Tatiana Hoerová; Ivo Marek

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Stiemke Theorem As a Tool for Some Mathematical Models of Economics

TATIANA HOEROVÁ and IVO MAREK

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We present some generalizations of very classical results of solving linear inequalities connected with the names Farkas, Minkowski and Tucker. We show also that those classical results as well as their generalizations are direct consequences of another classical result – the theorem of Stiemke. Our generalization is two fold. First, we admit the appropriate linear maps to operate in infinite dimensional Hilbert spaces and second, the spaces under consideration are partially ordered by quite general closed normal cones. The latter situation seems to lead to a generalization even for the case of finite dimensional spaces and linear inequalities involving matrices with real numbers as their elements.

1 Introduction

The main aim of this contribution is to generalize the Stiemke Theorem [16] and to show some of its applications. We also want to emphasize its importance in mathematical modeling and analysis. The original Stiemke theorem has been unjustifiably overlooked in the context of the theory of linear inequalities represented by results of the founders of the theory such as Farkas, Minkowski,

Faculty of Economics and Business, Technical University Liberec, Hálkova 6, 461 17 Liberec 1, Czech Republic, e-mail: tatiana.hoerova@vslib.cz.

Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 166 29 Praha 6, Czech Republic, e-mail: marek@ms.mff.cuni.cz.

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Kuhn, Tucker et al. It has been shown by H. Nikaido in his monograph [14] that the classical results known as Farkas, Minkowski, Tucker theorems are direct consequences of the Stiemke theorem. We are going to show that the situation is very similar in the case of the generalized Stiemke theorem. Our generalization is two fold. First, our analysis is provided in infinite dimensional Hilbert spaces and, second, the order in the spaces under consideration is induced by general normal cones. Our intention is to apply the generalized theory to more general models of Economics than are the classical models of Leontief and von Neumann. We believe that our generalizations tracing the classical mathematical tools seem to be more adequate in applications than those presented on a very abstract level in [3]. This concerns in particular the generalized Tucker theorem 5.7 which seems to be difficult even to formulate it in the spirit of [3].

As accepted by the scientific community the classical Leontief and von Neumann models of expanding economies have influenced the development of mathematical modeling very substantially and still are in use in some well defined situations. A kind of weakness or even a drawback might be seen in the fact that the amount of goods, technologies, etc. involved in the models is globally finite and thus excludes a possible creation of new productions and technologies. In order to weaken this restriction we are to extend these models to more general ones and such that they will be free of the above restrictive consequences. This means their realizations will be based on generally infinite dimensional state spaces. Once we admit infinite dimensional spaces as state spaces we have, on the one hand, much more freedom in modeling. On the other hand, however, the freedom will be accompanied by a difficulty connected with the phenomenon of infinity. A further consequence of this phenomenon is a necessity to generalize the appropriate theorems needed for foundation the classical models mentioned above. We are going to demonstrate the above remarks on some details of the von Neumann model of an expanding economy as desribed in [14, p. 145].

The mathematical problem is to guarantee existence of a solution to the following system of inequalities for the unknowns x, p, α, β

$$(B - \alpha A)\,x \geq 0, \tag{1.1}$$
$$(B^T - \beta A^T)\,p \leq 0, \tag{1.2}$$
$$[(B - \alpha A)\,x, p] = 0, \tag{1.3}$$
$$[x, (B^T - \beta A^T)\,p] = 0, \tag{1.4}$$
$$x \geq 0,\; p \geq 0, \tag{1.5}$$

where A, B denote matrices whose elements are nonnegative real numbers $a_{jk} \geq 0$, $b_{jk} \geq 0$ and A^T, B^T denote the appropriate transpose matrices.

Actually, the generalized von Neumann model can be formulated using the same symbols as shown in (1.2)$-$(1.5) with an obvious difference in interpreting the matrices A, B as bounded linear operators, A^T, B^T as the adjoints A^*, B^*, and the natural order relation in $\mathscr{R}^N$ as a partial order in the appropriate state space induced by a suitable cone.

An existence proof presented in [14, pp. 145 – 146] is based on a sophisticated application of the Stiemke and Tucker theorems. Since our aim is to extend the Leontief and von Neumann models to more general ones possessing a possibility to handle an infinite amount of goods in an infinite time interval, we are approached with a new mathematical problem whose solution is required prior to solve the problem of mathematical economics: To generalize the theorems of Stiemke and Tucker and some other related results in such a manner they become efficient tools for proving the correctness of the extended Leontief and von Neumann models.

2 Definitions and notation

Let $\mathscr{E}$ and $\mathscr{F}$ be Hilbert spaces over the field of real numbers equipped by inner products $[.,.]_{\mathscr{E}}$ and $[.,.]_{\mathscr{F}}$ be respectively. We use the notation $[.,.]$ for $[.,.]_{\mathscr{E}}$. Let $\mathbf{B}(\mathscr{E}, \mathscr{F})$ denote the space of bounded linear operators mapping $\mathscr{E}$ into $\mathscr{F}$. This latter introduced space is assumed to be equipped by standard norm and thus, it is a Banach space. If $\mathscr{F} = \mathscr{E}$ then we denote $\mathbf{B}(\mathscr{E}, \mathscr{F})$ simply by $\mathbf{B}(\mathscr{E})$

Let $A \in \mathbf{B}(\mathscr{E}, \mathscr{F})$. Then there exists a uniquely determined operator denoted by $A^* \in \mathbf{B}(\mathscr{E}, \mathscr{F})$ and called *adjoint operator* (with respect to A). This operator is defined via relations

$$[Ax, v]_{\mathscr{F}} = [x, A^*v]_{\mathscr{E}}$$

valid for all $x \in \mathscr{E}$ and $v \in \mathscr{F}$.

We say shortly that a set in $\mathscr{E}$ is closed if it is norm-closed.

Let $\mathscr{K} \subset \mathscr{E}$ and let us consider the following axioms (i) – (vi), where

(i) $\mathscr{K} + \mathscr{K} \subset \mathscr{K}$,

(ii) $a\mathscr{K} \subset \mathscr{K}$ for $a \in \mathscr{R}_+$,

(iii) $\mathscr{K} \cap (-\mathscr{K}) = \{0\}$,

A set satisfying conditions (i) – (iii) is called a *cone.*

(iv) $\bar{\mathscr{K}} = \mathscr{K}$,

where $\bar{\mathscr{K}}$ denotes the norm-closure, shortly closure of $\mathscr{K}$,

(v) $\mathscr{E} = \mathscr{K} - \mathscr{K}$

and

(vi) there exists a $\delta > 0$ such that $\|x + y\| \geq \delta\|x\|$, whenever $x, y \in \mathscr{K}$.

If cone $\mathscr{K}$ obeys property (vi) it is called *normal.*

(vii) For every pair $x, y \in \mathscr{K}$ there exist $x \wedge y = \inf\{x,y\}$ and $x \vee y = \sup\{x,y\}$ as elements of $\mathscr{K}$.

A cone $\mathscr{K}$ satisfying condition (vii) is called a *lattice cone* and the partial order on $\mathscr{E}$ a *lattice order.* In the terminology of H. H. Schaefer [15] $\mathscr{E}$ is called a *Banach lattice*. Our theory is free of hypothesis (vii).

A set $\mathscr{C} \subset \mathscr{E}$ possessing properties (i) and (ii) is called a *wedge*.

Let

$$\mathscr{K}^* = \{x' \in \mathscr{E} : [x', x] \geq 0 \text{ for all } x \in \mathscr{K}\}$$

and

$$\mathscr{K}^d = \{x \in \mathscr{K} : [x', x]_{\mathscr{E}} > 0 \text{ for all } 0 \neq x' \in \mathscr{K}^*\}.$$

We call $\mathscr{K}^*$ the *adjoint cone* of $\mathscr{K}$ and $\mathscr{K}^d$ the *quasiinterior* of $\mathscr{K}$, respectively.

A set $\mathscr{H}^* \subset \mathscr{K}^*$ is called $\mathscr{K}$*-total* if the following relations hold

$$[x, y] \geq 0 \; \forall y \in \mathscr{H}^* \Rightarrow x \in \mathscr{K}.$$

It is easy to see that $\mathscr{K}^*$ is $\mathscr{K}$-total. Our wish of very practical worth is to have a $\mathscr{K}$-total set as possible small with respect to the set inclusion.

A cone $\mathscr{K} \subset \mathscr{E}$ is called *solid,* if its interior $Int\mathscr{K}$ is not empty. Here $x \in Int\mathscr{K}$ if and only if $x + y \in \mathscr{K}$, $y \in \mathscr{E}$, whenever there exists a $\delta > 0$, such that $\|y\|_{\mathscr{E}} < \delta$.

We let

$$x \leq y \text{ or equivalently } y \geq x \Leftrightarrow (y - x) \in \mathscr{K}$$

and

$$y > x \text{ or equivalently } x < y \Leftrightarrow (y - x) \in \mathscr{K}^d.$$

In the following analysis we assume that the quasiinterior $\mathscr{K}^d$ is nonemty.

An operator $T \in \mathbf{B}(\mathscr{E})$ is called *regular* ([15, p. 228]) if there exist linear operators $T_1 \in \mathbf{B}(\mathscr{E})$ and $T_2 \in \mathbf{B}(\mathscr{E})$, $T_j\mathscr{K} \subset \mathscr{K}, j = 1, 2$, such that $T = T_1 - T_2$.

3 Auxilliary results

In this section we collect various results that will be useful in the next sections.

The following result can be proven in the same manner as it is done for a less general case in [14, Theorem 2.6, p. 20].

3.1 Proposition. *Let X be a convex set in a Hilbert space $\mathscr{E}$. Then*

1^0 *The norm closure $\bar{X}$ is also convex.*

2^0 *If $a \in X$ is an interior element and $b \in X$, then every element x of the segment $[a, b] = \{x \in \mathscr{E} : (1 - t)\, a + tb,\ t \in [0, 1]\}$, except possibly b is interior element with respect to X.*

3.2 Theorem. *Let $X \subset \mathscr{E}$ be a convex set. If element $w \notin \operatorname{Int} X$, where $\operatorname{Int} X$ denotes the norm interior of X, then there is an element $z \in \mathscr{E}$ such that*

$$X \subset \{y \in \mathscr{E} : [y - w, z] \geq 0\}.$$

3.3 Theorem. *Let $\emptyset \neq X \subset \mathscr{E}$ be a closed convex set. If $w \in \mathscr{E}$ is such that $w \notin X$, then the following assertions 1^0 and 2^0 hold, where*

1^0 *There exists an element $z \in \mathscr{E}$ and a real number $\beta \in \mathscr{R}^1$ such that*

$$X \subset \{u \in \mathscr{E} : [u, z] \geq \beta\}$$

and

$$w \in \{u \in \mathscr{E} : [u, z] < \beta\}.$$

2^0 *The element z mentioned above produces the hyperplane*

$$\{v \in \mathscr{E} : [v - w, z] = 0\}$$

such that

$$X \subset \{u \in \mathscr{E} : [u - w, z] > 0\}.$$

As in [14, pp. 28—31] one can show that each of the Theorems 3.2, 3.3 and 3.4 are equivalent to the others. It is enough to present a proof just of one of them. We have chosen the proof of Theorem 3.4.

3.4 Theorem. *Let $\mathscr{E}$ be a Hilbert space over the field of real numbers. Let $\emptyset \neq X \subset \mathscr{E}$ be an open convex set. If $a \in \mathscr{E}$ does not belong to X, then there exists an element $v \in \mathscr{E}$ such that*

$$X \subset \{x \in \mathscr{E} : [x - a, v] > 0\}.$$

Proof. We may assume that $0 \notin X$; otherwise we would translate X appropriately. Let us consider

$$\mathscr{M} = \bigcup_{0 < \lambda \in \mathscr{R}^1} \lambda X .$$

Here λX is open for each $0 < \lambda \in \mathscr{R}^1$. As a union of open sets $\mathscr{M}$ itself is open. Moreover. $\mathscr{M}$ possesses the following properties

1^0 $\alpha \mathscr{M} \subset \mathscr{M}$ whenever $\alpha \in \mathscr{R}^1, \alpha > 0$.

2^0 $\mathscr{M}$ is convex.

Only property 2^0 is to be checked. To show 2^0 holds, let $u, v \in \mathscr{M}$: $u = \lambda x$, $v = \mu y$, with $x, y \in X$ and $\alpha, \beta \in \mathscr{R}^1_+$, $\alpha + \beta = 1$, we have $\alpha\lambda + \beta\mu > 0$ and hence,

$$\alpha u + \beta v = (\alpha\lambda + \beta\mu)\left(\frac{\alpha\lambda}{\alpha\lambda + \beta\mu} x + \frac{\beta\mu}{\alpha\lambda + \beta\mu} y\right) \in (\alpha\lambda + \beta\mu) X \subset \mathscr{M},$$

implying convexity of $\mathscr{M}$. Thus, $\mathscr{M}$ is an open convex set satisfying 1^0 and not containing the origin 0.

To complete the proof of the theorem it is enough to show the existence of a subspace $\mathscr{L} \subset \mathscr{E}$ not intersecting $\mathscr{M}$, whose orthogonal complement $\mathscr{L}^\perp$ is one-dimensional. Such a subspace is determined by a fixed ray τv, $\tau \in \mathscr{R}^1_+$, $v \in \mathscr{E}$, in such a manner that $\mathscr{M}$ is contained in one of the halfspaces formed by $\mathscr{L}$.

Let us consider the collection of all linear subspaces of $\mathscr{E}$ not intersecting $\mathscr{M}$. Such a collection is nonempty because the 0-dimensional linear subspace in $\mathscr{E}$ consisting just of the origin 0 possesses this property. Since $\mathscr{E} \cap \mathscr{M} = \mathscr{M}$, $\mathscr{E}$ does not belong to the collection mentioned. It follows that there exists a maximal subspace $\mathscr{L}$ possessing the above nonintersection property. We can expect that the orthogonal complement $\mathscr{L}^\perp$ would have minimal dimension, say $d = \dim \mathscr{L}^\perp$. We know already that $d \geq 1$.

Let us assume that $d \geq 2$. Let P denote the orthogonal projection of $\mathscr{E}$ onto $\mathscr{L}^{\perp}$. This automatically implies that

$$P(\mathscr{L}) = \{0\}.$$

The linearity of P implies that property 1^0 and 2^0 remain valid also for $P(\mathscr{M})$. We show that $P(\mathscr{M})$ is open in $\mathscr{L}^{\perp}$. To see this, let $c_1 = P(c)$, $c \in \mathscr{M}$. Define,

$$\phi(x) = x - (c - c_1), x \in \mathscr{E}.$$

Since P is a projection, it is clear that

$$P(\phi(x)) = P(x), x \in \mathscr{E}.$$

If $y \in \phi(\mathscr{M}) \cap \mathscr{L}^{\perp}$, then $y = \phi(x)$ for some $x \in \mathscr{M}$ and $y = P(y) = P(\phi(x)) = P(x) \in P(\mathscr{M})$. Hence $\phi(\mathscr{M}) \cap \mathscr{L}^{\perp} \subset P(\mathscr{L}^{\perp})$. Since ϕ is a homeomorphism, $\phi(\mathscr{M})$ is open in $\mathscr{E}$. Therefore, $\phi(\mathscr{M}) \cap \mathscr{L}^{\perp}$ is an open set in $\mathscr{L}^{\perp}$ containing $c_1 \in P(\mathscr{M})$, which proves that $P(\mathscr{M})$ is open in $\mathscr{L}^{\perp}$. We also see immediately that $0 \neq P(\mathscr{M})$ since $\mathscr{M} \cap \mathscr{L} = \emptyset$ and $P^{-1}(0) = \mathscr{L}$.

Next we show that $\mathscr{L}^{\perp}\backslash\{0\}$ is connected. Actually, it is enough to show that $\mathscr{L}^{\perp}\backslash\{0\}$ is arcwise connected. Take two distinct elements $\tilde{x}, \tilde{y} \in \mathscr{L}^{\perp}\backslash\{0\}$. If the segment joining $\tilde{x}$ and $\tilde{y}$ does not contain the origin, this segment can serve as an arc joining these two elements in $\mathscr{L}^{\perp}\backslash\{0\}$. If 0 is an element of the segment joining $\tilde{x}$ and $\tilde{y}$, then take $\tilde{z} \in \mathscr{L}^{\perp}\backslash\{0\}$ such that the inner product $[\tilde{z}, (1 - t)\tilde{x} + t\tilde{y}] = 0$, $t \in [0, 1]$. The existence of vector $\tilde{z}$ is guaranteed by our hypothesis $\dim \mathscr{L}^{\perp} \geq 2$. Then the polygonal path consisting of the segments $(1 - t)\tilde{x} + t\tilde{z}$ and $(1 - s)\tilde{x} + s\tilde{z}$, $0 \leq s, t \leq 1$, can serve as an arc joining $\tilde{x}$ and $\tilde{y}$ in $\mathscr{L}^{\perp}\backslash\{0\}$. Now, $P(\mathscr{M}) \subset \mathscr{L}^{\perp}\backslash\{0\}$. Since $\mathscr{M} \neq \emptyset$ we see that $P(\mathscr{M}) \neq \emptyset$ as well. Moreover, $P(\mathscr{M}) \neq \mathscr{L}^{\perp}\backslash\{0\}$. If $P(\mathscr{M}) = \mathscr{L}^{\perp}\backslash\{0\}$, we would have (by convexity of $P(\mathscr{M})$)

$$0 = \frac{1}{2}\tilde{x} = \frac{1}{2}(-\tilde{x}) \in P(\mathscr{M}), \ \forall \tilde{x} \in \mathscr{L}^{\perp}\backslash\{0\},$$

which is a contradiction. We have just shown that $P(\mathscr{M})$ is a nonempty open proper subset of $\mathscr{L}^{\perp}\backslash\{0\}$. It follows that $P(\mathscr{M})$ cannot be closed in $\mathscr{L}^{\perp}\backslash\{0\}$ simultaneously. Otherwise, the complement $P(\mathscr{M})^c = \mathscr{E}\backslash P(\mathscr{M})$ would be also nonempty open proper subset of $\mathscr{L}^{\perp}\backslash\{0\}$. Followingly, $\mathscr{L}^{\perp}\backslash\{0\}$ would be a union of $P(\mathscr{M})$ and $P(\mathscr{M})^c$ which would contradict the connectedness of $\mathscr{L}^{\perp}\backslash\{0\}$. Since $\overline{P(\mathscr{M})}\backslash P(\mathscr{M}) \neq \emptyset$, where $\overline{P(\mathscr{M})}$ denotes the norm closure of $P(\mathscr{M})$, there must exist an element $\tilde{u} \in \mathscr{L}^{\perp}\backslash\{0\}$ such that $\tilde{u} \notin P(\mathscr{M})$. With this $\tilde{u}$ we construct a linear space

$$\mathscr{L}_1 = \{x + \lambda\tilde{u} : x \in \mathscr{L}, \lambda \in \mathscr{R}^1\}.$$

We have

$$P(x + \lambda\tilde{u}) = \lambda\tilde{u}, \forall x \in \mathscr{L}.$$

Hence

$$\mathscr{L}_1 \cap \mathscr{M} = \emptyset$$

if and only if $\lambda\tilde{u} \notin P(\mathcal{M})$ for any λ. Suppose, $\lambda\tilde{u} \in P(\mathcal{M})$ for some λ. Obviously, $\lambda \neq 0$. It follows that λ cannot be positive. If so, then

$$\tilde{u} \in \frac{1}{\lambda} P(\mathcal{M}) \subset P(\mathcal{M}),$$

a contradiction. If $\lambda < 0$, we have

$$-\tilde{u} \in -\frac{1}{\lambda} P(\mathcal{M}) \subset P(\mathcal{M}).$$

We note that

$$0 = \frac{1}{2}\tilde{u} + \frac{1}{2}(-\tilde{u}), \ \tilde{u} \in \overline{P(\mathcal{M})}, \ -\tilde{u} \in P(\mathcal{M}).$$

By Theorem 3.1 (ii) this implies that $0 \in P(\mathcal{M})$, a contradiction again. Hence, we have constructed a new subspace $\mathcal{L}_1$ with the nonintersection property such that $\mathcal{L}_1 \cap \mathcal{M} = \emptyset$ and

$$\mathcal{L}_1 \supset \mathcal{L}, \ \mathcal{L}_1 \neq \mathcal{L}.$$

We see the $\mathcal{L}$ is not maximal. This contradiction shows that $d = 1$ and completes the proof of Theorem 3.4. □

The following result can be proven same manner as Theorem 3.5 in [14, p. 14].

3.5 Theorem. *Let $\mathcal{C}$ be a convex wedge. Then*
(i) $\mathcal{C}^{**} \supset \mathcal{C}$.
(ii) $\mathcal{C}^{**} = \mathcal{C}$ if and only if $\mathcal{C}$ is closed.

3.6 Theorem. *Let $X \subset \mathcal{E}$ be a convex set containing no elements of $(\mathcal{K}^*)^d$, where the cone $\mathcal{K}$ is closed normal and generating $\mathcal{E}$.*
Then there is an element $0 \neq z \in \mathcal{K}$ such that

$$X \subset \{x \in \mathcal{E} : [x, z] \leq 0\}.$$

Proof. Let

$$M = \mathcal{K}^* - X = \{w = y - x : y \in \mathcal{K}^*, x \in X\}.$$

Then, since

$$(1-t)[y_1 - x_1] + t[y_2 - x_2] = [(1-t)y_1 + ty_2] - [(1-t)x_1 + tx_2] \in M$$

for $y_j \in \mathcal{K}^*$, $x_j \in X$, $j = 1, 2$, $0 \leq t \leq 1$, M is convex. We show that 0 does not belong to the interior $\operatorname{Int} M$. In contrary, let $\Omega(0, \delta) \subset M$, $\delta > 0$, be an open δ-neighborhood of 0. Let $-u \in \Omega(0, \delta)$, where $u \in (\mathcal{K}^*)^d$. It follows that

$$-u = y(u) - x(u), \ y(u) \in \mathcal{K}^*, \ x(u) \in X.$$

Consequently,

$$x(u) = y(u) + u \in (\mathcal{K}^*)^d,$$

a contradiction. Thus, $0 \notin \operatorname{Int} M$.

By Theorem 3.2 there is as element $z \in \mathscr{E}$ such that

$$M \subset \{w \in \mathscr{E} : [w, z] \geq 0\}$$

We interpret this situation as follows:

$$(3.1) \qquad [u, z] \geq [x, z] \text{ whenever } u \in \mathscr{K}^*, \ x \in X.$$

This means that the hyperplane

$$[y, z] = 0, \ y \in \mathscr{E},$$

separates X from $\mathscr{K}^*$. We show that
(a) $[x, z] \leq 0$ for any $x \in X$,
(b) $z \in \mathscr{K}$.

Since $0 \in \mathscr{K}^*$, the validity of (a) is obtained immediately by setting $u = 0$ in (3.1).

It remains to prove (b). The relations (3.1) imply also that the linear function $[y, z]$ is bounded from below for $y \in \mathscr{K}^*$. Let θ be one of the lower bounds. Then, by definition, $[y, z] \geq \theta$. Let $\mathscr{H}^*$ be a $\mathscr{K}$-total set. Obviously, $\lambda y \in \mathscr{K}^*$ for any $y \in \mathscr{K}^*$ and $0 < \lambda \in \mathscr{R}^1$. We see that

$$\theta \leq [\lambda h, z], \ h \in \mathscr{H}^*,$$

for all $0 < \lambda \in \mathscr{R}^1$ imply that

$$0 \leq [h, z] \ \forall h \in \mathscr{H}^*$$

and thus, $z \in \mathscr{K}$. □

3.7 Theorem. *Let $\mathscr{E}$ be a Hilbert space generated by a closed normal cone $\mathscr{K}$. Let $\mathscr{C} \subset \mathscr{E}$ be a closed convex wedge. Then* (3.2) $\Leftrightarrow$ (3.3)*, where*

$$(3.2) \qquad \mathscr{C} \cap \mathscr{K}^* = \{0\}$$

and

$$(3.3) \qquad -\mathscr{C}^* \cap \mathscr{K}^d \neq \emptyset.$$

Proof. Let in contrary $-\mathscr{C}^* \cap \mathscr{K} = \emptyset$. Applying Theorem 3.6 to $-\mathscr{C}^*$ we see that there exists an element $0 \neq z \in \mathscr{K}$ such that

$$-\mathscr{C}^* \subset \{x \in \mathscr{E} : [x, z] \leq 0\}.$$

It follows that

$$[u, z] \geq 0 \text{ for all } u \in \mathscr{C}^*$$

and therefore, $z \in \mathscr{C}^{**}$. Since $\mathscr{C}$ is closed and convex, Theorem 3.5 (ii) implies $\mathscr{C}^{**} = \mathscr{C}$. Thus, $z \in \mathscr{C}$, a contradiction.

Conversely, let $w \in -\mathscr{C}^* \cap \mathscr{K}$. Then, $-w \in \mathscr{C}^*$. It follows that for any $u \in \mathscr{K}$ we have $[u, -w] = -[u, w] < 0$, which implies that $w \notin \mathscr{C}^{**}$. Since $\mathscr{C}^{**} \supset \mathscr{C}$ by Theorem 3.5(i) $w \notin \mathscr{K}$. □

4 Stiemke theorem

This section is devoted to proving the following.

4.1 Theorem *[16]. Let $\mathscr{E}$ and $\mathscr{F}$ be Hilbert spaces over the field of real numbers. Let $\mathscr{E}$ be generated by a closed normal cone $\mathscr{K}$. Let A be a bounded linear operator mapping $\mathscr{E}$ into $\mathscr{F}$. Then the following two conditions (i) and (ii) are equivalent.*

(i) The equation

$$Ax = 0$$

possesses a solution $\hat{x} \in \mathscr{K}^d$.

(ii) The following implication

$$A^*x' \in \mathscr{K}^* \Rightarrow A^*x' = 0$$

holds.

Proof. First we prove that (i) $\Rightarrow$ (ii). Let $\hat{x} \in \mathscr{K}^d$ be such that $A\hat{x} = 0$. We see that $0 \neq A^*x' = y'$, $x' \in \mathscr{F}$, cannot belong to $\mathscr{K}^*$; otherwise,

$$0 < [\hat{x}, y']_{\mathscr{E}} = [A\hat{x}, x']_{\mathscr{F}} = 0,$$

a contradiction.

(ii) $\Rightarrow$ (i). Let

$$\mathscr{L} = \{A^*x' : x' \in \mathscr{F}\}.$$

One easily shows that $\mathscr{L}$ is a linear subspace of $\mathscr{E}$ and that (ii) expresses the fact that

$$\mathscr{K}^* \cap \mathscr{L} = \{0\}.$$

Now, we utilize Theorem 3.7 with

$$-\mathscr{L}^* = \{u \in \mathscr{F} : [u, x'] \geq 0,\ x' \in \mathscr{L}\}.$$

It follows that $-\mathscr{L}^* = \mathscr{L}$ is linear subspace of $\mathscr{F}$ and thus,

$$-\mathscr{L}^* = \mathscr{L}^{\perp}.$$

According to Theorem 3.7 there exists an element $\hat{x} \in \mathscr{L}^{\perp} \cap \mathscr{K}^d$. However,

$$\mathscr{L}^{\perp} = \mathscr{N}(A) = \{x \in \mathscr{E} : Ax = 0\}.$$

This completes the proof of implication (ii) $\Rightarrow$ (i) as well as of the Stiemke Theorem. □

5 Some consequences of the Stiemke theorem: Theorems of Tucker, Farkas and Minkowski

5.1 Hypothesis (H1). *Let $A \in \mathbf{B}(\mathscr{E}, \mathscr{F})$. Define*

(5.1) $$\mathscr{M}(u) = \{y \in \mathscr{K}^* : [A^*u, y]_{\mathscr{E}} > 0\}.$$

There exists an element $\hat{u} \in \mathcal{F}$ such that such that the set $\mathcal{M} = \mathcal{M}(\hat{u})$ is maximal with respect to any $u \in \mathcal{F}$:

$$\mathcal{M} = \mathcal{M}(\hat{u}) \supset \mathcal{M}(u), \ u \in \mathcal{F}.$$

Let

$$\mathcal{L} = Lin(\mathcal{M}) \tag{5.2}$$

be the linear hull of $\mathcal{M}$ closed in $\mathcal{E}$ and let

$$\mathcal{L}^c = \mathcal{E} \setminus \mathcal{L}. \tag{5.3}$$

Obviously,

$$\mathcal{L} = \mathcal{L} \cap \mathcal{K} - \mathcal{L} \cap \mathcal{K}.$$

The existence of an element $\mathfrak{p}$ required in 5.1 Hypothesis (H1) may depend on an interplay between the geometrical properties of the space $\mathcal{E}$ and some structural properties of the operator A under consideration. An example is demonstrated in 5.3.

5.2 Hypothesis (H1'). *Let $A \in \mathbf{B}(\mathcal{E}, \mathcal{F})$. It is assumed that*
(α) for every $u \in \mathcal{E}$ the image $Au \in \mathcal{G} \subset \mathcal{E}$, where $\mathcal{G}$ is a linear manifold dense in $\mathcal{E}$.
(β) $\mathcal{G}$ is a Banach space equipped with a norm $\|.\|_{\mathcal{G}}$ for which we have

$$[x, x]_{\mathcal{E}}^{1/2} \leq \|x\|_{\mathcal{G}}, \ x \in \mathcal{G}.$$

(γ) The space $(\mathcal{G}, \|.\|)$ is generated by a solid cone $\mathcal{K}_{\mathcal{G}}$.
(δ) There are operators A^+, A^- both in $\mathcal{B}(\mathcal{E}, \mathcal{F})$ such that

$$A = A^+ - A^-, \ A^{\pm}\mathcal{K}_{\mathcal{E}} \subset \mathcal{K}_{\mathcal{G}} \subset \mathcal{K}_{\mathcal{E}}.$$

5.3 Example. *Let $a = a(s, t)$ be a continuous function mapping $[0, 1] \times [0, 1]$ into $\mathcal{R}^1$. Let $\mathcal{E} = \mathcal{L}^2(0, 1)$, $\mathcal{G} = \mathcal{C}([0, 1])$. $\mathcal{K}_{\mathcal{E}} = \mathcal{L}^2_+(0, 1)$ and $\mathcal{K}_{\mathcal{G}} = \mathcal{C}_+([0, 1])$. Define*

$$(Ax)(s) = \int_0^1 a(s, t)\, x(t)\, dt.$$

Then

$$(Ax)(s) = \int_{\mathcal{D}^+} a_+(s, t)\, x(t)\, dt - \int_{\mathcal{D}^-} a_-(s, t)\, x(t)\, dt,$$

where $a_{\pm}(s, t) \geq 0$, $a(s, t) = a_+(s, t) - a_-(s, t)$, $s, t \in [0, 1]$ and

$$\mathcal{D}^+ = \{(s,t) \in [0, 1] \times [0, 1] : a(s, t) \geq 0\},$$
$$\mathcal{D}^- = \{(s,t) \in [0, 1] \times [0, 1] : a(s, t) < 0\}.$$

We see that in this case,

$$\hat{u} = \begin{cases} 1 & t \in \mathcal{D}^+ \\ -1 & t \in \mathcal{D}^- \end{cases}$$

and then

$$\mathcal{M}(\hat{u}) = \left\{ s \in [0, 1] : \int_{\mathcal{D}^+} a_+(s, t)\, dt + \int_{\mathcal{D}^-} a_-(s, t)\, dt > 0 \right\}.$$

5.4 Hypothesis (H2). *The manifolds $\mathscr{L}$ and $\mathscr{L}^c$ defined by (5.2) and (5.3) respectively are closed subspaces of the Hilbert space $\mathscr{E}$ and therefore,*

$$\mathscr{E} = \mathscr{L} \oplus \mathscr{L}^c. \tag{5.4}$$

We denote respectively by $[x_1, y_1]_{\mathscr{L}}$ and $[x_2, y_2]_{\mathscr{L}^c}$ the inner products on $\mathscr{L}$ and $\mathscr{L}^c$ introduced by the inner product $[x, y]$ on $\mathscr{E}$. This means

$$[x, y]_{\mathscr{E}} = [x_1, y_1]_{\mathscr{L}} + [x_2, y_2]_{\mathscr{L}^c},$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

5.5 Agreemnent. *We identify any element of the form*

$$\begin{pmatrix} y \\ 0 \end{pmatrix}$$

with $y \in \mathscr{L}$ and similarly

$$\begin{pmatrix} 0 \\ z \end{pmatrix}$$

with $z \in \mathscr{L}^c$.

Let

$$A = (B, C)$$

be a block representation of A according to the decomposition (5.4). Then

$$A^* = \begin{pmatrix} B^* \\ C^* \end{pmatrix},$$

where

$$B = A|_{\mathscr{L}}, \; C = A|_{\mathscr{L}^c}.$$

5.6 Hypothesis (H3). *Let $\mathrm{Lin}(\mathscr{M}) \neq \mathscr{E}$. For every $u \in \mathscr{F}$ for which $\mathscr{M}(u) = \mathscr{M} = \mathscr{M}(\hat{u})$ there exists a vector $v \in \mathscr{F}$ and a constant $\tau \in \mathscr{R}^1$such that*

$$u = \tau\hat{u} + v,$$

such that

$$B^*u \in (\mathscr{L} \cap \mathscr{K})^d$$

and

$$C^*u = \tau C^*\hat{u} + C^*v, \; C^*v \in \mathscr{K} \setminus \{0\}$$

and there is an $\tilde{u} \notin \mathscr{M}(\hat{u})$, $\tilde{u} \in \mathscr{K} \setminus \{0\}, [C^*v, \tilde{u}] > 0$.

5.7 Theorem *(Tucker [17]). Let $\mathscr{E}$ and $\mathscr{F}$ be Hilbert spaces over the field of real numbers. Let $\mathscr{E}$ be a Hilbert space generated by a self-dual cone $\mathscr{K}$, i.e. $\mathscr{E} = \mathscr{K} - \mathscr{K}$, $\mathscr{K} = \mathscr{K}^*$, where*

$$\mathscr{K}^* = \{y \in \mathscr{E} : [y, x]_{\mathscr{E}} \geq 0, \; \forall x \in \mathscr{K}\}. \tag{5.5}$$

Let $A \in \mathbf{B}(\mathscr{E}, \mathscr{F})$ and let Hypotheses (H1)–(H3) hold.

Then there exists a pair $\{\hat{y}, \hat{x}\}$ *in which* $\hat{y}$ *is a solution to the system of relations*

$$A^*y \in \mathscr{K} \tag{5.6}$$

and $\hat{x}$ *is a solution to the equation*

$$Ax = 0 \tag{5.7}$$

such that

$$\hat{x} + A^*\hat{y} \in \mathscr{K}^d. \tag{5.8}$$

Proof. Consider the maximal set $\mathscr{M}$ whose existence is guaranteed by Hypothesis 5.1. If $\mathscr{M} = \emptyset$, then (5.6) possesses no nontrivial solutions and the conditions of the Stiemke Theorem 4.1 are satisfied. According to this theorem there is an $\hat{x} \in \mathscr{K}^d$ such that (5.7) holds. The required pair is then given by $(0, \hat{x})$. Similarly, let $\mathscr{M} = \mathscr{H}^*$. Then $\hat{y} \in \mathscr{K}^d$ and the required pair is given by $(\hat{y}, 0)$. Thus, let

$$\emptyset \neq \mathscr{M} \neq \mathscr{H}^*$$

We check easily that

$$\{0\} \neq \mathscr{L} \neq \mathscr{E}. \tag{5.9}$$

It follows that

$$Lin(\mathscr{M}) = \mathscr{L} \subset \mathscr{R}(A^*) = \{v \in \mathscr{E} : v = A^*u, u \in \mathscr{E}\}$$

and

$$\mathscr{L}^c \supset \mathscr{N}(A) = \{x \in \mathscr{E} : Ax = 0\}.$$

To complete the proof of the Tucker theorem we just need to show the existence of a solution $\tilde{x} \in \mathscr{L} \cap \mathscr{K}$ fulfilling

$$[\tilde{x}, y] = 0 \;\; \forall y \in \mathscr{M} \tag{5.10}$$

and simultaneously

$$[\tilde{x}, y] > 0 \;\; \forall y \in \mathscr{H}^* \setminus \mathscr{M}. \tag{5.11}$$

Let $A = (B, C)$ be the representation of A with respect to the decomposition (5.4). Let us consider relations

$$C^*y \in \mathscr{K}.$$

We show that these relations have only trivial solution $C^*y = 0$. Assume the contrary, i.e. let $0 \neq z \in \mathscr{E}$ be such that $0 \neq C^*z \in \mathscr{K} \cap \mathscr{L}^c$. Recall that

$$0 < [y, A^*\hat{y}] = [y_1, B^*\hat{y}]_{\mathscr{L}} \;\; \forall y \in \mathscr{M} = \mathscr{M}(\hat{y}) \subset \mathscr{L},$$

where

$$y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \; A^*\hat{y} = \begin{pmatrix} B^*\hat{y} \\ C^*\hat{y} \end{pmatrix}$$

and

$$0 = [y, A^*\hat{y}] = [y_2, C^*\hat{y}]_{\mathscr{L}^c} \ \forall y \in \mathscr{L}^c$$

implying

$$C^*\hat{y} = 0.$$

We are able to define an element

$$w = \tau\hat{y} + z,$$

where τ is a suitable constant whose existence is guaranteed by Hypothesis 5.6. We see that

$$B^*w = \tau B^*\hat{y} + B^*z \in \mathscr{K} \cap \mathscr{M}(\hat{y}),$$

while

$$C^*w = \tau C^*\hat{y} + C^*z \in \mathscr{K} \setminus \{0\}.$$

Since $C^*w \in \mathscr{K} \cap \mathscr{L}^c$, $\mathscr{M}(\hat{y}) \subset \mathscr{L}$ and

$$\mathscr{L} \cap \mathscr{L}^c = \{0\},$$

we conclude that there is an element $\hat{v} \in \mathscr{H}^* \setminus \mathscr{M}(\hat{y})$ such that

$$[A^*w, \hat{v}]_{\mathscr{E}} > 0, \ \hat{v} \notin \mathscr{M}(\hat{y}).$$

This fact, however, contradicts maximality of $\mathscr{M}(\hat{y})$. Hence, operator C satisfies the conditions of the Stiemke theorem with respect to $\mathscr{L}$. Consequently, there exists an element

$$\hat{z} \in \{\mathscr{L}^c \cap \mathscr{K}\}^d$$

such that

$$\tilde{x} = \begin{pmatrix} 0 \\ \hat{z} \end{pmatrix}$$

is a solution to (5.7):

$$A\tilde{x} = \begin{pmatrix} 0 \\ C\hat{z} \end{pmatrix} = 0$$

We also see that $\tilde{x}$ is a solution to (5.7) satisfying (5.10) and (5.11). The proof of the Tucker theorem is complete. □

5.8 Theorem *Farkas [5], Minkowski [13]. Let $\mathscr{E}$ and $\mathscr{F}$ be Hilbert spaces over the reals with the inner products $[.,.]_{\mathscr{E}}$ and $[.,.]_{\mathscr{F}}$ respectively. Let $\mathscr{E}$ be generated by a closed normal cone $\mathscr{K}$ and let $\mathscr{K}$ be self-dual. Let $A \in \mathbf{B}(\mathscr{E}, \mathscr{F})$ and $b \in \mathscr{F}$.*

Then the following two conditions 1^0 and 2^0 are equivalent

1^0 *Equation*

$$Ax = b$$

possesses a solution $x^* \in \mathscr{K}$.

2^0 *Relation*

$$A^*u \in \mathscr{K}$$

implies that

$$[b, u]_{\mathscr{F}} \geq 0.$$

Proof. $1^0 \Rightarrow 2^0$. Let $x^* \in \mathscr{K}$ be a solution to $Ax = b$. Then from $A^*u \in \mathscr{K}$ it follows that

$$0 \leq [x^*, A^*u]_{\mathscr{E}} = [Ax^*, u]_{\mathscr{F}} = [b, u]_{\mathscr{F}}.$$

$2^0 \Rightarrow 1^0$. Define a new map $\mathscr{A} \in \mathbf{B}(\mathscr{E} \times \mathscr{R}^1, \mathscr{F})$ by setting

$$X = \begin{pmatrix} x \\ \xi \end{pmatrix}, \; x \in \mathscr{E}, \; \xi \in \mathscr{R}^1,$$

$$\mathscr{A}X = Y \Leftrightarrow Y = Ax - \xi b \in \mathscr{F}.$$

Then

$$\mathscr{A}^*u = \begin{pmatrix} A^*u \\ -[b, u]_{\mathscr{F}} \end{pmatrix}.$$

Now, we apply the Tucker theorem 5.7 to $\mathscr{A}$ in $\mathbf{B}(\mathscr{E} \times \mathscr{R}^1, \mathscr{F})$. It follows that there exists a pair (u, x) such that

$$\begin{pmatrix} A^*u \\ -[b, u]_{\mathscr{F}} \end{pmatrix} \in (\mathscr{E} \cap \mathscr{K}) \times \mathscr{R}^1_+,$$

(5.12) $$Ax - \xi b = 0, \; x \in \mathscr{K}, \; \xi \in \mathscr{R}^1_+,$$

(5.13) $$\begin{pmatrix} A^*u \\ -[b, u]_{\mathscr{F}} \end{pmatrix} + X \in (\mathscr{K} \times \mathscr{R}^1_+)^d = \mathscr{K}^d \times \operatorname{Int} \mathscr{R}^1_+.$$

From the first relation we deduce easily that

$$A^*u \in \mathscr{K}, \; [b, u] \leq 0.$$

However, since $A^*u \in \mathscr{K}$, we also have

$$[b, u]_{\mathscr{F}} \geq 0.$$

This implies $[b, u]_{\mathscr{F}} = 0$ and using (5.13) we derive

$$\xi > 0.$$

Relations (5.12) implies that

$$Ax^* = b, \; x^* \in \mathscr{K},$$

by identifying

$$x^* = \frac{1}{\xi} x.$$

This completes the proof. □

Next, we present two results which are generalizations of another of Tucker theorems whose very elegant application offers a nice tool to solving some problems in existence proofs in linear programming.

5.9 Theorem. *Let $\mathscr{K}^* = \mathscr{K}$ and let the Hypotheses* **H1 – H3** *hold. Let $T \in \mathbf{B}(\mathscr{E})$ be skew symmetric, i.e. $T^* = -T$.*

Then there exists an element $w \in \{y \in \mathscr{E} : Ty \in \mathscr{K}, \; y \in \mathscr{K}\}$ such that

$$Tw + w \in \mathscr{K}^d.$$

Proof. Let us consider the operator

$$(T^*, I),$$

I being the identity map on $\mathscr{E}$ mapping $\mathscr{E} \times \mathscr{E}$ into $\mathscr{E}$. By the Tucker Theorem 5.7 there exists a pair p and x such that

$$\begin{pmatrix} T \\ I \end{pmatrix} p \in \mathscr{K} \times \mathscr{K}, \ (T^*, I)x = 0, \ x \in \mathscr{K} \times \mathscr{K}$$

and

$$\begin{pmatrix} T \\ I \end{pmatrix} p + x \in \mathscr{K}^d \times \mathscr{K}^d.$$

We let

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ x_1, x_2 \in \mathscr{E}$$

and define

$$u = x_1, \ v = x_2.$$

Then from

$$\begin{pmatrix} T \\ I \end{pmatrix} p = \begin{pmatrix} Tp \\ p \end{pmatrix} \in \mathscr{K} \times \mathscr{K}$$

it follows that

$$Tp \in \mathscr{K}, \ p \in \mathscr{K}$$

and futhermore,

$$T^*u + v = 0, \ u \in \mathscr{K}, \ v \in \mathscr{K}, \ Tp + u \in \mathscr{K}^d, \ p + v \in \mathscr{K}.$$

In view of $T^* = -T$ we see that

$$T^*u + v \Rightarrow Tu = v.$$

Hence, if we let

$$w = p + u,$$

we get

$$\begin{aligned} Tw &= Tp + Tu = Tp + v \\ Tw + w &= (Tp + v) + (p + u) \\ &= (Tp + u) + (p + v) \in \mathscr{K}^d. \end{aligned}$$

The proof is complete. □

5.10 Corollary. *Under the hypotheses of Theorem 5.9 let $A \in \mathbf{B}(\mathscr{E}, \mathscr{F})$. Then there exist solutions to the relations*

$$A^*p \in \mathscr{K}, \ p \in \mathscr{K}, \ -Ax \in \mathscr{K}, \ x \in \mathscr{K},$$

such that

$$x + A^*p \in \mathscr{K}^d, \ p - Ax \in \mathscr{K}^d.$$

Proof. We apply Theorem 5.9 to the operator $\mathscr{T} \in \mathbf{B}(\mathscr{E} \times \mathscr{F})$ defined as

$$\mathcal{T} = \begin{pmatrix} 0_{\mathcal{E}} & A^* \\ -A & 0_{\mathcal{F}} \end{pmatrix}$$

where $0_{\mathcal{E}}$ and $0_{\mathcal{F}}$ denotes the zero-operator on $\mathcal{E}$ and $\mathcal{F}$ respectively. There exists an element $w \in \mathcal{K}$ satisfying

$$\mathcal{T}w \in \mathcal{K}, \ \mathcal{T}w + w \in \mathcal{K}^d \times \mathcal{K}^d.$$

Let

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and

$$x = w_1, \ p = w_2.$$

Then the required relations easily follow. □

6 Conclusion

Though our primary aim was to develop a tool to generalizing the classical Leontief and von Neumann models modeling some problems of Economics, our results appear as autonomous and independent of any application. The second aspect of our generalization, i.e. a cone order, stresses this fact indeed, because the problems of Economics are quite tightly connected with the standard natural order.

We also see that the infinite dimensional generalizations are quite natural and preserve most of the properties of the finite dimensional counterparts. This is caused also by our restriction to a Hilbert space approach.

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