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On Geodesic Motion around Magnetized Black Holes

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In this note we study equatorial motion of electrically neutral test particles around the magnetized Kerr-Newman black hole. We derive the form of the effective potential governing radial motion in a weak-field limit. The shape of the effective potential curves and existence of their local minimum are evidently affected by strength of the magnetic field.

Práce se zabývá pohybem neutrálních testovacích částic kolem magnetizované Kerrovy-Newmanovy černé díry. Je odvozen tvar efektivního potenciálu pro radiální složku pohybu v limitě slabého magnetického pole.

Исследуется движение нейтральных пробных частиц вокруг замагниченной черной дыры Керра-Ньюмена. Показана потенциальная кривая радиальной составляющей движения в пределе слабого поля.

1. Introduction

In this paper we shall deal with magnetized black hole solutions of Ernst and Wild [1], which were derived by applying the Ernst procedure [2] to the ordinary Kerr-Newman spacetime. Exact solutions representing a black hole immersed in an external magnetic field are important for several reasons. They enable us to elucidate the influence of the magnetic field on geometry of the horizon [3, 4, 5], to generalize well-known uniqueness theorems for black holes [6], and to study mutual interaction of the black hole's mass, rotation and charge with the magnetic field [7]. These solutions also provide us with a nontrivial comparison of the exact theory with approximate results [8, 9]. Astrophysicists are not very enthusiastic about magnetized models of black holes, because they are not asymptotically flat. (Asymptotical flateness is essential for the generally accepted Hawking's definition of a black hole.) Nevertheless, they still may be of some astrophysical importance – at least in a weak-field limit – in connection with the Blandford-Znajek [10] electrodynamical model of active galactic nuclei. (According to our present knowledge, these objects are probably energized by massive black holes rotating in an external

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magnetic field.) Besides, magnetized black holes have several very remarkable features, which make them interesting in their own right (for further details and references see Ref. 7). Some insight into interesting effects which occur in these spacetimes can be obtained, as usually, by studying the geodesic motion. This has already been done in special cases by several authors [11–14].

In this note we shall study the effective potential for geodetical motion in the equatorial plane of the black hole. We restrict ourselves to a weak-field limit for reasons given below. We start out by rewritting the line element of Ernst and Wild in a suitable form. Then, in Sec. 3, we derive and analyse in detail the form of the effective potential governing radial motion. A short summary concludes the paper. Independent investigation of a similar problem was recently given by Esteban [15].

2. The line element

The metric of the magnetized Kerr-Newman black hole [1] can be expressed in a standard form of the stationary axisymmetric line element in spheroidal coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$:

(1) $$g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -e^{2\nu}\,\mathrm{d}t^2 + e^{2\psi}(\mathrm{d}\phi - \omega\,\mathrm{d}t)^2 + e^{2\lambda}\,\mathrm{d}r^2 + e^{2\mu}\,\mathrm{d}\theta^2\,.$$

Functions

(2) $$e^{2\nu} = |\Lambda|^2\,\varrho_k^2 A^{-1}\,,\qquad e^{2\psi} = |\Lambda|^{-2}\,\varrho_k^{-2} A\sin^2\theta\,,$$
$$e^{2\lambda} = |\Lambda|^2\,\varrho_k^2 \Delta^{-1}\,,\qquad e^{2\mu} = |\Lambda|^2\,\varrho_k^2$$

depend on a magnetic field strength parameter B_0 through functions Λ and ω, which approach their Kerr-Newman values

(3) $$\Lambda = 1\,,\quad \omega = a(2Mr - Q^2)\,A^{-1}$$

for $B_0 = 0$. In Eq. (2)

$$\Delta = r^2 + a^2 - 2Mr + Q^2\,,\quad \varrho_k^2 = r^2 + a^2\cos^2\theta\,,$$
$$A = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta\,.$$

M, Q, and a are the parameters, which in the Kerr-Newman spacetime denote mass, electric charge, and angular momentum of the black hole, respectively. The spacetime (1) is supposed to contain a black hole with a horizon located at $r = r_+ \equiv M + (M^2 - Q^2 - a^2)^{1/2}$. The form of Λ can be derived by straightforward application of the Ernst magnetizing procedure. Approximate expression linearized in B_0 yields

(4) $$\Lambda = 1 + B_0 Qar\varrho_k^{-2}\sin^2\theta - \mathrm{i}B_0Q(r^2 + a^2)\,\varrho_k^{-2}\cos\theta\,.$$

Function ω is determined by a partial differential equation with a very complex solution [16, 17]. We shall not need a precise expression for ω in this paper. Let us only remark that the terms nonlinear in B_0 obscure physical interpretation of ω and parameters a and Q. For example, holding t, r, ϕ fixed, ω as a function of θ may change – even on the horizon – its sign, and, therefore, can hardly be considered as an angular velocity. (Of course, the main problem arises from the asymptotical nonflatness, as discussed, e.g., in Ref. 6.) Thus hereafter we linearize all expressions in B_0. Naturally this weak-field limit cannot be applied for values of the dimensionless parameter $\beta \equiv B_0 M \gtrsim 1$ and/or distance parameter $r \gtrsim |B_0|^{-1}$. An approximate expression for ω can be obtained by straightforward manipulation from Ref. 1:

(5) $$\omega = a(2Mr - Q^2)\,A^{-1} - 2B_0 Qr(r^2 + a^2)\,A^{-1}\,.$$

3. The effective potential

In this section we derive the form of the effective potential governing geodetical motion in the equatorial plane. According to the standard procedure [18] we start out from the normalization condition for the tangent four-vector p^μ of a timelike geodesic of a test particle with nonzero mass μ_0,

(6) $$p^\mu p_\mu = -\mu_0^2\,.$$

Introducing constants of motion $E = -p_t$, $\Phi = p_\phi$, putting $\mu_0 = 1$, $\theta = \pi/2$, $p_\theta = 0$, and solving Eq. (6) for p^r we arrive at the effective potential

(7) $$V = \omega\Phi \pm [(\omega^2 + \delta)\,\Phi^2 + \gamma]^{1/2}\,,$$

where

$$\delta = [r^6 - 2Mr^5 + (a^2 + Q^2)\,r^4 - 4M^2r^2a^2 + 4MQ^2a^2r - a^2Q^4 + 4B_0Qar(r^4 + a^2r^2 + 2Ma^2r - Q^2a^2)]\,A^{-2}\,,$$

$$\gamma = (r + 2B_0Qa)\,r\Delta A^{-1}\,.$$

The positive local energy condition for Bardeen's [19] zero-angular-momentum observers,

$$p^{(t)} = e^{-\nu}(E - \omega\Phi) \gtrsim 0\,,$$

excludes the negative sign solution in (7) from further consideration. We can easily verify that for $B_0 = 0$ Eq. (7) agrees with the effective potential for the Kerr-Newman black hole given, e.g., in the Box 33.5 of "MTW" [20].

Note that all terms in V containing B_0 vanish if $Q = 0$. For example in the Schwarzschild-Melvin spacetime (Eqs. (1–3) with $a = Q = 0$) our approximate expression (7) is reduced to the effective potential of the ordinary Schwarzschild black hole; corrections due to the magnetic field are of second order in B_0 [11]. Also the effective potential of the magnetized Kerr spacetime is identical to the corresponding non-magnetized case [20].

Fig. 1 shows the behaviour of the effective potential as a function of the dimensionless distance parameter r/r_+ for several typical values of the other parameters, a/M, Q/M, B_0M, and Φ/M. It is evident that the existence of bound trajectories

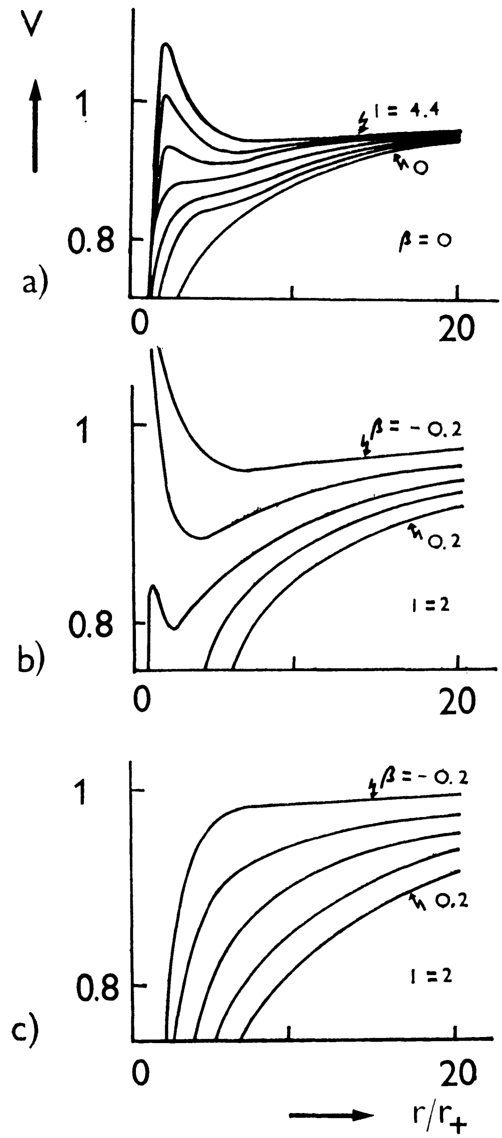

Fig. 1. The figure shows curves of the effective potential governing radial motion of electrically neutral test particles around the nonmagnetized ($B_0 = 0$) Schwarzschild black hole (a), and around the magnetized Kerr-Newman black hole with parameters $a/M = Q/M = 0.7$ (b), resp. $a/M = -Q/M = -0.7$ (c). The values of magnetic field parameter $\beta \equiv B_0M$ and specific angular momentum $l \equiv \Phi/M$ are given with the curves.

depends on the strength of the magnetic field. For $r/r_+ \gg 1$, due to the unbound amount of energy in the magnetic field concentrated around the polar axis, the effective potential should grow up to infinity (cf. [11]). However, the growing terms are not retained after linearization in B_0, indicating a failure of our approximation

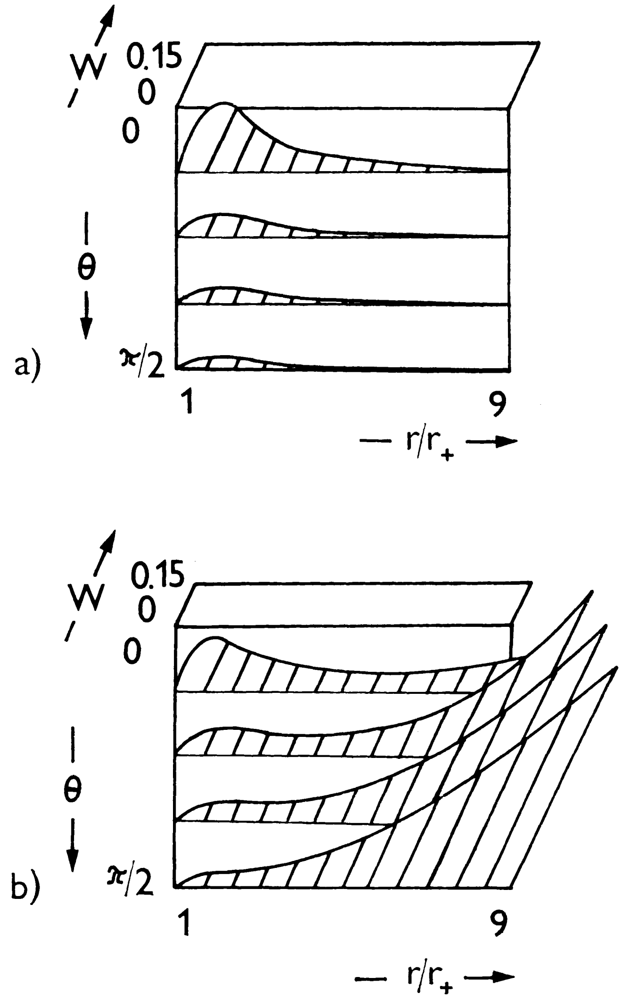

Fig. 2. The effective potential $W(r, \theta)$ for (generally nonradial) motion of particles with zero rest mass around the Schwarzschild black hole (a), and around the Schwarzschild-Melvin blackhole with parameter $\beta = 0.19$ (b) (which is a critical value for the equatorial null geodesics; see [11, 13]).

far from the black hole. This effect can easily be seen in the case of the Schwarzschild-Melvin spacetime. Now we show that even all photons with nonzero impact parameter $b \equiv \Phi/E$ are confined to a limited region around the polar axis; see also

[14]. Without restricting ourselves to the equatorial plane, Eq. (6) with $\mu_0 = a = Q = 0$ reads

(8) $$b^{-2} = W(r, \theta) + \Lambda^4 \left[\left(\frac{dr}{d\lambda} \right)^2 + r^2 \left(1 - \frac{2M}{r} \right) \left(\frac{d\theta}{d\lambda} \right)^2 \right],$$

where the effective potential (Fig. 2)

(9) $$W(r, \theta) = \Lambda^4 \left(1 - \frac{2M}{r} \right) r^{-2} \sin^{-2} \theta .$$

Clearly $W \to \infty$ whenever $B_0 r \sin \theta \to \infty$. For b given the effective potential determines the region of r and θ, where the particle may occur (Fig. 3).

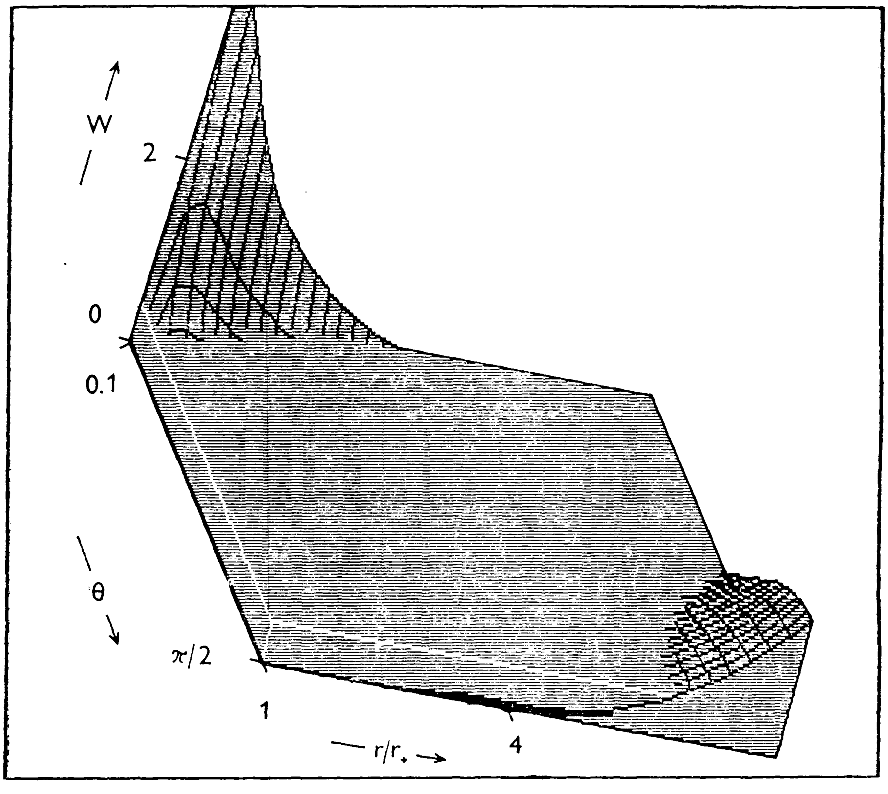

Fig. 3. Similarly to Fig. 2 the effective potential $W(r, \theta)$ is constructed. The motion is possible only under the "sea level", whose height depends on b ($b = 2$ in this figure).

4. Concluding remarks

In our note we derived the approximate form of the effective potential for geodesic motion around magnetized Kerr-Newman black hole. Our analysis shows that the effective potential is affected by the magnetic field already in the first order of the expansion in B_0. This should be contrasted to the Schwarzschild-Melvin case studied previously by several authors, where only the higher order terms in B_0 contribute to the effective potential and, therefore, a strong magnetic field is required to distort timelike geodesics significantly. The exception is the magnetized Kerr black hole, in which case the terms linear in B_0 also disappear.

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