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On Decomposition of Projections of Finite Order

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A map $f: X \to Y$ is said to be a *map of order* $\leq$k if for every $y \in Y$ the set $f^{-1}(y)$ consits of at most k points. The continuous maps of order ≤ 2 will be called *simple* following Borsuk and Molski [1].

Hurewicz established the formula

$$\dim f(X) \leq \dim X + k - 1$$

for continuous maps of order $\leq k$ between compact metric spaces; see, for instance [5], p. 97. Another theorem of Hurewicz implies that continuous maps of finite order defined on compact metric spaces cannot lower dimension (loc. cit., p. 114).

We say that a continuous map $f: X \to Y$ between metric spaces is a *superposition of m maps* (or *f decomposes into m maps*), if there exist metric spaces $X_0 = X$, $X_1, \ldots, X_{m-1}, X_m = Y$ and continuous maps $f_i : X_{i-1} \to X_i$, $i = 1, 2, \ldots, m$, such that $f = f_m \circ f_{m-1} \circ \ldots \circ f_1$.

Sieklucki [6] proved the following theorem:

Let X be a finite dimensional compact metric space and Y be a metric space. If $f : X \to Y$ is a continuous map of finite order, then it is a superposition of finite number of simple maps.

The present paper contains a as main theorem the following result:

Let K be a compact subset of the product $T \times \boldsymbol{R}^n$, where T is a metric space. If the projection $P : K \to T$ is a map of order $\leq k (k \geq 3)$, then it is a superposition of $3n$ continuous maps of order $\leq k - 1$.

This theorem implies Sieklucki's theorem. The paper also contains an example which indicates that in the case $n = 1$ the number $3n$ is the minimal one. Namely, there is constructed a compact subset K of $\boldsymbol{R}^2 \times \boldsymbol{R}$ such that the projection of K into $\boldsymbol{R}^2$ is a map of order ≤ 3 which does not decompose into two simple maps.

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It is worthwhile to add that the Sieklucki Theorem and the previously mentioned theorems of Hurewicz imply the following corollary in type of Л. В. Келдыш's theorems on decompositions (comp.: [2], Следствие 1, 3; [3], Следствие 1–3; [4], Теорема):

Let X, Y be compact metric spaces, $\dim X = n$, $\dim Y = n + m$. *If* $f: X \to Y$ *is a continuous map of order* $\leq k$, *then it can be given in the form*:

$$f = \psi_{m+1} \circ \varphi_m \circ \psi_m \circ \ldots \circ \varphi_1 \circ \psi_1 ,$$

where all ψ_i *are continuous maps of order* $\leq k$ *which do not raise dimension and* φ_i *are simple maps which raise dimension by one.*

The core of this paper lies in the following special case of the main theorem:

Theorem 1. *Let K be a compact subset of the Cartesian product* $T \times \boldsymbol{R}$ *of a metric space T and the real line* $\boldsymbol{R}$. *If the projection* $p: K \to T$ *is a map of order* $\leq k (k \geq 3)$, *then it is a superposition of three continuous maps of order* $\leq k - 1$.

Proof. (I) Let K, T and p satisfy the assumptions of the theorem. There is no loss in generality to assume, that p is onto T. Then T is a compact space. Let

$$K_t = \{x \in \boldsymbol{R} : (t, x) \in K\} \qquad \text{for } t \in T.$$

Let us agree that $K_t = \{x_1(t), x_2(t), \ldots, x_k(t)\}$, where $x_1(t) < x_2(t) < \ldots < x_l(t) = x_{l+1}(t) = \ldots = x_k(t)$. Let

$$r(t) = \min \{x_i(t) - x_{i-1}(t) : 2 \leq i \leq k\}.$$

Obviously, $r(t) > 0$ if and only if the number of elements of K_t is exactly k.

Identify points in K if they are of the form $(t, x_{i-1}(t))$ and $(t, x_i(t))$ with $x_i(t) - x_{i-1}(t) = r(t)$. This identification induces an equivalence relation R on K such that $(t, x) R (\bar{t}, \bar{x})$ if and only if $t = \bar{t}$, $\{x, \bar{x}\} = \{x_{j_1}(t), x_{j_2}(t)\}$ for some j_1, j_2, $j_1 \leq j_2$ and $x_j(t) - x_{j-1}(t) = r(t)$ for every $j \in \{j_1 + 1, \ldots, j_2\}$. Consider also a finer equivalence S on K induced by identification of points $(t, x_1(t))$, $(t, x_2(t))$ with $x_2(t) - x_1(t) = r(t)$.

(II). *The equivalences R and S are upper semicontinuous.* We will prove this only for the equivelence R; the proof for S is analogous.

Since K is compact it suffices to show that the set $R \subset K \times K$ is closed. In this purpose take any sequences (t^n, x^n), $(\bar{t}^n, \bar{x}^n)$ in K converging respectively to (t^0, x^0), $(\bar{t}^0, \bar{x}^0)$ and such that $(t^n, x^n) R (\bar{t}^n, \bar{x}^n)$ for every $n \in \boldsymbol{N}$. Obviously $t^0 = \bar{t}^0$. Passing to subsequences we can assume that all the sequences $x_i(t^n)$, $i = 1, 2, \ldots, k$ are convergent. We can also assume that for fixed $j_1, j_2, j_1 < j_2$ and every $n \in \boldsymbol{N}$ we have $\{x^n, \bar{x}^n\} = \{x_{j_1}(t^n), x_{j_2}(t^n)\}$ and $x_j(t^n) - x_{j-1}(t^n) = r(t^n)$ for each $j \in \{j_1 + 1, \ldots, j_2\}$. Consider two cases:

1. $r(t^n) \geq \varepsilon > 0$ from an index large enough. Then sequences $x_i(t^n)$ converge to different elements of K_{t^0}. The inequalities between $x_i(t^n)$ are preserved in the limit, so

$$\lim_{n\to\infty} x_i(t^n) = x_i(t^0) \qquad \text{for} \qquad i = 1, 2, \ldots, k\,,$$

in particular $\{x_{j_1}(t^0), x_{j_2}(t^0)\} = \{x^0, \bar{x}^0\}$.

Given $j \in \{j_1 + 1, \ldots, j_2\}$ notice that for every $i \in \{2, \ldots, k\}$

$$x_i(t^0) - x_{i-1}(t^0) = \lim_{n\to\infty} (x_i(t^n) - x_{i-1}(t^n)) \geq \lim_{n\to\infty} (x_j(t^n) - x_{j-1}(t^n)) = x_j(t^0) - x_{j-1}(t^0)$$

and hence $r(t^0) = x_j(t^0) - x_{j-1}(t^0)$. Therefore $(t^0, x^0)\, R(\bar{t}^0, \bar{x}^0)$.

2. $\liminf_{n\to\infty} r(t^n) = 0$. Then $x = \bar{x}$ and obviously $(t^0, x^0)\, R(\bar{t}^0\, \bar{x}^0)$.

Therefore R is closed in $K \times K$ and this implies upper semicontinuity of R.

(III). Since the equivalence R is upper semicontinuous, the quotient space K/R is metric and compact. Let $q : K \to K/R$ be the quotient map. The formula

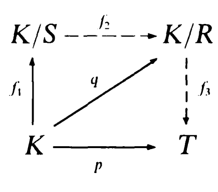
$$f_3([(t, x)]_R) = t \qquad \text{for} \qquad (t, x) \in K$$

defines a continuous map $f_3 : K/R \xrightarrow{onto} T$. Notice that f_3 is a map of order $\leq k - 1$. Indeed, even if $p^{-1}(t)$ consists of k elements, then $r(t) > 0$ and so there exists a pair of different R-equivalent elements of $p^{-1}(t)$. Therefore $f_3^{-1}(p)$ contains at most $k - 1$ equivalence classes of the relation R.

Similarly K/S is a compact metric space for S is an upper semicontinuous equivalence. Denote by $f_1 : K \to K/S$ the quotient map. The formula

$$f_2([(t, x)]_S) = [(t, x)]_R \qquad \text{for} \qquad (t, x) \in K$$

defines a continuous map $f_2 : K/S \xrightarrow{onto} K/R$. This map is of order $\leq k - 1$. Indeed, even if the number of elements in $[(t, x)]_R$ is k, then $r(t) > 0$, $x_1(t) \neq x_2(t)$ and $(t, x_1(t))\, S(t, x_2(t))$. Thus any $[(t, x)]_R$ contains at most $k - 1$ equivalence classes of the relation S.



The map $f_1 : K \xrightarrow{onto} K/S$ is simple and $f = f_3 \circ f_2 \circ f_1$. Thus, our proof is finished.

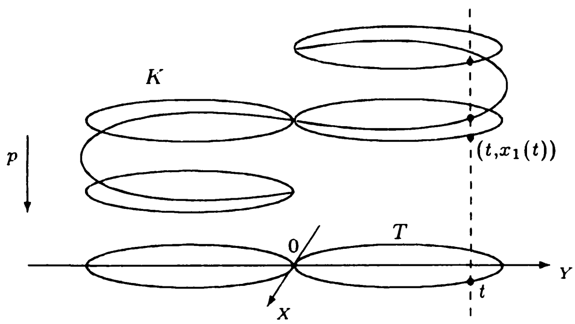
Example. Let

$$T_1 := \{(\sin\varphi, -\cos\varphi + 1) \in \boldsymbol{R}^2 : \varphi \in [0; 2\pi]\},$$
$$K_1 := T_1 \times \{0\} \subset \boldsymbol{R}^3, \qquad K_2 := T_1 \times \{2\pi\},$$
$$K_3 := \{(\sin\varphi, -\cos\varphi + 1, \varphi) \in \boldsymbol{R}^3 : \varphi \in [0; 2\pi]\},$$
$$T := T_1 \cup (-T_1),$$
$$K := K_1 \cup K_2 \cup K_3 \cup (-K_1) \cup (-K_2) \cup (-K_3).$$

We have $K \subset T \times \boldsymbol{R}$, the projection $p : K \xrightarrow{onto} T$ is a three-to-one map, i.e. $p^{-1}(t)$ consists of exactly 3 points for every $t \in T$.

Suppose, that $p = f_2 \circ f_1$, *where* f_2 *is a simple map,* f_1 *is continuous. We shall prove that* f_1 *cannot be simple.* Accept notations like in Theorem 1 i.e.: $K_t = \{x_1(t) < x_2(t) < x_3(t)\}$ for $t \in T$. Since f_2 is a map of order ≤ 2 we obtain the following property:

$$(*) \qquad \bigwedge_{t \in T} \bigvee_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} f_1[(t, x_i(t))] = f_1[(t, x_j(t))].$$



Suppose that f_1 is simple and denote $\mathbf{0} = (0,0) \in T$. We shall show that $f_1[(\mathbf{0}, x_2(\mathbf{0}))] = f_1[(\mathbf{0}, x_3(\mathbf{0}))]$. Indeed, let

$$I := T_1 \setminus \{\mathbf{0}\},$$
$$F_1 := \{t \in I : f_1[(t, x_1(t))] = f_1[(t, x_2(t))]\},$$
$$F_2 := \{t \in I : f_1[(t, x_2(t))] = f_1[(t, x_3(t))]\},$$
$$F_3 := \{t \in I : f_1[(t, x_3(t))] = f_1[(t, x_1(t))]\},$$

Since functions $x_l | I : I \to \boldsymbol{R}$, $l = 1, 2, 3$ are continuous, the sets F_l are closed in I. They are pairwise disjoint because of f_1's simplicity. On the other hand the property $(*)$ implies that they cover I. Since I is a connected space, $F_l = I$ for some $l \in \{1, 2, 3\}$. By continuity of f_1 we obtain $f_1[(\mathbf{0}, x_2(\mathbf{0}))] = f_1[(\mathbf{0}, x_3(\mathbf{0})]$.

The point $(\mathbf{0}, 0) \in \boldsymbol{R}^3$ is the symmetry center of K. Then we can repeat the above argumentation to obtain that $f_1[(\mathbf{0}, x_1(\mathbf{0}))] = f_1[(\mathbf{0}, x_2(\mathbf{0}))]$, which implies that f_1 is not a simple map.

Therefore *p does not decompose into two simple maps.*

Theorem 2. *Let K be a compact subset of the Cartesian product $T \times \boldsymbol{R}^n$ of a metric space T by $\boldsymbol{R}^n$. If the projection $P: K \to T$ is a map of order $\leq k$ $(k \geq 3)$, then P is a superposition of $3n$ continuous maps of order $\leq k - 1$.*

Proof. In the case $n = 1$ this was proved in Theorem 1. Assume that the theorem is proved for $n, n \geq 1$. Consider a compact subset $K \subset T \times \boldsymbol{R}^{n+1}$ such that the projection $P: K \to T$ is a map of order $\leq k$. We have $K \subset (T \times \boldsymbol{R}^n) \times \boldsymbol{R}$ so denote by $p: K \to T \times \boldsymbol{R}^n$ the projection $p(t, x, y) = (t, x)$, by $P_0: p(K) \to T$ — the projection $P_0(t, x) = t$.

$$\begin{array}{ccc} T \times \boldsymbol{R}^n \subset & p(K) & \\ & {\scriptstyle p}\uparrow & \searrow^{P_0} \\ (T \times \boldsymbol{R}^n) \times \boldsymbol{R} \subset & K & \xrightarrow{P} T \end{array}$$

We have $P = P_0 \circ p$, p and P_0 are maps of order $\leq k$ (If not, P would not be a map of order $\leq k$), $p(K) \subset T \times \boldsymbol{R}^n$ is compact. We may apply Theorem 1 to K, p and the inductive assumption to $p(K)$, P_0. Thus we obtain P as a superposition of $3 + 3n$ maps of order $\leq k$. The proof is finished.

Corollary 1. *Let $x \subset \boldsymbol{R}^n$ be a compact subset and Y be a metric space. If $f: X \to Y$ is a continuous map of order $\leq k$ $(k \geq 3)$, then it is a superposition of $3n$ maps of order $\leq k - 1$.*

Proof. Let $f: X \to Y$ be a map of order $\leq k$, let $\varphi: X \to Y \times \boldsymbol{R}^n$ be defined as follows:

$$\varphi(x) := (f(x), x) \qquad \text{for} \qquad x \in K\,.$$

φ is an embeding X into $Y \times \boldsymbol{R}^n$. We have $f = P \circ \varphi$, where $P: \varphi(X) \to Y$ is such a projection as in Theorem 2. This theorem implies that $P = f_{3n} \circ \ldots \circ f_2 \circ f_1$, where $f_i: X_{i-1} \to X_i$ $(i = 1, 2, \ldots, 3n)$ are continuous maps of order $\leq k - 1$, while $X_0 = \varphi(X), X_1, \ldots, X_{3n-1}, X_{3n} = Y$ are metric spaces. Then $f = f_{3n} \circ \ldots \circ f_2 \circ (f_1 \circ \varphi)$. The proof is complete.

Using Corollary 1 and the Menger-Nöbeling Theorem on embeding an n-dimensional compact metric space in Euclidean space $\boldsymbol{R}^{2n+1}$ (see for example [5], p. 116) we obtain.

Corollary 2. *Let X be an n-dimensional compact metric space and Y be a metric space. If $f: X \to Y$ is a map of order $\leq k$ $(k \geq 3)$, then it is a superposition of $3(2n + 1)$ maps of order $\leq k - 1$.*

Theorem 3 (Sieklucki). *Let X be a finite dimensional compact metric space and Y be a metric space. If $f: X \to Y$ is a continuous map of order $\leq k$, then it is a superposition of finite number of simple maps.*

Proof. We apply induction with respect to k. The theorem is obvious for maps of order ≤ 2. Assume that the theorem is established for maps of order $\leq k$, $k \geq 2$. Let $f: X \to Y$ be a continuous map of order $\leq k + 1$. It follows from Corollary 2 that $f = f_m \circ \ldots \circ f_2 \circ f_1$, where $m \in N$, $f_i: X_{i-1} \to X_i$ are continuous maps of order $\leq k$ and $X_0 = X, X_1, \ldots, X_{m-1}, X_m = Y$ are metric spaces. We can assume that all f_i, $i = 1, 2, \ldots, m - 1$ are onto X_i. Then for each $i = 1, 2, \ldots, m$ the superposition $(f_{i-1} \circ f_{i-2} \circ \ldots \circ f_1): X \xrightarrow{onto} X_{i-1}$ is a map of order $\leq k$. Thus every X_{i-1} is a finite dimensional compact metric space. Therefore by the inductive assumption each f_i is a superposition of finite number of simple maps. Hence the inductive conclusion is obvious and the theorem is proved.

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