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Continuous Restrictions of Linear Functionals

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We construct a linear map from \( l_1 \) to \( \mathbb{R} \) which does not have a continuous restriction to any closed infinite-dimensional subspace of \( l_1 \). This answers a question of Bogachev, Kirchheim and Schachermayer.

In [1], Bogachev, Kirchheim and Schachermayer proved that if \( X \) is a separable Banach space not containing \( l_1 \) isomorphically, and \( Y \) is any infinite-dimensional Banach space, then there is a linear map from \( X \) to \( Y \) without a continuous restriction to any closed infinite-dimensional subspace of \( X \). They asked what happens for \( X = l_1 \). More precisely, they asked the following question. Let \( Y \) be a Banach space, and let \( T \) be a linear map from \( l_1 \) to \( Y \). Does \( T \) always have a continuous restriction to some closed infinite-dimensional subspace of \( l_1 \)?

Our aim in this short note is to show that this is not the case. In fact, it is not the case even when \( Y = \mathbb{R} \). Our proof, which is based on a well-ordering argument, does not rely on any particular properties of \( l_1 \). Indeed, the same proof shows that if \( X \) is any separable Banach space then there is a linear map from \( X \) to \( \mathbb{R} \) without a continuous restriction to any closed infinite-dimensional subspace of \( X \). This strengthens the result of Bogachev, Kirchheim and Schachermayer mentioned above. The proof also works if \( X \) has a dense subset of size \( c \), the cardinality of \( \mathbb{R} \).

Our notation and terminology follow [2].

The following lemma is based on the existence of a family of \( c \) subsets of \( \mathbb{N} \) with pairwise finite intersections.

**Lemma 1.** Let \( X \) be an infinite-dimensional Banach space. Then the algebraic dimension of \( X \) is at least \( c \).

**Proof.** Choose a normalised basic sequence \( (x_n)_{n=0}^\infty \) in \( X \). Let \( \mathcal{A} \) be a family of \( c \) subsets of \( \mathbb{N} \) with pairwise finite intersections. To see the existence of such a family \( \mathcal{A} \), take, for example, the collection of all sets of the form \( \{ \sum_{i=0}^n 2^i : n \in \mathbb{N} \} \), where \( r_0 < r_1 < \ldots \) is an increasing sequence of natural numbers.

For \( A \in \mathcal{A} \), set \( x_A = \sum_{a \in A} 2^{-a} x_a \). Thus if \( A, A' \in \mathcal{A} \) with \( A \neq A' \) then the supports


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of the vectors $x_A$ and $x_{A'}$ have finite intersection, and hence the vectors $x_A, A \in \mathcal{A}$ are linearly independent.

We remark that, of course, in the presence of the Continuum Hypothesis, Lemma 1 follows immediately from the fact that a Banach space cannot have countably infinite algebraic dimension.

**Theorem 2.** Let $X$ be an infinite-dimensional separable Banach space. Then there is a linear map $T: X \to \mathbb{R}$ which does not have a continuous restriction to any closed infinite-dimensional subspace of $X$.

**Proof.** Since $X$ is separable, the cardinality of $X$ is $c$. Now, any closed infinite-dimensional subspace of $X$ is the closure of a countable subset of $X$, and hence there are at most $c$ such subspaces. Well-order the closed infinite-dimensional subspaces of $X$ as $(Y_\alpha)_{\alpha < \lambda}$, where the ordinal $\lambda$ is such that any predecessor of $\lambda$ has less than $c$ predecessors.

Let us construct, by transfinite induction on $\alpha$, a family of linearly independent vectors $x_{\alpha, n}, \alpha < \lambda, n \in \mathbb{N}$ such that $x_{\alpha, n} \in Y_\alpha$ for all $n$ and $\alpha$. That this is possible is immediate by Lemma 1, since, when we have chosen linearly independent $x_{\beta, n}, \beta < \alpha, n \in \mathbb{N}$, we have chosen less than $c$ vectors, while the dimension of $Y_\alpha$ is $c$.

Define a linear map $T: X \to \mathbb{R}$ by setting $T(x_{\alpha, n}) = n \|x_{\alpha, n}\|, \alpha < \lambda, n \in \mathbb{N}$ and taking an arbitrary linear extension to the whole of $X$. Then it is clear that, for each $\alpha$, the restriction of $T$ to $Y_\alpha$ is not continuous. □

In fact, one can weaken the assumption that $X$ is separable.

**Theorem 2'.** Let $X$ be an infinite-dimensional Banach space which has a dense subset of cardinality $c$. Then there is a linear map $T: X \to \mathbb{R}$ which does not have a continuous restriction to any closed infinite-dimensional subspace of $X$.

**Proof.** As before, the cardinality of $X$ is $c$, and so there are at most $c$ separable closed infinite-dimensional subspaces of $X$. Well-order these subspaces, and proceed as in the proof of Theorem 2. □

By considering the kernel of $T$, we obtain the following reformulation of Theorem 2'.

**Corollary 3.** Let $X$ be an infinite-dimensional Banach space which has a dense subset of cardinality $c$. Then there is a hyperplane of $X$ which does not contain any infinite-dimensional closed subspace of $X$.

We do not know what happens when $X$ is so large that it does not contain a dense subset of cardinality $c$.

**References**
