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Free Left Distributive Semigroups

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The number of elements of finitely generated free left distributive semigroups is found.

V článku se nachází počet prvků konečně generovaných volných levodistributivních pologrup.

В статье находится число элементов конечно порожденной свободной леводистрибутивной полугруппы.

1. Introduction

Let L denote the variety of left distributive semigroups, i.e. of semigroups satisfying $xyz = xyxz$. By [1], every finitely generated left distributive semigroup is finite. Hence, for every positive integer n and any subvariety K of L, we can denote by $a(K, n)$ the number of elements of the free K-semigroup of rank n. The aim of this short note is to find the numbers $a(K, n)$ for some significant subvarieties K of L (by [1], L contains just 88 subvarieties).

In this paper, let F be a free semigroup over an infinite set X of variables. For $r, s \in F$, let $\mathrm{Mod}(r = s)$ denote the variety of semigroups satisfying $r = s$ and let $\mathrm{M}(r = s) = L \cap \mathrm{Mod}(r = s)$.

2. The variety L

Consider the following subsets of F: $A = \{x, x^2, x^3; x \in X\}$,

$B = \{x_1x_2 \dots x_n;\ 2 \leqq n,\ x_1, \dots, x_n \in X \text{ pair-wise different}\}$,

$C = \{x_1^2x_2 \dots x_n;\ 2 \leqq n,\ x_1, \dots, x_n \in X \text{ pair-wise different}\}$,

$D = \{x_1x_2 \dots x_{n-1}x_n^2;\ 2 \leqq n,\ x_1, \dots, x_n \in X \text{ pair-wise different}\}$,

$E = \{x_1^2x_2 \dots x_{n-1}x_n^2;\ 2 \leqq n,\ x_1, \dots, x_n \in X \text{ pair-wise different}\}$,

$G = \{x_1x_2 \dots x_nx_k;\ 2 \leqq n,\ 1 \leqq k < n,\ x_1, \dots, x_n \in X \text{ pair-wise different}\}$,

$H = \{x_1^2x_2 \dots x_nx_k;\ 2 \leqq n,\ 1 \leqq k < n,\ x_1, \dots, x_n \in X \text{ pair-wise different}\}$.

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2.1. Lemma. (i) Let $r, s \in F$. Then there are $p, q \in A \cup B \cup C \cup D \cup E \cup G \cup H = M$ such that $\mathrm{M}(r = s) = \mathrm{M}(p = q)$.
(ii) Let $p, q \in M$ be such that $p \neq q$. Then L is not contained in Mod $(p = q)$.

Proof. See [1].

For all integers $0 \leqq m \leqq n$, let $a(n, m) = n(n-1)\dots(n-m)$, $a(n) = \sum_{m=0}^{n} a(n, m)$ and $z(n) = \sum_{m=0}^{n} m\, a(n, m)$. Clearly, $a(n+1, m+1) = (n+1)\, a(n, m)$, $a(n+1) = (n+1)(1 + a(n))$ and $z(n+1) = (n+1)(a(n) + z(n))$.

2.2. Proposition. $a(L, n) = 4a(n) + 2z(n) - n$ for every $n \geqq 1$.

Proof. Let X_n be an n-element subset of X and let F_n be the subsemigroup of F generated by X_n. Put $A_n = A \cap F_n$ and define similarly B_n, etc. With regard to 2.1, we have $a(L, n) = \mathrm{card}(A_n) + \mathrm{card}(B_n) + \mathrm{card}(C_n) + \mathrm{card}(D_n) + \mathrm{card}(E_n) + \mathrm{card}(G_n) + \mathrm{card}(H_n)$. However, $\mathrm{card}(A_n) = 3n$, $\mathrm{card}(B_n) = \mathrm{card}(C_n) = \mathrm{card}(D_n) = \mathrm{card}(E_n) = \sum_{m=2}^{n} \binom{n}{m} m! = \sum_{m=2}^{n} n(n-1)\dots(n-m+1) = \sum_{m=1}^{n} . a(n, m) = a(n) - n$, $\mathrm{card}(G_n) = \mathrm{card}(H_n) = \sum_{m=2}^{n} \binom{n}{m} m!\,(m-1) = \sum_{m=2}^{n} (m-1) . n(n-1)\dots(n-m+1) = z(n)$. Thus $a(L, n) = 3n + 4\,a(n) - 4n + 2\,z(n) = 4\,a(n) + 2\,z(n) - n$.

2.3. Remark.

n	1	2	3	4	5	6	7	8	9	10
$a(L, n)$	3	18	93	516	3255	23478	191793	1753608	17755371	197282010

For every $n \geqq 0$, let $b(n) = \sum_{m=0}^{n} 1/m!$. Hence $1 = b(0) < 2 = b(1) < 5/2 = b(2) < b(3) < \dots$ and $\lim(b(n)) = \mathrm{e}$. Put also $b(-1) = 0$.

2.4. Lemma. $a(n) = b(n-1)\, n!$ for every $n \geqq 0$.

Proof. By induction.
For every $n \geqq 0$, let $y(n) = \sum_{m=0}^{n} b(m)$. Put also $y(-1) = y(-2) = 0$.

2.5. Lemma. $z(n) = y(n-2)\, n!$ for every $n \geqq 0$.

Proof. By induction (use 2.4).

For every $n \geqq -1$, let $v(n) = \sum_{m=n+1}^{\infty} 1/m! = e - b(n)$. Further, for $n \geqq 1$, let $u(n) = \sum_{m=1}^{n} v(n)$, $u(0) = 0$. Then $u(1) < u(2) < \dots < 1$ and $\lim(u(n)) = 1$.

2.6. Proposition. $a(L, n) = 2\,y(n)\,n! - 2 - n$ for every $n \geqq 1$.

Proof. By 2.2, 2.4 and 2.5, $a(L, n) = 4\,b(n-1)\,n! + 2\,y(n-2)\,n! - n =$

$= 2n!\,(2\,b(n-1) + y(n-2)) - n = 2n!\,(b(n-1) + y(n-1)) - n = 2n!\,.$
$.\,(b(n-1) + 1/n! + y(n-1)) - 2 - n = 2n!\,(b(n) + y(n-1)) - 2 - n =$
$= 2n!\,y(n) - 2 - n.$

2.7. Proposition. $a(L, n) = 2n\,\mathrm{e}n! - 2 - n + 2(1 - u(n))\,n!$ for every $n \geqq 1$.

Proof. This follows from 2.6 ($y(n) = n\,\mathrm{e} + 1 - u(n)$).

2.8. Corollary. $a(L, n) = 2n\,\mathrm{e}n! - 2 - n$ for every $n \geqq 1$. Moreover, lim $(a(L, n)/(2n\,\mathrm{e}n! - 2 - n)) = 1$.

2.9. Remark.

n	1	2	3	4	5	6
$2n\,\mathrm{e}n!$-2-n	1,436...	17,746...	92,858...	515,910...	3254,938...	23477,856...

3. The varieties R and R_1, R_2

Put $R = \mathrm{M}(x^2y = x^2y^2)$, $R_1 = \mathrm{M}(xy = xyx)$ and $R_2 = \mathrm{M}(xy = xy^2)$.

3.1. Lemma. (i) $R_1 \subseteq R_2 \subseteq R$.
(ii) $R_1 = \mathrm{Mod}\,(xy = xyx)$.

Proof. Clearly, $R_2 \subseteq R$. Further, for $S \in R_1$ and $x, y \in S$, we have $xy = xyx = (xy)\,x = (xyx)\,(xy) = x(yxy) = (xy)\,(xy) = xy^2$, so that $S \in R_2$. The equality $R_1 = \mathrm{Mod}\,(xy = xyx)$ is evident.

3.2. Lemma. $R_1 \neq R_2 \neq R$.

Proof. Consider the following groupoid $A = \{a, b, c, d\}$: $ab = ba = c$ and $xy = d$ in the remaining cases. Then A is a semigroup which is nilpotent of class 3, and hence $A \in R$. Clearly, $A \notin R_2$. Now, consider the following groupoid $B = \{a, b\}$: $aa = ba = a$, $ab = bb = b$. Then B is a semigroup of right zeros, $B \in R_2$ and $B \notin R_1$.

Denote by V the set of the following terms from F: $x, x^2, x^3, x \in X$; xy, x^2y, xy^2, $x, y \in X$, $x \neq y$; $y_1^i y_2 \dots y_n$, $1 \leqq i \leqq 2$, $3 \leqq n$, $y_1, \dots, y_n \in X$ pair-wise different; $y_1^i y_2 \dots y_n y_k$, $2 \leqq n$, $1 \leqq k < n$, $1 \leqq i \leqq 2$, $y_1, \dots, y_n \in X$ pair-wise different.

3.3. Lemma. (i) Let $r, s \in F$. Then there are $p, q \in V$ such that $R \cap \mathrm{Mod}\,(r = s) = R \cap \mathrm{Mod}\,(p = q)$.
(ii) If $p, q \in V$ are such that $p \neq q$, then R is not contained in $\mathrm{Mod}\,(p = q)$.

Proof. Use 2.1 and 3.2.

3.4. Proposition. $a(R, n) = n^2 + 2\,a(n) + 2\,z(n)$, $a(R_2, n) = 2n - 2n^2 + 2\,a(n) + 2\,z(n)$ and $a(R_1, n) = 2\,a(n)$ for every $n \geqq 1$.

Proof. Similar to that of 2.2.

4. The varieties T, T_1 and $T \cap R$

Put $T = M(xy^2 = x^2y^2)$ and $T_1 = \mathrm{M}(xy - x^2y)$. Clearly, $T_1 \subseteq T$.

4.1. Lemma. $T_1 \neq T$.

Proof. Consider the semigroup A from 3.2. Then $A \in T$ and $A \notin T_1$.

4.2. Proposition. $a(T, n) = n^2 + 2\,a(n) + 2\,z(n)$, $a(T_1, n) = 2\,a(n) + 2\,z(n)$, $a(T \cap R, n) = n^2 + n + a(n) + z(n)$, $a(T_1 \cap R_2, n) = n + a(n) + z(n)$ and $a(T_1 \cap R_1, n) = n + a(n)$ for every $n \geqq 1$.

Proof. Similar to that of 2.2.

5. Varieties of idempotent left distributive semigroups

Put $I = \mathrm{M}(x = x^2) = I_9$, $I_0 = \mathrm{Mod}\,(x = y)$, $I_1 = \mathrm{Mod}\,(x = xy)$, $I_2 = \mathrm{Mod}\,(x = x^2, xy = yx)$, $I_3 = \mathrm{Mod}\,(x = yx)$, $I_4 = \mathrm{Mod}\,(x = x^2, xyz = xzy)$, $I_5 = \mathrm{Mod}\,(x = xyx)$, $I_6 = \mathrm{Mod}\,(x = x^2, xyz = yxz)$, $I_7 = \mathrm{Mod}\,(x = x^2, xy = xyx)$ and $I_8 = \mathrm{Mod}\,(x = x^2, xyzx = xzyx)$. As it is proved in [1], these varieties are pair-wise different and they are the only subvarieties of the variety I of idempotent left distributive semigroups.

5.1. Proposition. For every $n \geqq 1$, $a(I_0, n) = 1$, $a(I_1, n) = a(I_3, n) = n$, $a(I_2, n) = 2^n - 1$, $a(I_4, n) = a(I_6, n) = n\,2^{n-1}$, $a(I_5, n) = n^2$, $a(I_7, n) = a(n)$, $a(I_8, n) = (n + n^2)\,2^{n-2}$, $a(I_9, n) = n + z(n)$.

Proof. Easy.

Reference

[1] Kepka T., Varieties of left distributive semigroup, Acta Univ. Carolinae Math. Phys. 25/1 (1984), 3—18