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Continuous Restrictions of Linear Functionals

BÉLA BOLLOBÁS*) AND IMRE LEADER**)

England, USA

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We construct a linear map from l_1 to $\mathbb{R}$ which does not have a continuous restriction to any closed infinite-dimensional subspace of l_1. This answers a question of Bogachev, Kirchheim and Schachermayer.

In [1], Bogachev, Kirchheim and Schachermayer proved that if X is a separable Banach space not containing l_1 isomorphically, and Y is any infinite-dimensional Banach space, then there is a linear map from X to Y without a continuous restriction to any closed infinite-dimensional subspace of X. They asked what happens for $X = l_1$. More precisely, they asked the following question. Let Y be a Banach space, and let T be a linear map from l_1 to Y. Does T always have a continuous restriction to some closed infinite-dimensional subspace of l_1?

Our aim in this short note is to show that this is not the case. In fact, it is not the case even when $Y = \mathbb{R}$. Our proof, which is based on a well-ordering argument, does not rely on any particular properties of l_1. Indeed, the same proof shows that if X is any separable Banach space then there is a linear map from X to $\mathbb{R}$ without a continuous restriction to any closed infinite-dimensional subspace of X. This strengthens the result of Bogachev, Kirchheim and Schachermayer mentioned above. The proof also works if X has a dense subset of size c, the cardinality of $\mathbb{R}$.

Our notation and terminology follow [2].

The following lemma is based on the existence of a family of c subsets of $\mathbb{N}$ with pairwise finite intersections.

Lemma 1. *Let X be an infinite-dimensional Banach space. Then the algebraic dimension of X is at least c.*

Proof. Choose a normalised basic sequence $(x_n)_0^\infty$ in X. Let $\mathscr{A}$ be a family of c subsets of $\mathbb{N}$ with pairwise finite intersections. To see the existence of such a family $\mathscr{A}$, take, for example, the collection of all sets of the form $\{\sum_{i=0}^n 2^{r_i}: n \in \mathbb{N}\}$, where $r_0 < r_1 < \dots$ is an increasing sequence of natural numbers.

For $A \in \mathscr{A}$, set $x_A = \sum_{a \in A} 2^{-a} x_a$. Thus if $A, A' \in \mathscr{A}$ with $A \neq A'$ then the supports

*) Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, England.

**) Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana, USA. Research partially supported by NSF Grant DMS-8806097.

of the vectors x_A and $x_{A'}$ have finite intersection, and hence the vectors x_A, $A \in \mathscr{A}$ are linearly independent. $\square$

We remark that, of course, in the presence of the Continuum Hypothesis, Lemma 1 follows immediately from the fact that a Banach space cannot have countably infinite algebraic dimension.

Theorem 2. *Let X be an infinite-dimensional separable Banach space. Then there is a linear map $T: X \to \mathbb{R}$ which does not have a continuous restriction to any closed infinite-dimensional subspace of X.*

Proof. Since X is separable, the cardinality of X is c. Now, any closed infinite-dimensional subspace of X is the closure of a countable subset of X, and hence there are at most c such subspaces. Well-order the closed infinite-dimensional subspaces of X as $(Y_\alpha)_{\alpha<\lambda}$, where the ordinal λ is such that any predecessor of λ has less than c predecessors.

Let us construct, by transfinite induction on α, a family of linearly independent vectors $x_{\alpha,n}$, $\alpha < \lambda$, $n \in \mathbb{N}$ such that $x_{\alpha,n} \in Y_\alpha$ for all n and α. That this is possible is immediate by Lemma 1, since, when we have chosen linearly independent $x_{\beta,n}$, $\beta < \alpha$, $n \in \mathbb{N}$, we have chosen less than c vectors, while the dimension of Y_α is c.

Define a linear map $T: X \to \mathbb{R}$ by setting $T(x_{\alpha,n}) = n\|x_{\alpha,n}\|$, $\alpha < \lambda$, $n \in \mathbb{N}$ and taking an arbitrary linear extension to the whole of X. Then it is clear that, for each α, the restriction of T to Y_α is not continuous. $\square$

In fact, one can weaken the assumption that X is separable.

Theorem 2′. *Let X be an infinite-dimensional Banach space which has a dense subset of cardinality c. Then there is a linear map $T: X \to \mathbb{R}$ which does not have a continuous restriction to any closed infinite-dimensional subspace of X.*

Proof. As before, the cardinality of X is c, and so there are at most c separable closed infinite-dimensional subspaces of X. Well-order these subspaces, and proceed as in the proof of Theorem 2. $\square$

By considering the kernel of T, we obtain the following reformulation of Theorem 2′.

Corollary 3. *Let X be an infinite-dimensional Banach space which has a dense subset of cardinality c. Then there is a hyperplane of X which does not contain any infinite-dimensional closed subspace of X.* $\square$

We do not know what happens when X is so large that it does not contain a dense subset of cardinality c.

References

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[2] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces 1: Sequence Spaces*, Springer Verlag, 1977, xiii + 190 pp.