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Optimal Choice of Parameters in Machine-Time Scheduling Problems with Penalized Earliness in Starting and Lateness in Completing the Operations

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A parametrized version of the machine-time scheduling problem from [1] with penalized earliness in starting and lateness in completing the operations is considered. The optimal choice of parameters for this problem is investigated and a method for finding optimal parameters is suggested.

Uvažuje se parametrizovaný problém nalezení optimálního rozvrhu práce n strojů z práce [1] při penalizaci předčasného započetí a opožděného ukončení práce jednotlivých strojů. Zkoumá se optimální volba parametrů pro tento problém a navrhuje se metoda umožňující nalézt optimální vektor prametrů pro tento případ.

Рассматривается параметризованная проблема оптимального расписания работы n машин при штрафах наложенных на преждевременное начало и опоздавшее время окончания работы отдельных машин. Исследуется оптимальный выбор параметров для этой проблемы и предлагается метод дающий возможность найти оптимальный вектор параметров.

1. The concept of optimal choice of parameters

Let us consider the optimization problem of the form

$$\varphi(x) \to \min$$

subject to $(\mathscr{P}_1(p))$

$$x \in M(p)$$

where φ: $\mathbb{R}^n \to \mathbb{R}^1$ is a continuous function, $M(p) \subset \mathbb{R}^n$ is a compact set, and $p \in \mathbb{R}^l$ is a given vector-parameter, which can be chosen from a given set P, $P \subset \mathbb{R}^l$. Suppose that

$$\tilde{P} \equiv \{p \in P \mid M(p) \neq \emptyset\}$$

$$\hat{X}(p) = \{\hat{x} \in M(p) \mid \varphi(\hat{x}) \leqq \varphi(x) \text{ for all } x \in M(p)\} \quad \forall p \in \tilde{P}\,.$$

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Definition 1.1 (compare [3])

Vector $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_l) \in \tilde{P}$ is called optimal choice of parameters $p_1, \ldots, p_l$ on the set P for the problems $(\mathscr{P}_1(p))$, $p \in P$, if it holds

$$[\hat{x} \in \hat{X}(\hat{p}),\ \hat{y} \in \hat{X}(p)] \Rightarrow \varphi(\hat{x}) \leqq \varphi(\hat{y})$$

for an arbitrary $p \in \tilde{P}$.

Especially, if $P = \{p \in \mathbb{R}^l \mid p^{(1)} \leqq p \leqq p^{(2)}\}$ and

$$M(p) = \{x \mid f_i(x) = p_i,\ i = 1, \ldots, l,\ x \in U\}\,, \tag{1.1}$$

where $p^{(1)}$, $p^{(2)}$ are given vectors, f_i: $\mathbb{R}^n \to \mathbb{R}^1$ are given functions, U a given subset in $\mathbb{R}^n$, the problem of finding the optimal choice of $p_1, \ldots, p_l$ on P for the problems $(\mathscr{P}_1(p))$, $p \in P$ can be solved as follows.

Let us consider the problems

$$\left.\begin{array}{ll} & \varphi(x) \to \min \\ \text{subject to} & \\ & x \in M(p) \end{array}\right\} \qquad (\mathscr{P}_1(p))$$

for $p \in P = \{p \mid p^{(1)} \leqq p \leqq p^{(2)}\}$ and $M(p)$ defined as in (1.1), and let $x^{\text{opt}}(p)$ be the optimal solution of $\mathscr{P}_1(p))$ for all $p \in \tilde{P}$.

Let us consider the problem

$$\left.\begin{array}{lll} & \varphi(x) \to \min & \\ \text{subject to} & & \\ & f_i(x) \geqq p_i^{(1)} & \forall i = 1, \ldots, l \\ & f_i(x) \leqq p_i^{(2)} & \forall i = 1, \ldots, l \\ & x \in U & \end{array}\right\} \qquad (\mathscr{P}_2)$$

and let x^{opt} be the optimal solution of $(\mathscr{P}_2)$. Let us set further $p_i^{\text{opt}} \equiv f_i(x^{\text{opt}})$ for all $i = 1, \ldots, l$.

Theorem 1

(a) $\varphi(x^{\text{opt}}) = \varphi(x^{\text{opt}}(p^{\text{opt}}))$

(b) $\varphi(x^{\text{opt}}(p)) \geqq \varphi(x^{\text{opt}}(p^{\text{opt}}))$ for all $p \in \tilde{P}$.

Proof

(a) It is obviously $x^{\text{opt}} \in M(p^{\text{opt}})$. Suppose that $\varphi(x^{\text{opt}}) \neq \varphi(x^{\text{opt}}(p^{\text{opt}}))$. It must be therefore $\varphi(x^{\text{opt}}) > \varphi(x^{\text{opt}}(p^{\text{opt}}))$. On the other hand $x^{\text{opt}}(p^{\text{opt}})$ is a feasible solution of $(\mathscr{P}_2)$ so that it must hold that $\varphi(x^{\text{opt}}) \leqq \varphi(x^{\text{opt}}(p^{\text{opt}}))$, which is a contradiction.

(b) Let us remark that $\varphi(x^{\text{opt}}(p^{\text{opt}})) = \varphi(x^{\text{opt}})$ according to (a).
Let us suppose that there exists $p^0 \in \tilde{P}$ such that

$$\varphi(x^{\text{opt}}(p^0)) < \varphi(x^{\text{opt}}(p^{\text{opt}})) = \varphi(x^{\text{opt}})\,. \tag{1.2}$$

It holds obviously: $x^{\text{opt}}(p^0) \in M(p^0)$ so that $p^{(1)} \leqq f(x^{\text{opt}}(p^0)) = p^0 \leqq p^{(2)}$, $x^{\text{opt}}(p^0) \in U$ and $x^{\text{opt}}(p^0)$ is therefore a feasible solution of $(\mathscr{P}_2)$. It must be therefore $\varphi(x^{\text{opt}}(p^0)) \geqq \varphi(x^{\text{opt}})$, which is a contradiction with (1.2).

Remark 1.1

The fact that p^{opt} is the optimal choice of $p_1, \ldots, p_l$ in the set P for the problems $\mathscr{P}_1(p)$, $p \in P$, follows immediately from Theorem 1.1(b) (compare Definition 1.1).

Therefore if we have at our disposal a numerical procedure for solving the problem $(\mathscr{P}_2)$, the problem of determining p^{opt} reduces to the solution of this problem (i.e. finding x^{opt}). Vector p^{opt} is then defined by the formulae $p_i^{\text{opt}} = f_i(x^{\text{opt}})$ for all $i = 1, \ldots, l$.

In this paper, we shall use this idea to find the optimal choice of parameters in one class of machine-time scheduling problems with penalization of starting time earliness and completion time tardiness for the jobs. The corresponding problem of the form $(\mathscr{P}_2)$ will be solved using an appropriately modified version of the method suggested in [4].

2. Problem formulation

The basic assumptions are the same as in the machine-time scheduling problems considered in [1]. We assume that n machines are given, machine j carries out exactly one operation j, the corresponding processing time is t_j for $j \in N \equiv \{1, \ldots, n\}$. The machines work in cycles (cycle 1, 2, ...). Let x_j be the starting time of the machine j in cycle 1 (for all $j \in N$). Machine $i \in N$ can start its work in cycle 2 only after the machines in a given set $N^{(i)}, N^{(i)} \subset N$, had finished their work in the preceding cycle 1 (i.e. the operations j with the starting time x_j and processing time t_j for all $j \in N^{(i)}$ had been carried out in cycle 1). Let d_i, $i \in N$, be the earliest possible starting time for the machine i in cycle 2. It holds then

$$d_i = \max_{j \in N^{(i)}} (x_j + t_j) . \quad \forall i \in N \tag{2.1}$$

We shall assume that x_j must belong to a prescribed time-interval $[k_j, K_j]$ for all $j \in N$. The set of feasible starting times x_j, $j \in N$ for a given $d = (d_1, \ldots, d_n)$ is therefore described by the following system of equations and inequalities:

$$\left.\begin{aligned} &\max_{j \in N^{(i)}} (x_j + t_j) = d_i , && \forall i \in N \\ &k_j \leqq x_j \leqq K_j , && \forall j \in N \end{aligned}\right\} \tag{2.2}$$

We shall suppose that there are given recommended time intervals $[a_j, b_j]$, $j \in N$, in which the operation j should be carried out, i.e. it is recommended that

$$[x_j, x_j + t_j] \subset [a_j, b_j] \quad \forall j \in N . \tag{2.3}$$

The violation of the recommended constraints (2.3) will be penalized by a function

$$\varphi_j(x_j) = \max (\psi_j^{(1)}(x_j), \psi_j^{(2)}(x_j + t_j), 0) \quad \forall j \in N , \tag{2.4}$$

where $\psi_j^{(1)}: \mathbb{R}^1 \to \mathbb{R}^1$ is a decreasing continuous function such that $\psi_j^{(1)}(a_j) = 0$

and $\psi_j^{(2)}: \mathbb{R}^1 \to \mathbb{R}^1$ is an increasing continuous function such that

$$\psi_j^{(2)}(b_j) = 0 .$$

We shall consider the problem

$$\left.\begin{aligned} &\varphi(x) \equiv \max_{j\in N} \varphi_j(x_j) \to \min \\ \text{subject to} \quad &\max_{j\in N^{(i)}} (x_j + t_j) = d_i\,, \quad \forall i \in N \\ &k_j \leqq x_j \leqq K_j\,, \qquad \forall j \in N \end{aligned}\right\} \qquad (\mathscr{P}_3(d))$$

where $d = (d_1, \ldots, d_n)$ is a parameter, which can move within the set $D = \{d \mid d^{(1)} \leqq \leqq d \leqq d^{(2)}\}$. We shall investigate in the sequel the problem of determining the optimal choice of parameters $d_1, \ldots, d_n$ for the problems $\mathscr{P}_3(d)$, $d \in D$ in the sense of Definition 1.1.

Using the idea of the section 1 we shall solve the problem

Minimize

$$\varphi(x) \equiv \max_{j\in N} (\psi_j^{(1)}(x_j),\ \psi_j^{(2)}(x_j + t_j), 0) \tag{2.5}$$

subject to

$$\max_{j\in N^{(i)}} (x_j + t_j) \geqq d_i^{(1)}\,, \quad \forall i \in N \tag{2.6}$$

$$\max_{j\in N^{(i)}} (x_j + t_j) \leqq d_i^{(2)}\,, \quad \forall i \in N \tag{2.7}$$

$$k_j \leqq x_j \leqq K_j\,, \quad \forall j \in N\,. \tag{2.8}$$

If $\hat{x}$ is the optimal solution of (2.5)–(2.8), then $\hat{d}_i \equiv \max_{j\in N^{(i)}} (\hat{x}_j + t_j)\ \forall i \in N$ is the optimal choice of parameters $d_1, \ldots, d_n$ for $\mathscr{P}_3(d)$, $d \in D$.

Let $L_j \equiv \{i \in N \mid j \in N^{(i)}\}\ \forall j \in N$. The inequalities (2.7) are equivalent to the system of inequalities

$$x_j \leqq \bar{x}_j(d^{(2)}) \equiv \min_{i\in L_j} d_i^{(2)} - t_j \quad \forall j \in N \tag{2.9}$$

so that the system of inequalities (2.7), (2.8) can be replaced by new bounds posed on the variables x_j, $j \in N$:

$$h_j \leqq x_j \leqq H_j \quad \forall j \in N\,, \tag{2.10}$$

where

$$h_j \equiv k_j\,, \quad H_j \equiv \min (K_j, \bar{x}_j(d^{(2)})) \quad \forall j \in N$$

($\bar{x}_j(d^{(2)})$ is defined in (2.9)) and the problem (2.5)–(2.8) is equivalent to

$$\left.\begin{aligned} &\varphi(x) \to \min \\ &\max_{j\in N^{(i)}} (x_j + t_j) \geqq d_i^{(1)} \quad \forall i \in N \\ &h_j \leqq x_j \leqq H_j \qquad\qquad \forall j \in N \end{aligned}\right\} \qquad (\mathscr{P}_4)$$

The problem of optimal choice of parameters d_i, $i \in N$ is now in principle reduced to the solution of ($\mathscr{P}_4$). We shall solve this problem by an appropriate adaptation of the method suggested in [4].

Remark 2.1

It can happen that there exists $j_0 \in N$ such that $h_{j_0} > H_{j_0}$. In such a case the set of solutions of the problem (2.5)–(2.8) is empty. If we denote by $M(d)$ the set of feasible solutions of $(\mathscr{P}_3(d))$, we have in this case $M(d) = \emptyset$ for all $d \in D$, so that our problem of optimal choice of d_i, $i \in N$ has no solution.

Remark 2.2

Comparing the problem $\mathscr{P}_3(d)$ with the general formulation in section 1, we obtain: $l = n$, $p = d$, $P = D$.

Remark 2.3

The objective function (2.5) is a generalization of the objective function used in [2].

3. The solution procedure

We shall describe the method for solving the problem $(\mathscr{P}_4)$. We can assume w.l.o.g. that $h_j \leqq H_j$ $\forall j \in N$ (compare Remark 2.1). The method is the adaptation of the general procedure suggested in [4]. Let us introduce the following notations for all $i, j \in N$:

$$V_{ij} \equiv \begin{cases} \emptyset, & \text{if } j \notin M^{(i)} \\ \{x_j \mid h_j \leqq x_j \leqq H_j,\ x_j + t_j \geqq d_i^{(1)}\}, & \text{if } j \in N^{(i)} \end{cases}$$

$$R_i \equiv \{j \mid V_{ij} \neq \emptyset\} \quad \text{for all} \quad i \in N;$$

$x_j^{(i)} \equiv \arg\min \{\varphi_j(x_j) \mid x_j \in V_{ij}\}$ $\forall i \in N, j \in R_i$ (i.e. $x_j^{(i)}$ is an arbitrary element of V_{ij} with the property $\varphi_j(x_j^{(i)}) = \min \{\varphi_j(x_j) \mid x_j \in V_{ij}\}$ for all $i, j \in N$, for which $V_{ij} \neq \emptyset$).

We shall denote by $j(i)$ and arbitrary index from R_i, for which

$$\varphi_{j(i)}(x_{j(i)}^{(i)}) = \min_{j \in R_i} \varphi_j(x_j^{(i)}) \quad \forall i \in N, \quad R_i \neq \emptyset.$$

We shall set further for all $k \in N$:

$$Z_k \equiv \{i \in N \mid j(i) = k\}$$

$$X_k \equiv \begin{cases} \bigcap_{i \in Z_k} V_{ik}, & \text{if } Z_k \neq \emptyset \\ [h_k, H_k] & \text{otherwise} \end{cases}$$

Remark 3.1

We shall assume further w.l.o.g. that $R_i \neq \emptyset$ for all $i \in N$ (otherwise the set of feasible solutions of $(\mathscr{P}_4)$ is empty).

Theorem 3.1 (compare [4])

Suppose $R_i \neq \emptyset$ for all $i \in N$ (compare Remark 3.1), let

$$\hat{x}_k = \arg\min \{\varphi_k(x_k) \mid x_k \in X_k\} \quad \forall k \in N.$$

Then $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_k)$ is the optimal solution of $(\mathscr{P}_4)$.

The assertion of this theorem follows immediately from Theorem 2 in [4]. Let us remark the sets V_{ik}, X_k are closed intervals and φ_k are continuous functions so that all minima exist and the assumptions of the Theorem 2 from [4] are satisfied.

It follows now immediately from the consideration in section 1 that if $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ is defined as in Theorem 3.1, then

$$\hat{d}_i \equiv \max_{j \in N^{(i)}} (\hat{x}_j + t_j), \quad \forall i \in N$$

is the optimal choice of parameters $d_1, \ldots, d_n$ for the problems $(\mathscr{P}_3(d))$, $d \in D$.

4. Some explicit formulae

We shall use the special form of the problem $(\mathscr{P}_4)$ and derive explicit formulae for $\hat{x}$, $\hat{d}$ from the preceding section. Let us note that it is in our case:

$$V_{ij} = \{x_j \mid \tilde{h}_{ij} \leqq x_j \leqq H_j\}, \text{ where } \tilde{h}_{ij} = \max(h_j, d_i^{(1)} - t_j) \; \forall i \in N, \; j \in R_i \quad (4.1)$$

Further we have for all $k \in N$:

$$X_k = \{x_k \mid \tilde{h}_k \leqq x_k \leqq H_k\}, \quad (4.2)$$

where

$$\tilde{h}_k = \begin{cases} \max\limits_{i \in Z_k} d_i^{(1)} - t_k, & \text{if } Z_k \neq \emptyset \text{ and } \max\limits_{i \in Z_k} d_i^{(1)} - t_k > h_k \\ h_k & \text{otherwise}. \end{cases}$$

It follows immediately from the definition of the functions φ_k, $k \in N$ (compare (2.4)) that for all $i \in N$, $k \in R_i$:

$$x_k^{(i)} = \begin{cases} H_k, & \text{if } a_k > H_k \\ \tilde{h}_{ik}, & \text{if } b_k < \tilde{h}_{ik} \\ \in [a_k, \min(b_k - t_k, H_k)], & \text{if } a_k \leqq H_k \text{ and } b_k \geqq \tilde{h}_{ik} \end{cases} \quad (4.3)$$

Similarly it holds for all $k \in N$:

$$x_k^{(0)} \equiv \arg\min\{\varphi_k(x_k) \mid x_k \in [h_k, H_k]\} = \begin{cases} H_k, & \text{if } a_k > H_k \\ h_k, & \text{if } b_k < h_k \\ \in [a_k, \min(b_k - t_k, H_k)] & \text{if } a_k \leqq H_k \text{ and } b_k \geqq h_k \end{cases} \quad (4.4)$$

Let us set further

$$\hat{N} \equiv \{k \in N \mid Z_k \neq \emptyset\}$$

$$\hat{Z}_k = \{s \in Z_k \mid \tilde{h}_{sk} = \max_{i \in Z_k} \tilde{h}_{ik}\} \quad \text{for all} \quad k \in \hat{N}.$$

It is then for all $k \in N$:

$$\hat{x}_k = \begin{cases} x_k^{(s)} \text{ with } s \in \hat{Z}_k, & \text{if } Z_k \neq \emptyset \\ x_k^{(0)}, & \text{if } Z_k = \emptyset \end{cases} \quad (4.5)$$

Therefore the process of determining $\hat{x}, \hat{d}$ can be summarized as follows:[1])

(1) Determine the sets V_{ij}, R_i $\forall i \in N$, $j \in N$;

(2) If there exists $i_0 \in N$ such that $R_{i_0} = \emptyset$, then $(\mathscr{P}_4)$ has no feasible solution and thus $M(d) = \emptyset$ for all $d \in D$.

(3) If $R_i \neq \emptyset$ for all $i \in N$, determine $x_j^{(i)}$ according to the formulae (4.3).

(4) Determine the sets, Z_k $\forall k \in N$, $\hat{N}$ and $\hat{Z}_k$ $\forall k \in \hat{N}$.

(5) Determine $\hat{x}_k$, $k \in N$ according to the formulae (4.5).

(6) Set $\hat{d}_i \equiv \max_{j \in N(i)} (\hat{x}_j + t_j)$ $\forall i \in N$.

5. Numerical example

$m = n = 5$ so that $N = \{1, 2, 3, 4, 5\}$,

$t = (t_1, t_2, t_3, t_4, t_5) = (2, 3, 1, 4, 5)$

$k = (0, 0, 0, 0, 0)$, $K = (10, 10, 10, 10, 10)$

$d^{(1)} = (6, 5, 7, 8, 6)$, $d^{(2)} = (10, 10, 10, 10, 10)$

i	1	2	3	4	5
$N^{(i)}$	$\{1, 2, 3\}$	$\{2, 4\}$	$\{1, 2, 3\}$	$\{1, 4, 5\}$	$\{1, 2, 3, 5\}$

The inequalities

$$\max_{j \in N(i)} (x_j + t_j) \leqq 10 \quad \forall i \in N$$

imply that $x_1 \leqq 8$, $x_2 \leqq 7$, $x_3 \leqq 9$, $x_4 \leqq 6$, $x_5 \leqq 5$.

It is therefore

$$h = k = (0, 0, 0, 0, 0), \quad H = (8, 7, 9, 6, 5).$$

We shall assume further that

$$\varphi_j(x_j) \equiv \max (a_j - x_j,\ x_j + t_j - b_j,\ 0) \quad \text{for all} \quad j \in N,$$

where a_j, b_j are for all $j \in N$ given constants so that we have in our case for all $j \in N$:

$$\psi_j^{(1)}(x_j) \equiv a_j - x_j, \quad \psi_j^{(2)}(x_i + t_j) \equiv x_j + t_j - b_j.$$

We assume that $a = (1, 1, 1, 3, 3)$, $b = (4, 4, 5, 5, 5)$.

We shall solve now the problem $(\mathscr{P}_4)$, which has in our case the following form:

$$\max_{1 \leqq j \leqq 5} \max (a_j - x_j,\ x_j + t_j - b_j,\ 0) \to \min$$

subject to

$$\max_{j \in N(i)} (x_j + t_j) \geqq d_i^{(1)} \quad \forall i \in N$$

$$0 \leqq x_1 \leqq 8, \quad 0 \leqq x_2 \leqq 7, \quad 0 \leqq x_3 \leqq 9, \quad 0 \leqq x_4 \leqq 6, \quad 0 \leqq x_5 \leqq 5.$$

The sets V_{ij} look as follows:

$$V_{11} = [4, 8], \quad V_{12} = [3, 7], \quad V_{13} = [5, 9], \quad V_{14} = \emptyset, \quad V_{15} = \emptyset$$

[1]) The complexity of the procedure depends on the complexity of determining $x_j^{(i)}$. If $\varphi_j(x_j)$ is partially linear as in the next section, the procedure has a polynomial complexity.

$V_{21} = \emptyset$, $V_{22} = [2, 7]$, $V_{23} = \emptyset$, $V_{24} = [1, 6]$, $V_{25} = \emptyset$

$V_{31} = [5, 8]$, $V_{32} = [4, 7]$, $V_{33} = [6, 9]$, $V_{34} = \emptyset$, $V_{35} = \emptyset$

$V_{41} = [6, 8]$, $V_{42} = \emptyset$, $V_{43} = \emptyset$, $V_{44} = [4, 6]$, $V_{45} = [3, 5]$

$V_{51} = [4, 8]$, $V_{52} = [3, 7]$, $V_{53} = [5, 9]$, $V_{54} = \emptyset$, $V_{55} = [1, 5]$

Further we obtain for $x_j^{(i)}$ and $\varphi_j^{(i)} \equiv \varphi_j(x_j^{(i)})$:

i	
1	$x_1^{(1)} = 4$, $\varphi_1^{(1)} = 2$; $x_2^{(1)} = 3$, $\varphi_2^{(1)} = 2$; $x_3^{(1)} = 5$, $\varphi_3^{(1)} = 1$;
2	$x_2^{(2)} = 2$, $\varphi_2^{(2)} = 1$; $x_4^{(2)} = 2$, $\varphi_4^{(2)} = 1$;
3	$x_1^{(3)} = 5$, $\varphi_1^{(3)} = 3$; $x_2^{(3)} = 4$, $\varphi_2^{(3)} = 3$; $x_3^{(3)} = 6$, $\varphi_3^{(3)} = 2$;
4	$x_1^{(4)} = 6$, $\varphi_1^{(4)} = 4$; $x_4^{(4)} = 4$, $\varphi_4^{(4)} = 3$; $x_5^{(4)} = 3$, $\varphi_5^{(4)} = 3$;
5	$x_1^{(5)} = 4$, $\varphi_1^{(5)} = 2$; $x_2^{(5)} = 3$, $\varphi_2^{(5)} = 2$; $x_3^{(5)} = 5$, $\varphi_3^{(5)} = 1$; $x_5^{(5)} = \frac{3}{2}$, $\varphi_5^{(5)} = \frac{3}{2}$

The indices $j(i)$, for which $\varphi_{j(i)}(x_{j(i)}^{(i)}) = \min_{j \in R_i} \varphi_j(x_j^{(i)})$ will be defined as follows

i	1	2	3	4	5
$j(i)$	3	2	3	4	3

It is then

$$Z_1 = \emptyset, \quad Z_2 = \{2\}, \quad Z_3 = \{1, 3, 5\}, \quad Z_4 = \{4\}, \quad Z_5 = \emptyset$$

so that

$$X_1 = [0, 8], \quad X_2 = [2, 7], \quad X_3 = [6, 9], \quad X_4 = [4, 6], \quad X_5 = [0, 5]$$

and

$$\hat{x} = (\hat{x}_1, 2, 6, 4, \tfrac{3}{2}), \quad \text{where} \quad \hat{x}_1 \in [1, 2].$$

The optimal value of φ is thus $\varphi(\hat{x}) = 3$. Let us choose e.g. $\hat{x}_1 = 1$. We obtain then for the optimal choice of $d_1, \ldots, d_5$:

$$\hat{d}_1 = \max(3, 5, 7) = 7$$

$$\hat{d}_2 = \max(5, 8) = 8$$

$$\hat{d}_3 = \max(3, 5, 7) = 7$$

$$\hat{d}_4 = \max(3, 8, 6\tfrac{1}{2}) = 8$$

$$\hat{d}_5 = \max(3, 5, 7, 6\tfrac{1}{2}) = 7$$

References

[1] Cunninghame-Green, R. A.: Minimax Algebra, Lecture Notes in Economics and Mathematical Systems, 166, Springer-Verlag 1979.

[2] Vlach, M. - Zimmermann, K.: Optimalizace okamžiků zahajování operací při pevném pořadí a penalizaci předčasného zahájení a opožděného ukončení operací. Algoritmy řízení a rozvrhování výroby, 1988, pp. 82—89.

[3] Zlobec, S.: Input Optimization: I. Optimal realizations of matematical models, Mathematical Programming 31 (1985), pp. 245—268.

[4] Zimmermann, K.: On max-separable optimization problems, Annals of Discrete Mathematics 19 (1984), pp. 357—362.