $I/M \in \mathbb{S}$. By the regularity and coinductivity of $\mathbb{S}$ this is possible. Further, let H be the ideal of A generated by M. By [5] Proposition 2 $H^3 \subseteq M$ and $H^3 \lhd A$. Setting

$$A' = A/H^3\,, \quad I' = I/H^3 \text{ and } M' = M/H^3\,,$$

we have $A'/I' \in \mathbb{S}$, and $I'/M' \in \mathbb{S}$. It follows by Proposition 2 $\mathbb{R}(A') \subseteq M'$. Since

$$\frac{M'}{\mathbb{R}(A')} \lhd \frac{I'}{\mathbb{R}(A')} \lhd \frac{A'}{\mathbb{R}(A')} \in \mathcal{S}\mathcal{U}\mathbb{S}\,,$$

the hereditariness of $\mathcal{S}\mathcal{U}\mathbb{S}$ implies that $M'/\mathbb{R}(A') \in \mathcal{S}\mathcal{U}\mathbb{S}$. Now we will show that $N = M'/\mathbb{R}(A') \in \mathbb{S}$. As above, we can choose an ideal P of N and an ideal Q of B being minimal with respect to N/P, $P/Q \in \mathbb{S}$. Since $N^3 = \{0\}$, it is easy to see that $(Q \cap N^2) \lhd N$. For the sake of convenience, we set

$$Q' = \frac{Q}{Q \cap N^2}, \quad P' = \frac{P}{Q \cap N^2}, \quad N' = \frac{N}{Q \cap N^2}.$$

Similarly as above we can show that $\mathbb{R}(N') \subseteq Q'$ and $Q'/\mathbb{R}(N') \in \mathcal{S}\mathcal{U}\mathbb{S}$. Since $Q'^2 = 0$, using the fact that every involution subalgebra of Q' is an ideal of Q' it is straightforward to see that $Q'/\mathbb{R}(N') \in \mathbb{S}$. Applying Proposition 1 for the chain

$$\frac{Q'}{\mathbb{R}(N')} \lhd \frac{P'}{\mathbb{R}(N')} \lhd \frac{N'}{\mathbb{R}(N')}\,,$$

we get $N'/\mathbb{R}(N') \in \mathbb{S}$. Since $N/P \cong N'/P'$, the minimality of P implies that $\mathbb{R}(N') = P'$. Hence $P'/Q' \in \mathscr{S}\mathscr{U}\mathbb{S} \cap \mathbb{R} = \{0\}$, that is $P' = Q'$, and consequently $P = Q$. But $P \lhd N \in \mathscr{S}\mathscr{U}\mathbb{S}$. By the choice of Q: $P = Q$ is possible only in case $P = \{0\}$. Thus $N \in \mathbb{S}$, that is $M'/\mathbb{R}(A') \in \mathbb{S}$. Now we return to show that $A \in \mathbb{S}$. Using again Proposition 1 for the chain

$$\frac{M'}{\mathbb{R}(A')} \lhd \frac{I'}{\mathbb{R}(A')} \lhd \frac{A'}{\mathbb{R}(A')}$$

we have $A'/\mathbb{R}(A') \in \mathbb{S}$. Similarly to the above proof we can see that in this fact $\mathbb{R}(A') = P'$ and hence $P = Q = \{0\}$. This means that $A \in \mathbb{S}$. The proof is complete.

Remark. Considering associative or alternative rings, the semisimple class $\mathscr{S}\mathscr{U}\mathbb{S}$ is always hereditary, hence the assertion of Theorem 3 gives the non-trivial part of Sand's Theorem [7] characterizing semisimple classes. Let us observe that our proof differs substantially from that of [7] inasmuch as we did not make use of the associativity of the multiplication, but we used the Andrunakievich Lemma. This observation has been exploited in [2].

The Tangeman-Kreiling [8] lower radical construction carries over to involution algebras without difficulty (see also [5]). Let us give a homomorphically closed class $\mathbb{C}$ of involution algebras and define inductively

$$\mathbb{C}_1 = \mathbb{C}$$

$$\mathbb{C}_\lambda = \{A\colon \text{there exists an } I \lhd A \text{ such that } I \in \mathbb{C}_{\lambda-1} \text{ and } A/I \in \mathbb{C}_{\lambda-1}\}$$

if $\lambda - 1$ exists, and

$$\mathbb{C}_\lambda = \{A\colon A \text{ is the union of an ascending chain of ideals each belonging to one of the classes } \mathbb{C}_\mu,\ \mu < \lambda\}$$

if λ is a limit ordinal. Then the smallest radical class containing $\mathbb{C}$, called the *lower radical class* of $\mathbb{C}$, is given as

$$\mathscr{L}\mathbb{C} = \bigcup(\mathbb{C}_\lambda\colon \text{for all ordinals}).$$

In the following we give a criterion for upper and lower radical constructions satisfying A–D–S.

Proposition 4. *Let* $\mathbb{C}$ *be a class of involution algebras, which satisfies condition* (ID). *Then*

1) *If* $\mathbb{C}$ *is regular, then* $\mathscr{U}\mathbb{C}$ *satisfies* A–D–S.

2) *If* $\mathbb{C}$ *is closed under homomorphisms, then* $\mathscr{L}\mathbb{C}$ *satisfies* A–D–S.

Proof. 1) Suppose $\mathbb{C}$ is regular. Then $\mathscr{U}\mathbb{C}$ is a radical class. It suffices to show $A^{id} \in \mathscr{U}\mathbb{C}$ if and only if $A^{-id} \in \mathbb{C}$, whenever $A^2 = \{0\}$. Let us notice that for any algebra, such that $A^2 = \{0\}$, A^{id} and A^{-id} define involution algebras. Now let $A^2 = 0$ and $A^{id} \in \mathscr{U}\mathbb{C}$. Then every homomorphic image of A^{id} is also contained in $\mathscr{U}\mathbb{C}$. Since $\mathbb{C}$ satisfies condition (ID), it follows that every non-zero of A^{-id} is not in $\mathbb{C}$. Hence A^{-id} must be in $\mathscr{U}\mathbb{C}$. If $A^{-id} \in \mathscr{U}\mathbb{C}$, then a similar reasoning shows that $A^{id} \in \mathscr{U}\mathbb{C}$. By [5] Theorem 1 the radical class $\mathscr{U}\mathbb{C}$ satisfies A – D – S.

2) Since $\mathscr{L}\mathbb{C} = \bigcap(\mathbb{C}_\lambda$: for all ordinals), it suffices to show that $\mathbb{C}_\lambda$ satisfies condition (ID) for all ordinal λ. By the assumption $\mathbb{C}_1 = \mathbb{C}$ satisfies condition (ID). Let λ be any ordinal and suppose $\mathbb{C}_\mu$ satisfies condition (ID) for all $\mu < \lambda$. Let $A^{id} \in \mathbb{C}$. If λ is a limit ordinal, there is an ascending chain $\{I_\alpha\}$ of ideals of A^{id} such that

$$I_\alpha^{id} = I_\alpha \in \mathbb{C} \quad \text{and} \quad A^{id} = \bigcup_{\alpha<\lambda} I_\alpha^{id} .$$

The involution algebra A^{-id} can also be written in the form $A^{-id} = \bigcup_{\alpha<\lambda} I_\alpha^{-id}$ Since $\mathbb{C}$ satisfies condition (ID) it follows that $I_\alpha^{-id} \in \mathbb{C}^k$. Hence also $A^{-.d} \in \mathbb{C}_\lambda$ holds. Now suppose that $\lambda - 1$ exists, then by the construction A contains an ideal $I = I^{id}$ so that I^{id} and $(A/I)^{id}$ are in $\mathbb{C}_{\lambda-1}$. Also by the hypothesis I^{-id} and $(A/I)^{-id}$ are contained in $\mathbb{C}_{\lambda-1}$. Hence $A^{-id} \in \mathbb{C}_\lambda$. Similarly, if $A^{-id} \in \mathbb{C}$ we can show that $A^{id} \in \mathbb{C}_\lambda$, too. Thus $\mathscr{L}\mathbb{C} = \bigcup\{\mathbb{C}_\lambda\}$ satisfies also condition (ID). By [5] Theorem 1 $\mathscr{L}\mathbb{C}$ satisfies A – D – S.

For the rings the assumption that $\mathbb{C}$ is homomorphically closed is not essential, inasmuch as if $\mathbb{C}$ is not homomorphically closed, then define

$$\mathbb{C}_1 = \{A\colon A \text{ is a homomorphic image of } B \in \mathbb{C}\} .$$

Considering involution algebras, however, there is some subtle difference. Starting from a class $\mathbb{C}$ which is not homomorphically closed, we can define the lower radical as we did for rings. Nevertheless in Proposition 4, 2) we cannot drop the assumption $\mathbb{C}$ is homomorphically closed. For instance, let

$$\mathbb{C} = \{pZ/p^3Z\colon p \text{ is a given prime number}\} .$$

Since $\mathbb{C}$ does not contain zero-algebras, $\mathbb{C}$ satifies condition (ID). As one easily sees, $(pZ/p^2Z)^{id} = (pZ/p^2Z) \in \mathscr{L}\mathbb{C}$ but $(pZ/p^2Z)^{-id} \notin \mathscr{L}\mathbb{C}$.

Let $\mathbb{F}$ be any class of involution algebras. As in the case of associative rings (cf. [7]) we construct the following classes inductively

$$\mathbb{F}_\lambda = \{A\colon A \text{ contains an ideal } I \text{ of } A, \text{ with } I, A/I \in \mathbb{F}_{\lambda-1}\}$$

if $\lambda - 1$ exists, and

$$\mathbb{F}_\lambda = \{A\colon \text{there is a descending chain of ideals of } A\colon I_1 \supseteq I_k \supseteq \ldots \text{ such that } \bigcap I_i = \{0\} \text{ and } A/I_i \in \mathbb{F}_{\lambda_i} \text{ for ordinal } \lambda_i < \lambda\}$$

for each limit ordinal λ. Finally define

$$\mathbb{F}^\cup = \bigcup(\mathbb{F}_\lambda \text{ for all ordinals}) .$$

It is clear that $\mathbb{F}^\cup$ is regular, coinductive and closed under extensions, whenever $\mathbb{F}$ is regular. Furthermore, if $\mathbb{F}$ is hereditary, then $\mathbb{F}^\cup$ is a coradical class.

Corollary 5. *Let* $\mathbb{F}$ *be a regular class of involution algebras, which satisfies condition* (ID), *then* $\mathbb{F}^{\cup}$ *is a hereditary semisimple class and* $\mathcal{U}\mathbb{F}^{\cup} = \mathcal{U}\mathbb{F}$ *satisfies* A−D−S.

Proof. By Proposition 4 $\mathcal{U}\mathbb{F} = \mathcal{U}\mathbb{F}^{\cup}$ satisfies A−D−S, hence $\mathcal{S}\mathcal{U}\mathbb{F}^{\cup}$ is hereditary. By Theorem 3 we have $\mathbb{F}^{\cup} = \mathcal{S}\mathcal{U}\mathbb{F}^{\cup}$. Thus $\mathbb{F}^{\cup}$ is a hereditary semisimple class.

2.

In the following we shall give necessary and sufficient conditions for a semisimple class, whose radical satisfies A−D−S. This will exhibit the decisive role of the behaviour of nilpotent involution algebra. We need some technique.

Lemma 6. *Let A be an involution algebra and let*

$$B = (ann\ A : A) = \{x\colon\ Ax + xA \subseteq ann\ A\}\,.$$

Then $B/ann\ A = ann(A/ann\ A)$ *and* $(B/ann\ A)^{id}$ *is a subdirect sum, whose direct summands are isomorphic to ideals of* $(ann\ A)^{id}$.

Proof. By definition of B it is clear that $B/ann\ A = ann(A/ann\ A)$. Let B_{∇} be the zero-algebra defined on the additive group of B. Then B_{∇} can be considered an involution algebra with the identical operation id: $b^{id} = b$ for all $b \in B_{\nabla}$. Let us denote this involution algebra by B^{id}_{∇}. Consider the mappings

$$r_a,\ l_a\colon\ B^{id}_{\nabla} \to (ann\ A)^{id}$$

such that $r_a(x) = ax$, $l_a(x) = xa$ for all $x \in B^{id}_{\nabla}$ and a given element a of A. Let us notice that $(ann\ A)$ is an involution algebra. We can show by a straightforward calculation that they are homomorphisms of involution algebras. Since $ann\ A$ is a zero-algebra, $r_a(B^{id}_{\nabla})$ and $l_a(B^{id}_{\nabla})$ as involution sub-algebra of $(ann\ A)^{id}$ are also ideals of $(ann\ A)^{id}$. On the other hand we have

$$\bigcap_{a\in A} (ker\ r_a \cap ker\ l_a) = \{x \in B \mid ax = 0 = xa \text{ for every } a \in A\} = (ann\ A)^{id}\,.$$

Therefore $(B/ann\ A)^{id}$ is the subdirect sum of all $\dfrac{(B/ann\ A)^{id}}{ker\ r_a}$, $\dfrac{(B/ann\ A)^{id}}{ker\ l_a}$ whenever $a \in A$. Since

$$\frac{(B/ann\ A)^{id}}{ker\ r_a} \cong r_a(B^{id}_{\nabla}) \lhd (ann\ A)^{id}$$

and

$$\frac{(B/ann\ A)^{id}}{ker\ l_a} \cong l_a(B^{id}_{\nabla}) \lhd (ann\ A)^{id}\,,$$

the assertion is fulfilled.

Theorem 7. *Let $\mathbb{F}$ be a class of involution algebras, which is regular, coinductive and closed under extensions. The following conditions are equivalent:*

1) *$\mathscr{U}\mathbb{F}$ satisfies* A−D−S,

2) *$\mathbb{F}$ satisfies condition* (ID),

3) *If $A^* \in \mathbb{F}$ and $A^2 = \{0\}$, then $A^{-*} \in \mathbb{F}$,*

4) *If $A^* \in \mathbb{F}$ and $A^2 = \{0\}$, then $A^\circ \in \mathbb{F}$ for any involution $\circ$ built on A,*

5) *If $A^* \in \mathbb{F}$ and A is nilpotent, then every nilpotent involution algebra which is built on the additive group of A, belongs to $\mathbb{F}$.*

Moreover, if $\mathbb{F}$ satisfies any one (and hence all) of conditions 1) → 5) then $\mathbb{F}$ is a semisimple class.

Proof. The implications 5) ⇒ 4) ⇒ 3) ⇒ 2) are obvious. The implication 2) ⇒ 1) follows from Proposition 4. Thus the theorem will be proved, if we show the implications 1) ⇒ 4) ⇒ 5). Before the proof let us notice that if $\mathscr{U}\mathbb{F}$ satisfies A−D−S, then $\mathscr{S}\mathscr{U}\mathbb{F}$ is hereditary (cf. [5] and [1]), and hence by Theorem 3 $\mathbb{F}$ is a hereditary semisimple class, that is $\mathbb{F} = \mathscr{S}\mathscr{U}\mathbb{F}$. Now we return to the proof.

1) ⇒ 4). Let us consider the involution algebra A^*, such that $A^* \in \mathbb{F}$ and $A^2 = \{0\}$. Since $\mathscr{U}\mathbb{F}$ satisfies A−D−S, by the above remark $\mathbb{F}$ is hereditary. Thus every ideal of A^* is also contained in $\mathbb{F}$. Hence the set

$$K^* = \{x + x^* : x \in A^*\}$$

as an ideal of A^* is in $\mathbb{F}$. Let ϱ be a mapping of A^* into K^*, such that $\varrho(x) = x + x^*$. It is straightforward to see that ϱ is a homomorphism onto. The set

$$ker\, \varrho = \{y : y + y^* = 0,\ y \in A\}$$

is clearly also an ideal of A^*, hence $ker\, \varrho \in \mathbb{F}$. Moreover by [3] Corollary 2 we have

$$(ker\, \varrho)^{id} = (ker\, \varrho)^{-*} \in \mathbb{F}\,.$$

Since $(ker\, \varrho)^{id} \in \mathbb{F}$ and

$$A^{id}/(ker\, \varrho)^{id} \cong A^*/ker\, \varrho \cong K^* = K^{id} \in \mathbb{F}\,,$$

it follows that $A^{ia} \in \mathbb{F}$. Now let $\circ$ be any involution built on A. We can show similarly that the sets

$$K^\circ = \{x + x^\circ : x \in A\}$$
$$H^\circ = \{x \mid x + x^\circ = 0,\ x \in A\}$$

can be considered as ideals of A^{id} and A^{-id} respectively. Thus K° and H° are in $\mathbb{F}$. Since $K^\circ \cong A^\circ/H^\circ$, it follows $A^\circ \in \mathbb{F}$. Thus (4) holds.

4) ⇒ 5). Let $A^* \in \mathbb{F}$ be any nilpotent involution algebra. Since A^* is nilpotent there is a descending chain

$$A^* = I_n \supseteq \ldots \supseteq I_1 \supseteq I_0 = \{0\}$$

of ideals of A^* such that $I_{i+1}/I_i = ann(A^*/I_i)$. First we show that A^{id}_{∇}, the zero-algebra with involution *id* defined on additive group of A, is contained in $\mathbb{F}$. Since $ann\, A^* = I_1$ is an ideal of A^* and since $\mathbb{F}$ is hereditary, it follows $I_1 \in \mathbb{F}$. Furthermore by $I_1^2 = \{0\}$, condition 4) implies $I_1^{id} \in \mathbb{F}$. By Lemma 6 we have that $(I_k/I_2)_{p}$, is the subdirect sum, whose direct summands are isomorphic to ideals of $I_1^{id} \in \mathbb{F}$. Since every semisimple class is closed under subdirect sums, it follows $(I_2/I_1)^{id} \in \mathbb{F}$. Using induction we can show that $(I_k/I_{k-1})^{id}$ for all $k \geqq 1$. By Proposition 1 we have $A^{id} \in \mathbb{F}$. Now let A° be any nilpotent involution algebra built on the additive group of A. Since A is nilpotent, there is a descending chain

$$A^\circ = K_m \supseteq K_{m-1} \supseteq \dots \supseteq K_1 \supseteq K_0 = \{0\}$$

of ideals of A° such that $K_{i+1}/K_i \cong ann(A^\circ/K_i)$. Since $(ann\, A^\circ)^{id}$ is an ideal of $A^{id}_{\nabla} \in \mathbb{F}$, also $(ann\, A^\circ)^{id} \in \mathbb{F}$. By condition 4) we have $ann\, A^\circ \in \mathbb{F}$. Similarly as above we can show that $(K_{i+1}/K_i)^{id} \in \mathbb{F}$ for every $i \leqq m-1$. Since $K_{i+1}^2 \subseteq K_i$ condition 4) implies $K_{i+1}/K_i \in \mathbb{F}$. Using again Proposition 1 we have $A^\circ \in \mathbb{F}$. Thus 5 is valid. The last assertion of the theorem has already been proved.

Lemma 8. *Let A be an involution algebra such that $A^n = 0$. Then $(A^{n-1})^{id}$, the algebra with identical involution id on A^{n-1}, is the sum of its ideals, which are homomorphic images of $(A/A^2)^{id}$.*

Proof. Let us consider the mapping

$$f_a\colon (A/A^2)^{id} \to (A^{n-1})^{id}$$

defined by $f_a(x) = ax$ for $x = x + A^2 \in (A/A^2)^{id}$ where a is a given element of A^{n-2}. It is straightforward to see that f_a is a homomorphism. Since

$$(A^{n-1})^{id} = (A^{n-2}A)^{id} = \Big(\sum_{a\in A^{n-2}} aA\Big)^{id} \cong \sum_{a\in A^{n-2}} f_a((A/A^2)^{id}),$$

the assertion is valid.

Proposition 9. *Let $\mathbb{R}$ be a radical class satisfying* A–D–S, *and A be any nilpotent involution algebra contained in $\mathbb{R}$. Then $A^n \in \mathbb{R}$ for all natural numbers n.*

Proof. Since $\mathbb{R}$ satisfies A–D–S, by [5] Theorem 1 we have $(A/A^2) \in \mathbb{R}$, and this implies $(A/A^2)^{id} \in \mathbb{R}$. Applying Lemma 8 we get that $(A^{n-1})^{id}$ is the sum of its ideals, which are homomorphic images of $(A/A^2)^{id}$. Since $(A/A^2)^{id} \in \mathbb{R}$, it follows $(A^{n-1})^{id} \in \mathbb{R}$. Using again [5] Theorem 1 we have $A^{n-1} \in \mathbb{R}$. By $(A/A^{n-1})^{n-1} = \{0\}$ and $(A/A^{n-1})^{d} \in \mathbb{R}$, Lemma 8 yields that $(A^{n-2}/A^{n-1})^{id} \in \mathbb{R}$. Since $\mathbb{R}$ is closed under extension we get $(A^{n-2})^{id} \in \mathbb{R}$. Furthermore, since $\mathbb{R}$ satisfies A–D–S, also $A^{n-2} \in \mathbb{R}$ holds. By induction we can show $A^n \in \mathbb{R}$ for all n.

Theorem 10. *Let $\mathbb{R}$ be any radical class of involution algebras. The following conditions are equivalent:*

1) $\mathbb{R}$ *satisfies* A – D – S,

2) $\mathbb{R}$ *satisfies condition* (ID),

3) *If* $A^* \in \mathbb{R}$ *and* A *is nilpotent, then every nilpotent involution algebra, which is built on the additive group of* A, *belongs to* $\mathbb{R}$.

Proof. The equivalence of 1) and 2) has been proved in Theorem 8. The implication 3) $\Rightarrow$ 2) is obvious. We have only to show the implication 2) $\Rightarrow$ 3). Let A^* be any nilpotent involution algebra in $\mathbb{R}$. Then there is a natural number n such that $A^n = \{0\}$ by Proposition 9 $A^k \in \mathbb{R}$ for all $k \leqq n$. Hence $A^k/A^{k+1} \in \mathbb{R}$ for all $k \leqq n$. By [3] Theorem 1 we have $(A^k/A^{k+1})^{id} \in \mathbb{R}$ for all $k \leqq n$. Since $\mathbb{R}$ is closed under extensions using Proposition 1 we have A^{id}_{∇}, the zero-algebra with identical involution *id* built on the additive group of A, is in $\mathbb{R}$. Suppose A° is any nilpotent involution algebra built on the additive group of A. By Theorem 7 $(A^\circ/\mathbb{R}(A^\circ)) \in \mathcal{S}\mathbb{R}$ implies $(A^\circ/\mathbb{R}(A^\circ))^{id}_{\nabla} \in \mathcal{S}\mathbb{R}$. On the other hand

$$(A^\circ/\mathbb{R}(A^\circ))^{id}_{\nabla} \cong (A^{id}_{\nabla}/\mathbb{R}(A^\circ)^{id}_{\nabla})^{id} \in \mathbb{R} .$$

This implies that $(A^\circ/\mathbb{R}(A^\circ))^{id}_{\nabla} \in \mathbb{R} \cap \mathcal{S}\mathbb{R} = \{0\}$. Thus $\mathbb{R}(A^\circ) = A^\circ$, that is, $A^\circ \in \mathbb{R}$. This completes our proof.

Corollary 11. *Let* $\mathbb{R}$ *be any radical class satisfying* A – D – S *and let* A *be any nilpotent involution algebra. Then*

1) $A \in \mathbb{R}$ *if and only if* $A^{id}_{\nabla} \in \mathbb{R}$,

2) $A \in \mathcal{S}\mathbb{R}$ *if and only if* $ann\, A \in \mathcal{S}\mathbb{R}$.

Moreover if $A \in \mathcal{S}\mathbb{R}$ *then* $A/ann\, A \in \mathcal{S}\mathbb{R}$.

Proof. Assertion 1) follows from (3) of Theorem 10.

2) Since $\mathbb{R}$ satisfies A – D – S, $\mathcal{S}\mathbb{R}$ is hereditary. Thus if $A \in \mathcal{S}\mathbb{R}$, then *ann* A as an ideal of A is also in $\mathcal{S}\mathbb{R}$. Conversely, let A be a nilpotent involution algebra, such that $ann\, A \in \mathcal{S}\mathbb{R}$. Since A is nilpotent, there is a descending chain

$$A = I_n \supseteq I_{n-1} \supseteq \ldots \supseteq I_1 \supseteq I_0 = \{0\}$$

of ideals of A such that $I_{i+1}/I_i = ann\,(A/I_i)$. Using Lemma 6 and Theorem 7 again we can show that $I_{i+1}/I_i \in \mathcal{S}\mathbb{R}$ for all $i \leqq n$. Thus by Proposition 1 the assertion 2) holds. Furthermore let $J_i = I_i/I_1$ we have

$$J_{i+1}/J_i \cong \frac{I_{i+1}/I_1}{I_i/I_1} \cong I_{i+1}/I_i \in \mathcal{S}\mathbb{R} .$$

Applying Proposition 1 for the chain $\{0\} = J_1 \lhd J_2 \lhd \ldots \lhd J_n = A/I_1$ we obtain that $A/I_1 = A/ann\, A \in \mathcal{S}\mathbb{R}$.

Remark. The assertions of Corollary 11 are valid also for associative rings with similar proof.

Corollary 12. *Let $\mathbb{V}$ be a variety of all involution algebras over a commutative ring K of characteristic 2. Then a subclass $\mathbb{F}$ of $\mathbb{V}$ is a semisimple class if and only if $\mathbb{F}$ is regular, coinductive, and is closed under extensions.*

Proof. It follows immediately from [5] Corollary 3 and from Theorem 8.

3.

In this final section we shall exhibit the difference between semisimple and coradical classes.

Let A^* be any involution algebra over a commutative ring K $(1 \in K)$ such that $A^2 = \{0\}$. The ring K can be considered as an involution algebra K^{id}. Using A and K. a new involution algebra can be defined as in [5] in the following way:

Let us consider the cartesian product $E = K \times K \times A \times A$, on E we define operators by the following rules:

$$(a, b, x, y) + (c, d, u, v) = (a + c,\ b + d,\ x + u,\ y + v),$$

$$(a, b, x, y)(c, d, u, v) =$$

$$= (ac - bd,\ ad + bc,\ au - bv + cx - dy,\ av + bu + cy + dx),$$

$$k(a, b, x, y) = (ka, kb, kx, ky),$$

$$(a, b, x, y)^\circ = (a, -b, x^*, -y^*),$$

for all $a, b, c, d, k \in K$ and $x, y, u, v \in A$. Let us denote this involution algebra by $E^\circ(K, A)$. In particular, if $A = \{0\}$ we write shortly by $E^\circ(K)$.

The following result gives a new sufficient criterion for such a semisimple class, whose radical satisfies A$-$D$-$S.

Proposition 13. *Let K be any commutative ring with identity and let $\mathbb{F}$ be any hereditary semisimple class of involution algebras over K. If $\mathbb{F}$ contains the involution algebra $E^\circ(K)$ then $\mathcal{U}\mathbb{F}$ satisfies* A$-$D$-$S.

Proof. We will show that $\mathbb{F}$ satisfies condition (ID). Indeed, let $A \in \mathbb{F}$ be any involution algebra, such that $A^3 = \{0\}$. Suppose $A = A^{id}$. Considering the algebra $E^\circ(K, A)$ let us denote

$$I^\circ = \{(0, 0, x, y) \in E^\circ(K, A) : x, y \in A\}$$

$$L^\circ = \{(0, 0, x, 0) \in E^\circ(K, A) : x \in A\}$$

and

$$L^{-\circ} = \{(0, 0, 0, y) \in E^\circ(K, A) : y \in A\}.$$

By [5] Proposition 7 we have $I \lhd E^\circ(K, A)$, $I^\circ \cong A^{id} \oplus A^{-id}$, and $L^\circ \cong A^{id}$, but L° is not ideal of $E^\circ(K, A)$. Moreover, it is straightforward to see that $L^{-\circ} \lhd I^\circ \lhd$ $\lhd E^\circ(K, A)$, but $L^{-\circ}$ is not an ideal of $E^\circ(K, A)$, and $E^\circ(K, A)/I^\circ \cong E^\circ(K)$. Notice that $L^{-\circ} \cong A^{-id}$ and $I^\circ/L^{-\circ} \cong A^{id} \cong L^\circ$. Since $\mathbb{F}$ contains $E^\circ(K)$ and A^{id}, Proposition 2 is applicable for the chain $L^{-\circ} \lhd I^\circ \lhd E^\circ(K, A)$. Thus for the radical $\mathbb{R} = \mathcal{U}\mathbb{F}$ we have

$$\mathbb{R}(E^\circ(A, K)) \subseteq \mathbb{R}(L^{-\circ}) \subseteq L^{-\circ} .$$

If $\mathbb{R}(E^\circ(A, K)) = 0$, then the hereditariness of $\mathbb{F}$ implies $A^{-id} \in \mathbb{F}$. In the other case, there exists $y \in A$, $y \neq 0$, such that $(0, 0, 0, y) \in (E^\circ(K, A))$. Since $\mathbb{R}(E^\circ(K, A))$ is an ideal of $E^\circ(K, A)$, we get

$$0 \neq (0, 0, -y, 0) = (0, 1, 0, 0)\,(0, 0, 0, y) \subseteq E^\circ(K, A)\,.\,\mathbb{R}(E^\circ(K, A)) \subseteq$$
$$\subseteq E^\circ(K, A) \subseteq L^{-\circ} .$$

But $(0, 0, -y, 0) \in L^\circ$ and hence $(0, 0, -y, 0) \in L^\circ \cap L^{-\circ} = \{0\}$ – contradiction. Similarly, if $A^{-.d} \in \mathbb{F}$, $A^2 = \{0\}$, then we can show that $A^{.d} \in \mathbb{F}$, too. Since $\mathbb{F}$ satisfies condition (ID), by Theorem 7 $\mathcal{U}\mathbb{F}$ satisfies A – D – S.

It is well-known that in the case of associative and alternative rings a class is a semisimple class if and only if it is a coradical class. In the case of not necessarily associative rings, however, a semisimple class need not be a coradical class (cf. [3]) and a coradical class need not be a semisimple class (cf. [4]). For involution algebras the situation is similar to that of not necessarily associative rings. As semisimple classes are not always hereditary, they are not necessarily coradical classes. Furthermore, we have also the following

Proposition 14. *There exists a coradical class, which is not semisimple.*

Proof. Let K be a field and let $\mathbb{S}$ be a class consisting of $E^\circ(K)$ and K_∇^{-id} (where K_∇ is a zero-algebra built on additive group of A). It is easy to see that both K_∇^{-id} and $E^\circ(K)$ are simple involution algebras. Thus $\mathbb{S}$ is hereditary. Let us consider the following construction

$$\mathbb{C}_0 = \mathbb{S}$$
$$\mathbb{C}_\lambda = \{A\colon \text{there exists } I \lhd A \text{ such that } I, A/I \in \mathbb{C}_{\lambda-1}\}$$

if $\lambda - 1$ exists, and

$$\mathbb{C}_\lambda = \{A\colon \text{there exists ideals } I_{\lambda_i} \text{ of } A \text{ such that}$$
$$\textstyle\bigcap I_{\lambda_i} = 0, \text{ and } A/I_{\lambda_i} \in \mathbb{C}_i, \text{ where } i < \lambda\}\,,$$

if λ is a limit ordinal. Finally let

$$\mathbb{C} = \textstyle\bigcup(\mathbb{C}_\lambda\colon \text{for all ordinals})\,.$$

As in the case of rings we get that $\mathbb{C}$ is closed under extensions, subdirect sums and is hereditary (cf. [7]), that is, $\mathbb{C}$ is a coradical class. Now we will show that $\mathbb{C}$ is not a semisimple class. Indirectly, suppose that $\mathbb{C}$ is semisimple. Then $\mathbb{C}$ is a hereditary semisimple class. Since $E^\circ(K) \in \mathbb{C}$, Proposition 13 yields that $\mathbb{R} = \mathcal{U}\mathbb{C}$ satisfies

A$-$D$-$S. This means that for any ideal I of an involution algebra A: $\mathbb{R}(I) \lhd A$. In particular, for the ideal I° of $E^\circ(K, K_\nabla^{id})$ we get $\mathbb{R}(I^\circ) = L^\circ$. But by [5] Proposition 7 L° is not an ideal of $E^\circ(K, K_\nabla^{id})$ $-$ a contradiction. Thus $\mathbb{C}$ is not semisimple.

Corollary 15. *If the class* $\mathbb{C}$ *is coradical but not semisimple, then* $\mathcal{S}\mathcal{U}\mathbb{C}$ *is not hereditary, and hence* $\mathcal{U}\mathbb{C}$ *does not satisfy* A$-$D$-$S.

The proof is straightforward by Theorem 3.

References

[1] Anderson T., Divinsky N. and Suliński: Hereditary radicals in associative and alternative rings, Canad. J. Math. 17 (1965), 594—603.

[2] Anh P. N., Loi N. V. and Wiegandt R.: To the radical theory of Andrunakievich varieties, Bull. Austral. Math. Soc. (to appear).

[3] Armendariz E. P. and Leavitt W. G.: Nonhereditary semisimple classes, Proc. Amer. Math. Soc. 18 (1967), 1114—1117.

[4] Leavitt W. G. and Wiegandt R.: Torsion theory for not necessarily associative rings, Rocky Mountain J. Math. 9 (1979), 259—271.

[5] Loi N. V. and Wiegandt R.: Involution algebras and the Anderson-Divinsky-Suliński property, Acta Sci. Math. Szeged, (to appear).

[6] Salavová K.: Радикалы колец с инволуцией, 1, 2, Comment. Math. Univ. Carolinae 18 (1977), 367—381, 455—467.

[7] Sands A. D.: A characterization of semisimple classes, Proc. Edinburgh Math. Soc. 24 (1981), 5—7.

[8] Tangeman R. and Kreiling D.: Lower radicals in non-associative rings, J. Austral. Math. Soc. 14 (1972), 419—423.

[9] Wiegandt R.: Radical and semisimple classes of rings, Queen's papers in pure and appl. math. 37, Kingston, Ontario, 1974.

[10] Wiegandt R.: Radical theory of rings, Research Report UPWF 83/1.