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Multiplication Groups of Quasigroups and Loops III

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Groups possessing connected transversals to Abelian subgroups are studied in more detail.

Podrobněji se studují grupy se spojenými transversálami k abelovským podgrupám.

1. Preliminaries

1.1 Lemma. *Let H be a non-normal finitely generated subgroup of a finitely generated group G. Then they exists a subgroup N of G maximal with respect to $N \trianglelefteq G$ and $NH \ntrianglelefteq G$.*

Proof. Let $\mathfrak{M}$ be the set of subgroups M such that $M \trianglelefteq G$ and $MH \ntrianglelefteq G$. Then the unit subgroup is in $\mathfrak{M}$ and $\mathfrak{M}$ is ordered by inclusion. Moreover, if M_i, $i \in I$, is a chain of subgroups from $\mathfrak{M}$, then certainly $M = \bigcup_I M_i \trianglelefteq G$. Now, suppose for a moment that $MH \trianglelefteq G$. If A and B are finite subsets of G such that $H = \langle A \rangle$ and $G = \langle B \rangle$, then $b^{-1}ab \in MH$ for all $a \in A$ and $b \in B \cup B^{-1}$. Consequently there is $i \in I$ such that all these elements $b^{-1}ab$ are in M_i. But then $M_iH \trianglelefteq G$, a contradiction.

We have proved that the ordered set $\mathfrak{M}$ is upwords-inductive.

1.2 Remark. *Consider the situation from 1.1 and put $\overline{G} = G/N$. Then $\overline{H} = HN/N$ is not normal in $\overline{G}$ and, if $\overline{M} \neq 1$ is normal subgroup of $\overline{G}$, then $\overline{M} . \overline{H} \trianglelefteq \overline{G}$.*

1.3 Lemma. *Let H be a subgroup of group G such that H is nilpotent and $\mathbb{N}_G(K) \subseteq H$ for every subgroup $K \neq 1$ of H. Then $H \cap H^x = 1$ for every $x \in G \setminus H$.*

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Proof. Suppose that $L = H \cap H^x \neq 1$ for some $x \in G$. Now, $\mathbb{N}_G(L) \subseteq H \cap H^x = L$ and since H and H^x are nilpotent, we conclude that $L = H = H^x$. Then $x \in \mathbb{N}_G(H) = H$.

1.4 Lemma. *Let A, B, C be subsets of a group G such that $G = BC$, C is finite $(\mathrm{card}(C) = n \geqq 1)$ and $[a, b] \in C$ for all $a \in A$ and $b \in B$. Then $[G : \mathbb{C}_G(a)] \leqq n^2$ for every $a \in A$.*

Proof. Let $a \in A$. For $c \in C$, put $B_c = \{b \in B, [a, b] = c\}$. Now, $B = \bigcup_{c \in C} B_c$ and this union is disjoint. If $b_1, b_2 \in B_c$, then $b_1^{-1}ab_1 = b_2^{-1}ab_2$ i.e., $b_2b_1^{-1} \in \mathbb{C}_G(a)$ and $\mathbb{C}_G(a)b_1 = \mathbb{C}_G(a)b_2$.

For every $c \in C$ such that $B_c \neq \emptyset$, choose $b_c \in B_c$ and put $D = \{b_c; c \in C\}$. Ten $G = \mathbb{C}_G(a)DC$. Indeed, if $x \in G$, then $x = bd$ for some $b \in B$ and $d \in C$. Of course, $b \in B_c$ for some $c \in C$ and we have $x(b_cd)^{-1} = bdd^{-1}b_c = bb_c^{-1} \in \mathbb{C}_G(a)$. Clearly, $\mathrm{card}(DC) \leqq n^2$.

1.5 Lemma. *Let H be a finite subgroup of a group G and let A, B be H-connected pseudotransversals to H in G such that $G = \langle A, B \rangle$. Then the index $[G : \mathbb{C}_G(H)]$ is finite. Moreover, if $\mathbb{L}_G(H) = 1$, then the index $[G : \mathbb{Z}(G)]$ is also finite.*

Proof. Since H is finite, $H \subseteq \langle C \rangle$ for a finite subset $C \subseteq A \cup B$. Now, put $K = \bigcap_{c \in C} \mathbb{C}_G(c)$. By 1.4, each of the subgroups $\mathbb{C}_G(c)$ is of finite index in G. Consequently, the index $[G : K]$ is also finite. Clearly, $K \subseteq \mathbb{C}_G(H)$.

Now, assume that $\mathbb{L}_G(H) = 1$. By the preceding observation and [1,3.18], $\mathbb{N}_G(H) = H\mathbb{Z}(G)$ is of finite index in G. But H is finite, $H \cap \mathbb{Z}(G) = 1$ and $[G : \mathbb{Z}(G)] = [G : H\mathbb{Z}(G)][H\mathbb{Z}(G) : \mathbb{Z}(G)] = [G : \mathbb{N}_G(H)]\,\mathrm{card}(H)$ is finite.

1.6 Remark. *Let H be a finite subgroup of a group G such that there exist H-connected pseudotransversals A, B to H in G.*

(i) If $G = \langle A,B \rangle$, then the index $[G : K]$ is finite, where $K/\mathbb{L}_G(H) = \mathbb{Z}(G/\mathbb{L}_G(H))$.

(ii) Put $G_1 = \langle A,B \rangle$ and $H_1 = H \cap G_1$. Then A, B are H_1-connected pseudotransversals to H_1 in G_1 and by 1.5, the index $[G_1 : \mathbb{C}_{G_1}(H_1)]$ is finite. On the other hand, since H is finite also the index $[G : G_1]$ is finite. Consequently $[G : \mathbb{C}_{G_1}(H_1)]$ is finite. In particular, there is a normal subgroup $K \trianglelefteq G$ such that $K \subseteq \mathbb{C}_{G_1}(H_1)$ and $[G : K]$ is finite.

1.7 Lemma. *Let H be an Abelian subgroup of a group G such that $\mathbb{N}_G(H) = H$. If $x \in G$ and $\mathbb{N}_G(T) \subseteq H$, where $T = H \cap H^x$, then $x \in H$ and $T = H$.*

Proof. We have $H \cup H^x \subseteq \mathbb{N}_G(T) \subseteq H$. Hence $H^x \subseteq H$ and $T = H^x$. Further, $H \subseteq H^{x^{-1}}$ is an Abelian group, $H^{x^{-1}} \subseteq \mathbb{N}_G(H) = H$ and $H \subseteq H^x$. Thus $H = H^x = T$ and $x \in \mathbb{N}_G(H) = H$.

1.8 Lemma. *Let $H \subseteq R \subseteq G$ be subgroups of a group G such that H is Abelian, $H \ntrianglelefteq R$ and $R \subseteq K$, whenever K is a subgroup of G properly containing H. Then:*

(i) $\mathbb{N}_G(H) = H$ and $\mathbb{Z}(R) \subseteq H$.

(ii) If T is a subgroup of H such that $\mathbb{N}_G(T) \nsubseteq H$, then $T \subseteq \mathbb{Z}(R)$.

(iii) $\mathbb{L}_G(H) \subseteq \mathbb{Z}(R) \subseteq \mathbb{L}_R(H)$.

Proof. (i) If $\mathbb{N}_G(H) \neq H$, then $H \subseteq R \subseteq \mathbb{N}_G(H)$ and $H \trianglelefteq R$, a contradiction.

(ii) We have $H \subseteq \mathbb{C}_G(T) \trianglelefteq \mathbb{N}_G(T)$. Since $\mathbb{N}_G(H) = H$ and $\mathbb{N}_G(T) \neq H$, H is not normal in $\mathbb{N}_G(T)$, and hence $H \neq \mathbb{C}_G(T)$. Consequently $R \subseteq \mathbb{C}_G(T)$ and $T \subseteq \mathbb{Z}(R)$.

(iii) By (i) and (ii), $\mathbb{L}_G(H) \subseteq \mathbb{Z}(R) \subseteq H$. Clearly $\mathbb{Z}(R) \subseteq \mathbb{L}_R(H)$. On the other hand, $\mathbb{N}_R(\mathbb{L}_R(H)) = R \nsubseteq H$, and so $\mathbb{L}_R(H) \subseteq \mathbb{Z}(R)$ again by (ii) (where we take $G = R$).

2. Connected transversals to cyclic subgroups

2.1 In this section, let H be a cyclic subgroup of a group G and let A, B be H-connected pseudotransversals to H in G.

2.2 Theorem. *Suppose that H is a cyclic p-group for a prime p and that $\mathbb{L}_G(H) = 1$. Then $A = B$ is an Abelian subgroup of G.*

Proof. First, A and B are transversals [1, 3.9]. Now, for every $a \in A$ there exists a (uniquely determined) $f(a) \in B$ with $f(a)^{-1} a \in H$ and we have $f(a)^{-1} a^2 H = f(a)^{-1} a f(a) H = f(a)^{-1} a f(a) f(a)^{-1} a^{-1} f(a) a H = aH$ (since $f(a)^{-1} a^{-1} f(a) a \in H$).

Let $d \in A$ and $c = f(d)^{-1} d \in H$. Further, let $b \in A$ and let K denote the subgroup $\langle c, f(b)^{-1} b \rangle$. Then $K \subseteq H$, K is a cyclic p-group, and hence either $K = \langle c \rangle$ or $K = \langle f(b)^{-1} b \rangle$. In the latter case, $cbH = (f(b)^{-1} b)^n bH = (f(b)^{-1} b)^{n-1} f(b)^{-1} b^2 H = (f(b)^{-1} b)^{n-1} bH = \dots = bH$ (see the above observation) for some $n \geq 1$ and we have $b^{-1} cb \in H$.

Now, assume that $K = \langle c \rangle$. Then $f(b)^{-1} bdH = c^n dH = dH = f(d) H$. Consequently, $d^{-1} f(b)^{-1} bd \in H$ and thus $b^{-1} d^{-1} bd = b^{-1} f(b) . f(b)^{-1} d^{-1} f(b) d . d^{-1} f(b)^{-1} bd \in H$. Moreover, $f(d)^{-1} f(b)^{-1} bd \in H$, and hence $f(b)^{-1} f(d)^{-1} bd = f(b)^{-1} b . b^{-1} f(d)^{-1} b f(d) f(d)^{-1} f(b)^{-1} bd . d^{-1} b^{-1} f(b) d \in H$. Finally $f(b)^{-1} cb = f(b)^{-1} f(d)^{-1} db = f(b)^{-1} f(d)^{-1} bd . d^{-1} b^{-1} db \in H$, and therefore $b^{-1} cb = b^{-1} f(b) . f(b)^{-1} cb \in H$.

We have shown that (in both cases) $b^{-1} cb \in H$ for every $b \in A$. Since $G = AH$, we conclude that $c \in \mathbb{L}_G(H) = 1$. Thus $c = 1$ and $f(d) = d$. But this means that $A = B$. Now A is an H-selfconnected transversal and, in particular, $abH = baH$ for all $a, b \in A$.

In the remaining part of the proof, we are going to show that A is a subgroup of G. For every ordered pair $(a, b) \in A \times A$ there exists a (unique)

$g(a,b) \in A$ such that $h(a,b) = g(a,b)^{-1}ab \in H$. Now, $g(a,b) = g(b,a)$ and $h(a,b)b^{-1}a^{-1}ba = h(b,a)$. Moreover, $h(a,b)aH = g(a,b)^{-1}abaH = g(a,b)^{-1}a^2bH = g(a,b)^{-1}a\,g(a,b)H = aH$ (since $a^{-1}g(a,b)^{-1}ag(a,b) \in H$) and $a^{-1}h(a,b)a \in H$.

Let $a, b, c \in A$ and $R = \langle h(a,b), h(c,b), h(c,c)\rangle$. Again, either $R = \langle h(a,b)\rangle$ or $R = \langle h(c,b)\rangle$ or $R = \langle h(c,c)\rangle$. In the latter two cases we have $c^{-1}h(a,b)c \in H$, since $c^{-1}h(c,b)c$, $c^{-1}h(c,c)c \in H$ (see the above observation).

Next assume that $R = \langle h(a,b)\rangle$. Then $u = a^{-1}h(c,b)a \in H$, $v = g(c,b)^{-1}a^{-1}cba = g(c,b)^{-1}a^{-1}g(c,b)a.u \in H$, $w = b^{-1}c^{-1}a^{-1}cba = h(c,b)^{-1}v \in H$, $z = c^{-1}b^{-1}a^{-1}cba = c^{-1}b^{-1}cbw \in H$, $r = c^{-1}b^{-1}a^{-1}cg(a,b) = za^{-1}b^{-1}g(a,b) = zh(b,a)^{-1} \in H$, $s = c^{-1}h(a,b)^{-1}c = rg(a,b)^{-1}c^{-1}g(a,b)c \in H$ and, finally, $t = s^{-1} = c^{-1}h(a,b)c \in H$.

We have shown that $c^{-1}h(a,b)c \in H$ for every $c \in A$ and it follows that $h(a,b) \in \mathbb{L}_G(H) = 1$ and $g(a,b)^{-1}ab = h(a,b) = 1$, i.e., $ab = g(a,b) \in A$. We conclude that A is a subsemigroup of G. On the other hand, if $a \in A$, then $b^{-1}a^{-1} \in H$ for some $b \in A$ and we have $ab \in H \cap A = 1$ [1, 3.12(i)] and $a^{-1} = b \in A$. This shows that A is a subgroup of G. Since $a^{-1}b^{-1}ab \in H \cap A$ for all $a, b \in A$, we get $ab = ba$. The proof is complete.

2.3 Corollary. *Suppose that H is a cyclic p-group. Then $A\mathbb{L}_G(H) = B\mathbb{L}_G(H) = K$ is a subgroup of G and $K/\mathbb{L}_G(H)$ is an Abelian group (and consequently $[A,B] \subseteq \mathbb{L}_G(H)$).*

2.4 Remark. *The preceding results remain true for H being the Prüfer quasicyclic p-group (the same proof).*

2.5 Lemma. *Suppose that $\mathbb{L}_G(H) = 1$, $[A,B] = 1$ and $G = \langle A,B\rangle$. Then $G = A = B$ is an Abelian group.*

Proof. Clearly, $C = \langle A\rangle \cap H$ is a normal in G, hence $C = 1$ and $\langle A\rangle = A$ is a normal subgroup of G. Quite similarly, $\langle B\rangle = B$ is normal in G. And we have $G = AB$. In particular, $H = \langle ab\rangle$ for some $a \in A$, $b \in B$ and $G = AH = A\langle b\rangle$. If $c \in B$, then $c = db^n$, $d \in A$, and since $d \in A \cap B$ it follows that $d \in \mathbb{Z}(G)$. We conclude that B is a Abelian, $H = 1$ and $G = A = B$.

2.6 Colorally. *Suppose that $[A,B] \subseteq \mathbb{L}_G(H)$ and $G = \langle A,B\rangle$. Then $G' \subseteq H$ and $H \trianglelefteq G$.*

2.7 Theorem. *If $G = \langle A,B\rangle$, then $G' \subseteq H$ and $H \trianglelefteq G$.*

Proof. Clearly, $G' \subseteq H$ iff $H \trianglelefteq G$. Now assume that $H \ntrianglelefteq G$ and let $H = \langle u\rangle$. Then $G' \nsubseteq H$ and there are $x, y \in G$ such that $[x,y] \notin H$. Further, since $G = \langle A,B\rangle$, there is a finite subset E of $A \cup B$ such that $x, y, u \in G_1 = \langle E\rangle$. Now, put $C = G_1 \cap A$ and $D = G_1 \cap B$. Then C, D are H-connected transversals to H in G_1 and $G_1 = \langle C,D\rangle$. Moreover $G_1' \nsubseteq H$. From now on we assume that G is finitely generated. With respect to 1.1 and 1.2, we can also assume that $MH \trianglelefteq G$ whenever $1 \neq M \trianglelefteq G$. The rest of the proof is divided into four parts:

(i) If $\mathbb{L}_G(H) \neq 1$, then $H = \mathbb{L}_G(H) \trianglelefteq G$, a contradiction. Thus $\mathbb{L}_G(H) = 1$.

(ii) Next we show that $\mathbb{Z}(G) = 1$. Assume this be not true and put $L = \cap\langle H, z\rangle$, where z goes trough non-identical elements of $\mathbb{Z}(G)$. We have $\langle H, z\rangle = H\langle z\rangle \trianglelefteq G$, and henceforth $L \trianglelefteq G$. Put also $V = \cap\langle z\rangle$, so that $V \trianglelefteq G$. Since $H \cap \mathbb{Z}(G) = 1$, we have $L = VH$ and, since $L \neq H$, it follows that $V \neq 1$. We conclude that $\mathbb{Z}(G)$ is either a cyclic p-group or quasicyclic p-group and, anyway, V is a (cyclic) group of prime order p. Moreover, $L = L^x = H^xV$ and $H^x \cap V = 1$ for every $x \in G$. Now, we see that $L = H^x \times V$.

Let T be a left transversal to L in G with $1 \in T$. Put $f = \prod f_t$, $t \in T$, where $f_t : L = H^t \times V \to V$ is the natural projection, so that f is a homomorphism of L into the cartesian product W of card$(T) = [G : L]$ copies of V. If $v \in \operatorname{Ker}(f)$, then $v \in H^t$ for every $t \in T$ and thus $v \in \mathbb{L}_G(H) = 1$. It follows that f is injective and then L (and, in particular, H) can be imbedded into L. But W is an Abelian elementary p-group, the same is true for H and H is a (cyclic) group of order p. Now, by 2.2, $A = B = \langle A, B\rangle = G$ is an Abelian group and $G' = 1 \subseteq H$, which is not true. We have proved that $\mathbb{Z}(G) = 1$. By [1, 3.18], $\mathbb{N}_G(H) = H$.

(iii) Let K be a subgroup of H such that $K \neq 1$ and $H \neq \mathbb{N}_G(K)$. Let $a \in (A \cap \mathbb{N}_G(K))\setminus H$ and $T = \langle H, a\rangle$. We have $\mathbb{N}_T(H) \subseteq \mathbb{N}_G(H) = H$, so that $\mathbb{N}_T(H) = H$. Now, $H \subseteq \mathbb{C}_T(K) \trianglelefteq \mathbb{N}_T(K) \neq H$ and consequently $H \neq \mathbb{C}_T(K)$. Let $b \in (A \cap \mathbb{C}_T(K))\setminus H$ and $S = \langle H, b\rangle$. By [1, 3.11(i)], $b \in \mathbb{L}_G(S) \neq 1$ and hence $S = \mathbb{L}_G(S)\, H \trianglelefteq G$. Further, $\mathbb{C}_S(K) = S$, and so $K \subseteq \mathbb{Z}(S)$. Clearly, $\mathbb{Z}(S) \subseteq H$ (since $\mathbb{N}_G(H) = H$) and since $S \trianglelefteq G$, we have also $\mathbb{Z}(S) \trianglelefteq G$. Thus $\mathbb{Z}(S) \subseteq \mathbb{L}_G(H) = 1$ and $K = 1$, a contradiction.

We have proved that $\mathbb{N}_G(K) = H$ for every non-trivial subgroup K of H.

(iv) By (iii) and 1.3, we have $H \cap H^x = 1$ for every $x \in G\setminus H$. Now, by [1, 3.20], $A = B = \langle A, B\rangle = G$ is an Abelian group, and hence $G' = 1$, which is the final contradiction.

2.8 Theorem. *Suppose that $\mathbb{L}_G(H) = 1$. Then $G'' = 1$ and $A = B$ is an Abelian subgroup of G.*

Proof. Put $K = \langle A, B\rangle$ and $E = K \cap H$. Now, A and B are E-connected transversals to E in K and $\mathbb{L}_K(E) = 1$. By 2.7, $K' \subseteq E$, and hence $K' = 1$. Then K is an Abelian group, $E = 1$, $A = B = K$ and $G'' = 1$ by [3].

2.9 Corollary. *$G'' \subseteq \mathbb{L}_G(H)$, $G''' = 1$, $K = A\mathbb{L}_G(H) = B\mathbb{L}_G(H)$ is a subgroup of G and $K' \subseteq \mathbb{L}_G(H)$ (and consequently $[A, B] \subseteq \mathbb{L}_G(H)$).*

3. Products of Abelian groups

3.1 In this section, let G be a group such that $G = KH$, where both K and H are Abelian subgroups of G, $H \neq G$, $K \neq 1$ and $K \trianglelefteq G$.

The following four lemmas are obvious:

3.2 Lemma.

(i) $H \cap K \subseteq H \cap \mathbb{C}_G(K) = H \cap \mathbb{Z}(G) \subseteq \mathbb{L}_G(H)$.

(ii) $\mathbb{Z}(G) = (K \cap \mathbb{Z}(G))(H \cap \mathbb{Z}(G))$.

(iii) If $\mathbb{L}_G(H) = 1$, *then* $H \cap K = 1 = H \cap \mathbb{C}_G(K)$ *and* $\mathbb{Z}(G) \subseteq K$.

(iv) If $\mathbb{Z}(G) = 1$ *then* $H \cap K = 1 = H \cap \mathbb{C}_G(K)$.

(v) If $H \cap K = 1$, *then* $\mathbb{L}_G(H) = H \cap \mathbb{C}_G(K) = H \cap \mathbb{Z}(G)$.

3.3 Lemma.

(i) If E is a subgroup of G such that $H \subseteq E \subseteq G$, *then* $E = (E \cap K)H$ *and* $E \cap K \trianglelefteq G$.

(ii) If no non-trivial proper subgroup of K is normal in G, then $H \cap K = 1$ *and H is maximal in G.*

3.4 Lemma. *Suppose that H is a maximal subgroup of G.*

(i) If L is a subgroup of K and $L \trianglelefteq G$, *then either* $L \subseteq H \cap K$ *or* $K = (H \cap K)L$.

(ii) If $H \cap K = 1$, *then no non-trivial proper subgroup of K is normal in G.*

(iii) If $H \ntrianglelefteq G$, *then* $\mathbb{Z}(G) \subseteq \mathbb{L}_G(H)$.

3.5 Lemma. *The following conditions are equivalent:*

(i) H is maximal in G and $H \cap K = 1$.

(ii) No non-trivial proper subgroup of K is normal in G.

3.6 In the remaining part of this section, we shall assume that the equivalent conditions of 3.5 are satisfied. Now, by 3.2(v), $\mathbb{L}_G(H) = H \cap \mathbb{C}_G(K) = H \cap \mathbb{Z}(G)$.

If $H \ntrianglelefteq G$, then $\mathbb{Z}(G) \subseteq H$ and $\mathbb{L}_G(H) = \mathbb{Z}(G)$.

If $H \trianglelefteq G$, then $G \cong K \times H$ is Abelian and K is (cyclic) of prime order.

For every $u \in H$, the map $q_u : a \to a^u = u^{-1}au$ is an automorphism of K and we denote by F the subring generated by all these automorphisms q_u in the endomorphism ring of K. Moreover, we put $q = -1_F \in F$; then $q(a) = a^{-1}$ for every $a \in K$ and $q^2 = 1_F$ $(= id_K)$.

3.7 Lemma.

(i) F is a field and K, as a vector space over F, is of dimension 1. In particular, the groups K and $F(+)$ *are isomorphic.*

(ii) If H is finitely generated, then both F and K are finite. If, moreover, $\mathbb{L}_G(H) = 1$, *then H is finite cyclic and G is finite.*

Proof. (i) Since H is abelian, F is a commutative ring. If $f \in F$, $f \neq 0_F$, then both $f(K)$ and $\mathrm{Ker}(f)$ are subgroups of K, they are normal in G, and hence $f(K) = K$ and $\mathrm{Ker}(f) = 1$, i.e., f is an automorphism of K.

Now, let $a \in K$, $a \neq 1$. Then $F(a)$ is a subgroup of K (use the fact that $q \in F$) and $F(a) \trianglelefteq G$. Since $a \in F(a)$, we have $F(a) = K$. If $f \in F$, $f \neq 0_F$, then $f^{-1}(a) = g(a)$ for some $g \in F$, $a = fg(a)$ and the equality $F(a) = K$ implies $fg = id_K = 1$. Consequently, $f^{-1} = g \in F$.

We have proved that F is a field.

(ii) Any field, finitely generated as a ring, is finite. Now, if $\mathbb{L}_G(H) = 1$, then the mapping $u \to q_u^{-1}$ is an injective homomorphism of H into the multiplicative group F^* of non-zero elements from F; the group F^* is cyclic.

3.8 Let A be a left pseudotransversal to H in G such that $[A, A] = 1$.

3.9 Lemma.
(i) $A \subseteq K\mathbb{L}_G(H)$.
(ii) If $\mathbb{L}_G(H) = 1$, *then* $A = K$.

Proof. There is a uniquely determined subset S of $K \times H$ such that $A = \{au, (a, u) \in S\}$. Now, fix an element $r \in K$, $r \neq 1$. For every $a \in K$, there is a unique $p_a \in F$ with $a = p_a(r)$; we have $p_a \neq 0_F$ iff $a \neq 1$.

Assume that $b \neq 1$ and $u \notin L = \mathbb{L}_G(H)$ for some $(b, u) \in S$ and put $p = (q + q_u^{-1})\, p_b^{-1} \in F$. Since $u \notin L = H \cap \mathbb{C}_G(K)$, we have $u \notin \mathbb{C}_G(K)$ and $q + q_u^{-1} \neq 0_F$. Thus $p \neq 0_F$ and $e^{-1} = p^{-1}(r)$ for some $e \in K$, $e \neq 1$. Now, $p_e(r) = e = (p^{-1}(r))^{-1} = p^{-1}(r^{-1}) = p^{-1}q(r)$, and so $p_e = p^{-1}q$ and $p_e^{-1} = q^{-1}p = qp$. On the other hand, $G = AH$, and hence $(e, v) \in S$ for some $v \in H$. The equality $[A, A] = 1$ implies $bueu^{-1}uv = buev = evbu = evbv^{-1}uv$ and $bueu^{-1} = evbv^{-1}$. From this, $(q + q_v^{-1})\, p_b(r) = b^{-1}vbv^{-1} = e^{-1}ueu^{-1} = (q + q_u^{-1})\, p_e(r)$ and $(q + q_v^{-1})\, p_b = (q + q_u^{-1})\, p_e$, $p = (q + q_u^{-1})\, p_b^{-1} = (q + q_v^{-1})\, p_e^{-1} = (q + q_v^{-1})\, qp$, $1_F = (q + q_v^{-1})\, q = 1_F + q_v^{-1}q$ and $0_F = q_v^{-1}q$, a contradiction.

We have proved that $A \subseteq H \cup KL$. But, if $w \in A \cap H$ and $c \in K$, then $cz \in A$ for some $z \in H$ and $wcz = czw = cwz$, $wc = cw$ and $w \in L \subseteq KL$. Thus $A \subseteq KL$ and the rest is clear.

3.10 Lemma.
(i) $G' \subseteq K$.
(ii) If $H \ntrianglelefteq G$, *then* $G' = K$.
(iii) If $\mathbb{L}_G(H) = 1 \neq H$, *then* $A = G'$.

Proof. (i) $G/K \cong H$.

(ii) Since $H \ntrianglelefteq G$, we must have $G' \neq 1$. But $G' \trianglelefteq G$ and $G' \subseteq K$.

(iii) Combine (ii) and 3.9(ii).

4. Connected transversals to Abelian maximal subgroups

4.1 In this part, let H be a (proper) maximal subgroup of a group G such that H is Abelian and not normal in G. Further, let A, B be H-connected pseudotransversals to H in G. By 1.8(iii), $\mathbb{Z}(G) = \mathbb{L}_G(H) \subseteq H$.

4.2 Theorem. *Suppose that* $(\mathbb{Z}(G) =)\, \mathbb{L}_G(H) = 1$. *Then* $A = B = G'$ *is a normal Abelian subgroup of* G.

Proof. First, let $a \in A$. Then $b^{-1}a \in H$ for some $b \in B$, and hence $b^{-1}a \in H \cap Hb^{-1} = H \cap bHb^{-1} = T$. If $\mathbb{N}_G(T) \subseteq H$, then $b \in H$ by 1.7, and so $a \in A \cap H = 1$, $b \in B \cap H = 1$ and $a = b = 1$ ([1, 3.12(i)]). On the other hand, if $\mathbb{N}_G(T) \nsubseteq H$, then $\mathbb{N}_G(T) = G$ (since H is maximal in G), $T \trianglelefteq G$, $T \subseteq \mathbb{L}_G(H) = 1$ and, again, $a = b$. We have proved that $A \subseteq B$. Quite similarly, $B \subseteq A$, and so $A = B$.

Now, let $a, b \in A$. There is $c \in A$ such that $c^{-1}ab \in H$. We have $c^{-1}ba = c^{-1}abb^{-1}a^{-1}ba \in H$ and $a^{-1}c^{-1}aba = a^{-1}c^{-1}ac \,.\, c^{-1}ba \in H$. Consequently, $c^{-1}ab \in T = H \cap aHa^{-1}$. Again, if $\mathbb{N}_G(T) \subseteq H$, then $a \in H$ by 1.7, and hence $a = 1$ and $ab = b \in A$. If $\mathbb{N}_G(T) \nsubseteq H$, then $T = 1$ and $ab = c \in A$. We have proved that $AA \subseteq A$, i.e., A is a subsemigroup of G. Further, if $a \in A$, then $b^{-1}a^{-1} \in H$ for some $b \in A$, and then $ab \in H \cap A$, and $a^{-1} = b \in A$. Thus A is a subgroup of G. Since $[A, A] \subseteq A \cap H = 1$, A is an Abelian group.

Now, $G = AH$, and hence $G'' = 1$ by [3]. Since $H \ntrianglelefteq G$, we have $G' \nsubseteq H$ and then $G = HG'$. By 3.10(iii) $A = G'$.

4.3 Corollary. (cf. 2.3 and 2.9.) *$A\mathbb{Z}(G) = B\mathbb{Z}(G) = G'\mathbb{Z}(G) = K$ is a normal subgroup of G, $K' \subseteq \mathbb{Z}(G)$, $K/\mathbb{Z}(G)$ is an Abelian group, K is nilpotent of class at most 2, $\langle A, B\rangle \subseteq K \neq G$ and $[A, B] \subseteq \mathbb{Z}(G)$. Finally, $G'' \subseteq \mathbb{Z}(G)$ and $G''' = 1$.*

5. Connected transversals to finite Abelian subgroups

5.1 In this section, let H be a finite Abelian subgroup of a group G such that there exist H-connected pseudotransversals A, B to H in G.

5.2 Theorem. *The group G is soluble.*

Proof. (i) First, assume that G is finite and proceed by induction on card(G).

If $H = G$, then G is Abelian. Hence, let $H \neq G$ and let G_1 be a subgroup of G such that H is a (proper) maximal subgroup of G_1. It follows from [1, 3.11(i)] that $\mathbb{L}_G(G_1) \neq 1$ and then $G/\mathbb{L}_G(G_1)$ is soluble by induction. Now it remains to show that G_1 is soluble.

If $H \trianglelefteq G_1$, then $G_1' \subseteq H$ and $G_1'' = 1$. If $H \ntrianglelefteq G$, then $G_1''' = 1$ by 4.3.

(ii) Next, let $G = \langle A, B\rangle$. Since $\mathbb{L}_G(H)$ is a normal Abelian subgroup of G, we may assume that $\mathbb{L}_G(H) = 1$. Now, $\overline{G} = G/\mathbb{Z}(G)$ is a finite group by 1.5 and $\overline{G}$ is soluble by (i).

(iii) Finally, consider the general case. Let $G_1 = \langle A, B\rangle$ and $H_1 = H \cap G_1$. Then A, B are H_1-connected pseudotransversals to H_1 in G_1 and the subgroup G_1 is soluble by (ii). On the other hand, the index $[G : G_1]$ is finite and there is a subgroup K of G such that $K \trianglelefteq G$ and $[G : K]$ is finite. Consequently, K is soluble (since G_1 is so) and G/K is soluble (by (i)).

5.3 Remark. *Suppose that there exists a subgroup R of G such that $H \subseteq R$ and $H \ntrianglelefteq R$ and $R \subseteq K$, wherever K is a subgroup of G properly containing H.*

(i) We have $R \nsubseteq \mathbb{N}_G(H)$, *and hence* $\mathbb{N}_R(H) = \mathbb{N}_G(H) = H$. *Consequently,* $\mathbb{Z}(G) \subseteq H$ *and* $\mathbb{Z}(R) \subseteq H$.

(ii) If T *is a subgroup of* H *such that* $\mathbb{N}_G(T) \nsubseteq H$, *then* $T \subseteq \mathbb{Z}(R)$.

(iii) $\mathbb{L}_G(H) \subseteq \mathbb{Z}(R) = \mathbb{L}_R(H)$.

(iv) Let $a \in A$. *Then* $b^{-1}a \in T = H \cap bHb^{-1}$ *for some* $b \in B$ *(see the proof of 4.2) and either* $\mathbb{N}_G(T) \subseteq H$ *and* $b \in B \cap H \subseteq \mathbb{L}_G(H) \subseteq \mathbb{Z}(R)$, $a \in A \cap H \subseteq \mathbb{L}_G(H) \subseteq Z(R)$ *(by 1.7 and [1, 3.11(i)]) or* $\mathbb{N}_G(T) \nsubseteq H$ *and* $b^{-1}a \in \mathbb{Z}(R)$ *by (ii).*

We have proved that $A \subseteq B\mathbb{Z}(R)$. *Quite similarly,* $B \subseteq A\mathbb{Z}(R)$, *and hence* $A\mathbb{Z}(R) = B\mathbb{Z}(R) = E$. *Clearly,* $H \cap E = \mathbb{Z}(R)$.

(v) Let $a \in A$ *and* $b \in B$. *Then* $c^{-1}ab \in H$ *for some* $c \in B$ *and we have* $c^{-1}ab \in T = H \cap aHa^{-1}$ *(see the proof of 4.2). If* $\mathbb{N}_G(T) \subseteq H$, *then* $a \in A \cap H \subseteq \mathbb{L}_G(H) \subseteq Z(R)$, $b^{-1}ab \in \mathbb{L}_G(H)$ *and* $ab \in b\mathbb{L}_G(H) \subseteq b\mathbb{Z}(R) \subseteq E$. *If* $\mathbb{N}_G(T) \nsubseteq H$, *then* $c^{-1}ab \in \mathbb{Z}(R)$ *and, again* $ab \in E$.

We have proved that $AB \subseteq E$. *Quite similarly,* $BA \subseteq E$.

(vi) Let $a \in A$. *Then* $b^{-1}a^{-1} \in H$ *for some* $b \in H$, *and so* $ab \in AB \cap H \subseteq E \cap H = \mathbb{Z}(R)$. *Now,* $b^{-1}a^{-1} \in \mathbb{Z}(R)$ *and* $a^{-1} \in b\mathbb{Z}(R) \subseteq E$.

We have proved that $A^{-1} \subseteq E$. *Quite similarly* $B^{-1} \subseteq E$.

(vii) Let $a \in A$ *and* $b \in B$. *Then* $c^{-1}a^{-1}b \in H$ *for suitable* $c \in B$ *and* $c^{-1}a^{-1}baH = c^{-1}a^{-1}abH = c^{-1}aa^{-1}bH = c^{-1}acH = c^{-1}caH = aH$ *(since* $a^{-1}b^{-1}ab \in H$ *and* $c^{-1}a^{-1}ca \in H$*). Now* $a^{-1}c^{-1}a^{-1}ba \in H$ *and* $c^{-1}a^{-1}b \in T = H \cap aHa^{-1}$. *Proceeding similarly as in (v), we check that* $ab^{-1} \in E$. *Thus* $A^{-1}B \subseteq E$ *and, symetrically,* $B^{-1}A \subseteq E$.

(viii) $AE = AB\mathbb{Z}(R) \subseteq E\mathbb{Z}(R) = A\mathbb{Z}(R)\mathbb{Z}(R) = E$. *Similary* $BE \subseteq E$, $A^{-1}E \subseteq E$ *and* $B^{-1}E \subseteq E$, *and hence* $SS \subseteq E$ *and* $SE \subseteq E$ *where* $S = A \cup A^{-1} \cup B \cup B^{-1}$. *Further, by induction on* $n \geq 1$, $n -$ *times*$SS \dots S \subseteq E$. *On the other hand,* $\langle A, B \rangle = \bigcup_{n \geq 1} SS \dots S$, *so that* $\langle A, B \rangle \subseteq E$.

(ix) Since $H \ntrianglelefteq R$, *we have* $H \neq \mathbb{Z}(R)$ *and we take* $u \in H \setminus \mathbb{Z}(R)$. *If* $u \in E$, *then* $u = ar$ *for same* $a \in A$, $r \in \mathbb{Z}(R)$, *and then* $a \in A \cap H \subseteq \mathbb{Z}(R)$, $u = ar \in \mathbb{Z}(R)$, *a contradiction. Thus* $u \notin E$ *and, in particular,* $\langle A, B \rangle \subseteq E \neq G$.

5.4 Theorem. *If* $G = \langle A, B \rangle$, *then* H *is subnormal in* G.

Proof. (i) First, assume that G is finite and proceed by induction on card(G). Let K be a subgroup of G such that $H \subseteq K$ and $H \neq K$. Then $L = \mathbb{L}_G(K) \neq 1$ and $K = HL$ ([1, 3.11(i)]). Further, by induction, K is subnormal in G and, since G is finite, also the subgroup $R = \bigcap K$ is subnormal in G. If $H \ntrianglelefteq R$ then $G = \langle A, B \rangle \neq G$ by 5.3(ix), a contradiction. Thus $H \trianglelefteq R$ and H is subnormal in G.

(ii) Next, let G be (possibly) infinite. We can assume that $\mathbb{L}_G(H) = 1$. Now, by 1.5, the index $[G : \mathbb{Z}(G)]$ is finite, and so $H\mathbb{Z}(G)$ is subnormal in G by (i). However, $H \trianglelefteq H\mathbb{Z}(G)$.

5.5 Corollary. *If* $\mathbb{L}_G(H) = 1 \neq G = \langle A, B \rangle$, *then* $\mathbb{Z}(G) \neq 1$.

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