Josef Kolomý

Maximal monotone mappings in banach spaces

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 33 (1992), No. 1, 63--67

1992 ACTA UNIVERSITATIS CAROLINAE—MATHEMATICA ET PHYSICA VOL. 33. NO. 1

Maximal Monotone Mappings in Banach Spaces

JOSEF KOLOMÝ

Czechoslovakia*)

Received 25 April 1991

1. Introduction

In this note, there is given an alternative and simpler proof of the interesting Fabian theorem (see [3] and compare [4]) concerning the set of all points where a monotone multivalued mapping acting in Banach spaces is singlevalued and norm-to-norm upper semicontinuous. We show that if X is a Banach space and $T: X \to 2^{X^*}$ a monotone mapping with $D = \operatorname{int} D(T) \neq \emptyset$ such that T is singlevalued and hemiclosed at $u_0 \in D$, then T is norm-to-weak* upper semicontinuous at u_0. Moreover, we give another and simple proof of the fact that a subdifferential map of a convex continuous function is maximal monotone.

Recall that Kenderov [10], [11], proved the following important results: (i) If X is a Banach space which admits an equivalent norm such that its dual norm on X^* is rotund and $T: X \to 2^{X^*}$ is monotone with $D = \operatorname{int} D(T) \neq \emptyset$, then there exists a dense G_δ subset $C(f)$ of D such that T is singlevalued at the points of $C(f)$, where $C(f)$ is a set of all points of D, where the function f of the minimum modulus of T is continuous; (ii) A Banach space X is an Asplund space if and only if for each monotone mapping $T: X \to 2^{X^*}$ with $D = \operatorname{int} D(T) \neq \emptyset$, there exists a dense G_δ subset $D_0 \subset D$ such that T is singlevalued and norm-to-norm upper semicontinuous on D_0.

Note that a rather different result has been proved by Kenderov and Robert [12]. A short survey of the recent results concerning the topological properties of monotone mappings is contained, among others, in [13].

2. Notions and notations

Let X be a normed linear space, X^* its dual, 2^{X^*} the system of all subsets of X^*, $T: X \to 2^{X^*}$ a mapping, $D(T) = \{u \in X: T(u) \neq \emptyset\}$ its domain, $G(T) = \{(u, u^*) \in X \times X^*: u \in D(T), u^* \in T(u)\}$ its graph in the space $X \times X^*$. Recall that a mapping $T: X \to 2^{X^*}$ is said to be: (i) monotone if for each $u, v \in D(T)$ and each $u^* \in T(u)$

*) Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia.

and $v^* \in T(v)$, there is $\langle u^* - v^*, u - v\rangle \geqq 0$; (ii) maximal monotone (see [1]), if T is monotone and its graph $G(T)$ is not contained as a proper subset in the graph of any other monotone map; (iii) upper semicontinuous at $u_0 \in D(T)$, if for each open subset W of X^* such that $T(u_0) \subset W$, there exists an open neighborhood U of u_0 such that $T(u) \subset W$ for each $u \in U \cap D(T)$; (iv) locally bounded at $u_0 \in D(T)$, if there exists a neighborhood V of u_0 such that $T(V) = \{T(u): u \in V \cap D(T)\}$ is bounded in X^*; (v) $T = \partial f$ is a subdifferential map on a convex subset $M \subset X$, if $f: M \to R$ is a convex continuous function and $M \ni u \to \partial f(u)$, where $\partial f(u) = \{u^* \in X^*: \langle u^*, v - u\rangle \leqq f(v) - f(u)$ for each $v \in M\}$. We shall say that X^* is an (H)-space, if for each $(u_n^*) \subset X^*$, $u_n^* \to u^*$ weakly*, $u^* \in X^*$, $\|u_n^*\| \to \|u^*\|$, we have that $u_n^* \to u^*$ in the norm of X^*. For a given set $A \subset X$, $\operatorname{int} A$, $\operatorname{int}_a A$, denote the interior of A and the algebraic interior of A, respectively. Furthermore, we shall use the standard notions given in Diestel [2], Giles [7] and Phelps [14].

3. Maximal monotone mappings

Let X be a Banach space, $T: X \to 2^{X^*}$ a maximal monotone mapping with $D = \operatorname{int} D(T) \neq \emptyset$. According to Kenderov [10], we define a function $f: X \to R_+ \cup \{+\infty\}$ by

$$f(u) = \inf\{\|u^*\|: u^* \in T(u)\}, \quad u \in X. \tag{1}$$

Since $T(u)$ is weakly* closed for each $u \in D(T)$ and T is locally bounded on D, we have that $T(u)$ is weakly* compact for each $u \in D$ and hence $f(u) = \min\{\|u^*\|: u^* \in T(u)\}$ for each $u \in D$. The function f is lower semicontinuous on X and finite on D and therefore f is continuous on a dense G_δ subset, say $C(f)$, of D.

Next, we shall use the following

Lemma 1 [6]. *Let X be a Banach space, $T: X \to 2^{X^*}$ a maximal monotone mapping with $D = \operatorname{int} D(T) \neq \emptyset$. If $u_0 \in C(f)$, $u_n \in D$, $u_n \to u_0$, $u_n^* \in T(u_n)$, $u_0^* \in T(u_0)$, then $\|u_n^*\| \to \|u_0^*\|$.*

Proof. Since T is norm-to-weak* upper semicontinuous on D and the norm of X^* is weakly* lower semicontinuous, we have that $\liminf_{n\to\infty} \|u_n^*\| \geqq \|u_o^*\|$. We shall prove that $\limsup_{n\to\infty} \|u_n^*\| \leqq \|u_o^*\|$. Suppose, on the contrary, that $\limsup_{n\to\infty} \|u_n^*\| > \|u_o^*\|$. Without loss of generality one can assume that $\|u_n^*\| > \|u_o^*\| + \alpha$ for infinitely many indexes and some $\alpha > 0$. Choose $v_n \in X$, $\|v_n\| = 1$ such that $\langle u_n^*, v_n\rangle \geqq \|u_n^*\| - n^{-1}$. Then $\langle u_n^*, v_n\rangle > \|u_o^*\| + \alpha - n^{-1}$ for each n. We have that $u_n + n^{-1}v_n \in D$ for sufficiently large n. Now, choose $v_n^* \in T(u_n + n^{-1}v_n)$ such that $\|v_n^*\| = f(u_n + n^{-1}v_n)$. By monotonicity of T, $\langle u_n^*, v_n\rangle \leqq \langle v_n^*, v_n\rangle$. Therefore $f(u_0) + \alpha - n^{-1} \leqq \|u_o^*\| + \alpha - n^{-1} < \langle u_n^*, v_n\rangle \leqq \langle v_n^*, v_n\rangle \leqq \|v_n^*\| = f(u_n + n^{-1}v_n)$, a contradiction to the fact that f is continuous at u_0.

Under the assumptions of Theorem 1, the Fabian theorem ([3]) asserts that the set $C(T)$ of all points of D, where T is singlevalued and norm-to-norm upper semicontinuous, is a dense G_δ subset of D. In fact, Fabian has proved the following.

Theorem 1. *Let X be a Banach space which admits an equivalent norm on X such that X^* in its dual norm is a rotund (H)-space. If $T: X \to 2^{X^*}$ is maximal monotone with $D = \operatorname{int} D(T) \neq \emptyset$, then the set $C(T)$ of all points of D, where T is singlevalued and norm-to-norm upper semicontinuous, is equal to the dense G_δ subset $C(f)$ of D, where the function f defined by (1) is continuous in D.*

Proof. Assume that $\|\cdot\|$ is a dual norm on X^* such that $(X^*, \|\cdot\|)$ is a rotund (H)-space. First of all, the function f is continuous on a dense G_δ subset $C(f)$ of D. According to the Kenderov theorem, T is singlevalued at the points of a dense G_δ subset $C(f)$ of D. Let $u_0 \in C(f)$ be arbitrary, $(u_n) \subset D(T)$, $u_n \to u_0$ and $u_n^* \in T(u_n)$. Since T is norm-to-weak* upper semicontinuous on D, we have that $u_n^* \to T(u_0)$ weakly* in X^*. By Lemma 1 and our hypothesis, we conclude that $u_n^* \to T(u_0)$ in the norm of X^* and therefore $u_0 \in C(T)$, which proves that $C(f) \subseteq C(T)$. Assume now that $u_0 \in C(T)$. Since f is lower semicontinuous on X, it is sufficient to prove that f is upper semicontinuous at u_0. Let $(u_n) \subset D(T)$, $u_n \to u_0$, $u_n^* \in T(u_n)$. By our hypothesis for a given $\varepsilon > 0$, there exists an integer n_0 such that $\|u_n^* - T(u_0)\| \leqq \varepsilon$ for each $n \geqq n_0$. Moreover, $f(u_0) = \|T(u_0)\|$ and $f(u_n) \leqq \|u_n^*\|$ for each n. Therefore $f(u_n) \leqq \|u_n^*\| \leqq \|T(u_0)\| + \varepsilon = \varepsilon + f(u_0)$ for each $n \geqq n_0$. Hence $u_0 \in C(f)$ and therefore $C(f) = C(T)$, which completes the proof.

Let us remark that a Theorem 1 gives at once the following result. Let X be a Hilbert space, $T: X \to 2^X$ a strongly monotone mapping with $D = \operatorname{int} D(T) \neq \emptyset$. Then the set of all points of D, where T is singlevalued and norm-to-norm upper semicontinuous, is equal to the dense G_δ subset $C(f)$ of D, where the function f is defined by (1).

Corollary 1. *Let X be a Banach space which admits an equivalent norm such that X^* in its dual norm is a rotund (H)-space. Assume that $M \subseteq X$ is an open convex subset and that $\varphi: M \to R$ is a convex continuous function on M. Then the set of all points of M, where φ is Fréchet differentiable, is equal to a dense G_δ subset $C(f)$ of M of the points of the continuity of the function $f: M \to R_+$ defined by $f(u) = \min\{\|u^*\|: u^* \in \partial\varphi(u)\}$, $u \in M$.*

Proof. First of all, $T = \partial\varphi$ is maximal monotone on M with $D(\partial\varphi) = M$ (see [1], [14]): Now, the result follows at once from Theorem 1 and the fact that φ is Fréchet differentiable at some $u_0 \in M$ if only if T is singlevalued and norm-to-norm upper semicontinuous at u_0 (see [14], [5]).

Recall that a Banach space X satisfies the assumption of Theorem 1 if one of the following three conditions is fulfilled: (i) X is reflexive; (ii) X^* is separable; (iii) X and X^* are both weakly compact generated (see [2], [7]).

Definition 1. *Let X be a normed linear space. A mapping $T: X \to 2^{X^*}$ is said to be hemiclosed at $u_0 \in \operatorname{int}_a D(T)$ if for every $z \in X$ and each null-sequence of positive numbers (t_n) and $u_n^* \in T(u_n)$ such that $\|u_n^*\| \leqq C$ for some constant $C > 0$, where $u_n = u_0 + t_n z$ and $u_n \in D(T)$ for sufficiently large n, then there exists a subnet $(u_{n_\alpha}^*)$ of (u_n^*) having the weak* limit point u_o^*, and $u_o^* \in T(u_0)$.*

Proposition 1. *Let X be a Banach space $T: X \to 2^{X^*}$ a monotone mapping with $D = \operatorname{int} D(T) \neq \emptyset$. If T is singlevalued and hemiclosed at $u_0 \in D$, then T is norm-to-weak* upper semicontinuous at u_0.*

Proof. Fix $z \in X$ and let (t_n) be a null-sequence of positive numbers. Then $u_n \in D$ for sufficiently large n, where $u_n = u_0 + t_n z$. Choose $u_n^* \in T(u_n)$. Since T is locally bounded on D, we conclude that there exist a subnet $(u_{n_\alpha}^*)$ of (u_n^*) and a point $u_o^* \in X^*$ such that $u_{n_\alpha}^* \to u_o^*$ weakly*. By our hypotheses, $u_o^* = T(u_0)$, and therefore the whole sequence (u_n^*) converges weakly* to $T(u_0)$. Now, using similar arguments as inKato [8] for multivalued monotone mappings, we conclude that T is norm-to-weak* upper semicontinuous at u_0.

Definition 2. *Let X be a normed linear space. We shall say that a mapping $T: X \to 2^{X^*}$ has a property (P) at $u_0 \in D(T)$ if the following condition is satisfied: If $(u_\alpha) \subset D(T)$ is a net, $u_\alpha \to u_0$ in the norm of X, $u_\alpha^* \in T(u_\alpha)$ is such that $\|u_\alpha^*\| \leqq C$ for some constant $C > 0$, then there exists a subnet $(u_{\alpha_j}^*)$ of (u_α^*) with the weak* limit point u_o^* such that $u_o^* \in T(u_0)$.*

Lemma 2. *Let X be a normed linear space, $T: X \to 2^{X^*}$ a mapping with $D(T) \subseteq X$. If T is locally bounded and possesses the property (P) at $u_0 \in D(T)$, then T is norm-to-weak* upper semicontinuous at u_0.*

Proof: standard, compare [9].

Lemma 3 ([14, § 6]). *Let X be a normed linear space, $M \subset X$ an open subset, $T: M \to 2^{X^*}$ a monotone and norm-to-weak* upper semicontinuous mapping. If $T(u)$ is nonempty, convex and weak* closed for all $u \in M$, then T is maximal monotone in M.*

Proposition 2. *Let X be a Banach space, $M \subset X$ and open convex subset, $f: M \to R$ a convex continuous function. Then the subdifferential map ∂f is maximal monotone in M.*

Proof. Since ∂f is locally bounded on M (see [7]), according to Lemmas 2, 3, it suffices to prove that ∂f satisfies the condition (P) at the points of M. Indeed, let $u_0 \in M$ be arbitrary, $(u_n) \subset M$, $u_n \to u_0$ and $u_n^* \in \partial f(u_n)$. Then (u_n^*) is bounded and hence there exist a subnet $(u_{n_\alpha}^*)$ of (u_n^*) and a point $u_o^* \in X^*$ such that $u_{n_\alpha}^* \to u_o^*$ weakly* in X^*. We have $\langle u_{n_\alpha}^*, v - u_{n_\alpha} \rangle \leqq f(v) - f(u_{n_\alpha})$ for each $v \in M$. Passing to the limit, we get that $\langle u_o^*, v - u_0 \rangle \leqq f(v) - f(u_0)$, for each $v \in M$, i.e. $u_o^* \in \partial f(u_0)$, which finishes the proof.

References

[1] Deimling K., Nonlinear Functional Analysis, Springer, Berlin, 1985.

[2] Diestel J., Geometry of Banach Spaces - Selected Topics, Lecture Notes in Math., 485, Springer, Berlin, 1975.

[3] Fabian M., On single-valuedness (and strong continuity) of maximal monotone mappings, Comment. Math. Univ. Carolinae, 18 (1977), 19—39.

[4] Fabian M., Once more on continuity of maximal monotone mappings, Comment. Math. Univ. Carolinae 18 (1977), 105—114.

[5] Fabian M., On strong continuity of monotone mappings in reflexive Banach spaces, Acta Polytechnica, 6 (1977), 85—98.

[6] Fitzpatrick S. P., Continuity of non-linear monotone operators, Proc. Amer. Math. Soc., 62 (1977), 111—116.

[7] Giles J. R., Convex Analysis with Application in the Differentiation of Convex Functions, Pitman, London, 1982.

[8] Kato T., Demicontinuity, hemicontinuity and monotonicity II, Bull. Amer. Math. Soc., 73 (1967), 886—889.

[9] Kendcrov P. S., Semicontinuity of set-valued monotone mappings, Fund. Math., 88 (1975), 61—69.

[10] Kenderov P. S., Multivalued monotone mappings are almost everywhere singlevalued, Studia Math., 56 (1976), 199—203.

[11] Kenderov P. S., Monotone operators in Asplund spaces, C.R. Acad. Bulgare Sci., 30 (1977), 263—264.

[12] Kenderov P. S., Robert R., Nouveaux résultats génériques sur les opérateurs monotones dans les espaces de Banach, C.R. Acad. Sci. Paris, 282 (2976), 845—847.

[13] Kolomý J., Set-valued mappings and structure of Banach spaces, Suppl. Rend. Circ. Mat. Palermo, 14 (1987), 345—351.

[14] Phelps R. R., Convex Functions, Monotone Operators and Differentiability, Lecture Notes in Math., 1364, Springer, 1989.