Petr Lachout
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A Solution of an Equation for Indexed Functions

PETR LACHOUT

Praha

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We present a characterization of indexed real functions (4) fulfilling an equation (5). Consequently, we are receiving a description of those linear transformations of a Wiener process which result in a time-changed Wiener process, Brownian bridge or Ornstein–Uhlenbeck process.

1. Problem setting and examples

Assymptotic investigation of statistical estimators and test statistics often leads to a linear transformation of a Wiener process that turns out to be a time-changed Wiener process, a Brownian bridge or Ornstein–Uhlenbeck process. Let us recall some typical examples of such transformations.

Given a Wiener process \((W(t), t \geq 0)\), the process \((tW(t), t > 0)\) is a Wiener processes, \((W(t) - tW(1), t \in [0, 1])\) and \((tW(t^{-1}), t \in (0, 1))\) are Brownian bridges and \((e^{-t}W(e^{2t}), t \in \mathbb{R})\) is an Ornstein–Uhlenbeck process.

In [3] we treated a collection of stochastic integrals of non-random real functions w.r.t. a Wiener process \((W(t), t \geq 0)\), i.e.

\[
\left( \int a_\lambda \, dW, \lambda \in \Lambda \right), \text{ where } a_\lambda \in L_2(m) \quad \forall \lambda \in \Lambda
\]

and \(m\) denotes the Lebesgue measure on \([0, +\infty)\).
We seek conditions under which (1) satisfies
\[
\int a_{\lambda} \, dW = V(\xi(\lambda)) \quad \text{a.s.} \quad \forall \lambda \in \Lambda, \tag{2}
\]
where \( V \) is a prescribed Gaussian process, e.g. a Wiener process, a Brownian bridge, an Ornstein–Uhlenbeck process, \( \Lambda \) is a non-empty set and \( \xi : T \to \mathbb{R}_+ \) is an appropriate function.

Because (1) is always a Gaussian process, one can verify (2) computing the covariance function of (1), only. Applying that, we have proved in [3] that (2) with \( V \) being a Wiener process is equivalent to
\[
\int a_{\lambda} a_{\psi} \, dm = \min \left\{ \int a_{\lambda}^2 \, dm, \int a_{\psi}^2 \, dm \right\} \quad \forall \lambda, \psi \in \Lambda. \tag{3}
\]

Of course in (2) we set \( \xi(\lambda) = \int a_{\lambda}^2 \, dm \).

In [3] we also present some examples of function families satisfying (3). Especially, we consider families of functions which are constant till a point and zero after that. For these families we succeeded to determine a complete description to satisfy (3). Two particular families keeping (3) are shown in [1], also.

Inspired by (3) we consider a measure space \((E, \mathcal{F}, \mu)\) in this paper and indexed real functions
\[
(f_{\lambda}, \lambda \in \Lambda), \quad \text{where} \quad f_{\lambda} : (E, \mathcal{F}) \to (\mathbb{R}, \mathbb{B}) \in \mathcal{L}_2(\mu), \tag{4}
\]
fulfilling
\[
\forall \lambda, \psi \in \Lambda : \int f_{\lambda} f_{\psi} \, d\mu = \min \left\{ \int f_{\lambda}^2 \, d\mu, \int f_{\psi}^2 \, d\mu \right\}. \tag{5}
\]

Let us start with two examples of indexed real functions fulfilling (5).

**Example 1.** Let \( f \in \mathcal{L}_2(\mu) \), \( \Lambda \subset \mathbb{R} \) and \( A_\lambda \in \mathcal{F} \) for each \( \lambda \in \Lambda \). If \( A_{\lambda} \subset A_{\psi} \) whenever \( \lambda, \psi \in \Lambda, \lambda \leq \psi \) then the collection of restrictions \( (f_{\lambda}, \lambda \in \Lambda) \) fulfills (5).

Evidently, \( f_{\lambda} \in \mathcal{L}_2(\mu) \) for each \( \lambda \in \Lambda \). The property (5) can be also easily checked since for \( \lambda, \psi \in \Lambda, \lambda \leq \psi \) we are receiving
\[
\int f_{\lambda} f_{\lambda} \, d\mu = \int f_{\lambda}^2 \, d\mu \leq \int f_{\psi}^2 \, d\mu. \quad \triangle
\]

**Example 2.** Let \( \mu \) be a probability measure, \( f \in \mathcal{L}_2(\mu) \), \( \Lambda \subset \mathbb{R} \) and \( \mathcal{A}_\lambda \subset \mathcal{F} \) be a \( \sigma \)-algebra for each \( \lambda \in \Lambda \). Further let \( \mathcal{A}_\lambda \subset \mathcal{A}_{\psi} \) whenever \( \lambda, \psi \in \Lambda, \lambda \leq \psi \).

Hence, the collection of conditional mean \( (\mathbb{E}[f | \mathcal{A}_\lambda], \lambda \in \Lambda) \) fulfills (5).

It is known that \( \mathbb{E}[f | \mathcal{A}_\lambda] \in \mathcal{L}_2(\mu) \) whenever \( f \in \mathcal{L}_2(\mu) \).

The condition (5) follows properties of the conditional mean, especially Jensen inequality. Taking \( \lambda, \psi \in \Lambda, \lambda \leq \psi \) we are receiving
\[
\int \mathbb{E}[f|\mathcal{A}_t] \, \mathbb{E}[f|\mathcal{A}_s] \, d\mu = \int (\mathbb{E}[f|\mathcal{A}_t])^2 \, d\mu \leq \int (\mathbb{E}[f|\mathcal{A}_s])^2 \, d\mu. \quad \triangle
\]

2. A solution

We start the section with observations allowing a simplification of the problem.

**Lemma 1.** Let a collection of indexed real functions (4) fulfills (5). Then
1. For $\lambda \in \Lambda$, $\int f_\lambda^2 \, d\mu = 0$ implies $\mu(f_\lambda \neq 0) = 0$.
2. For $\lambda, \psi \in \Lambda$, $\int f_\lambda^2 \, d\mu = \int f_\psi^2 \, d\mu$ implies $\mu(f_\lambda \neq f_\psi) = 0$.
3. For a net $\lambda_i \in \Lambda$, $i \in I$
   
   \[ f_{\lambda_i}, i \in I \text{ is convergent in } L_2(\mu) \text{ iff } \int f_{\lambda_i}^2 \, d\mu, i \in I \text{ is convergent.} \]
4. If $\lambda, \psi, \varphi \in \Lambda$, $\int f_\lambda^2 \, d\mu < \int f_\psi^2 \, d\mu < \int f_\varphi^2 \, d\mu$ then $f_\varphi - f_\psi, f_\lambda$ are orthogonal in $L_2(\mu)$.

**Proof.** The first statement is evident. The other statements need short proofs.
(a) For $\lambda, \psi \in \Lambda$, we have
\[
\int (f_\lambda - f_\psi)^2 \, d\mu = \int f_\lambda^2 \, d\mu - 2 \int f_\lambda f_\psi \, d\mu + \int f_\psi^2 \, d\mu
\]
\[
= \left| \int f_\lambda^2 \, d\mu - \int f_\psi^2 \, d\mu \right|, \quad \text{accordingly to (5)}. \]

Hence, the property 2 is evident and, clearly, a convergence of indexed real functions in $L_2(\mu)$ is equivalent with convergence of their second powers integrals w.r.t. to $\mu$.
(b) Let $\lambda, \psi, \varphi \in \Lambda$, $\int f_\lambda^2 \, d\mu < \int f_\psi^2 \, d\mu < \int f_\varphi^2 \, d\mu$ then
\[
\int (f_\varphi - f_\psi) f_\lambda \, d\mu = \int f_\lambda^2 \, d\mu - \int f_\varphi^2 \, d\mu = 0 \quad \text{according to (5)}. \quad \text{Q.E.D.} \]

The solution we want to present is based on an integration w.r.t. a process with orthogonal increments.

**Definition 2.** A mapping $U : [0, +\infty) \to L_2(\mu) : t \mapsto U_t$ being right-continuous in $L_2(\mu)$ and having $U_v - U_s, U_t$ orthogonal in $L_2(\mu)$ whenever $0 \leq t < s < v$ will be called on o.i.-process in $L_2(\mu)$.

The process possesses the reference function defined by
\[
F_U : [0, +\infty) \to [0, +\infty) : t \to \int U_t^2 \, d\mu. \quad (6)
\]

(The abbreviation “o.i.-process” stands for “process with orthogonal increments”.)
Lemma 3. The reference function of an o.i.-process is always a non-decreasing non-negative right-continuous function.

Proof. Non-negativity is evident. The reference function is non-decreasing since for each $0 \leq t < s$,

$$F_U(s) = \int U_s^2 \, d\mu = \int U_t^2 \, d\mu + 2 \int (U_s - U_t) \, U_t \, d\mu + \int (U_s - U_t)^2 \, d\mu$$

$$= F_U(t) + \int (U_s - U_t)^2 \, d\mu.$$ 

Right-continuity follows the same equality and the fact that the o.i.-process is right-continuous in $L_2(\mu)$ by definition. Q.E.D.

Hence, we can employ Lebesgue–Stieltjes integral w.r.t. $F_U$ and integrate w.r.t. an o.i.-process $U$.

Proposition 4. Let $U$ be an o.i.-process in $L_2(\mu)$. Then an integral w.r.t. $U$ can be defined such that

1. The integral is defined for all functions from $L_2(F_U)$ and its values are in $L_2(\mu)$.
2. $\forall f, g \in L_2(F_U), a, b \in \mathbb{R}$: $\int a f + b g \, dU = a \int f \, dU + b \int g \, dU \quad \mu$-a.e.
3. $\forall t, s \in [0, +\infty)$: $\int_{[t, s]} dU = U_s - U_t \quad \mu$-a.e.
4. $\forall f, g \in L_2(F_U)$: $\int (\int f \, dU)(\int g \, dU) \, d\mu = \int fg \, dF_U$.

The integral is correctly defined and its values are modulo $\mu$ uniquely determined.

A proof for a finite (probability) measure $\mu$ is given in [2], Chap. 2, § 3. The same arguments are also valid for an arbitrary measure. The crucial point of the proof, i.e. 12. lemma in [2], concludes the proof for an arbitrary measure $\mu$, too, since $L_2(\mu)$ is always a Banach space.

Now, we formulate a solution of the considered problem.

Theorem 1. Indexed functions (4) fulfill (5) iff there are an o.i.-process $U$ in $L_2(\mu)$, a function $h : [0, +\infty) \rightarrow \mathbb{R}$ and a collection of sets $A_\lambda \subset (0, +\infty), \lambda \in \Lambda$ such that

$$h_{|_{A_\lambda}} \in L_2(F_U) \quad \forall \lambda \in \Lambda,$$

$$A_\lambda \subset A_\psi \quad \text{whenever} \quad \int f_\lambda^2 \, d\mu \leq \int f_\psi^2 \, d\mu \quad \forall \lambda, \psi \in \Lambda,$$

$$f_\lambda = \int h_{|_{A_\lambda}} \, dU \quad \mu$-a.e. \quad \forall \lambda \in \Lambda.$$

Proof. 1. Let (9) be fulfilled, $\lambda, \psi \in \Lambda$ and $\int f_\lambda^2 \, d\mu \leq \int f_\psi^2 \, d\mu$. Hence,
\[ \int f_\lambda f_\psi \, d\mu = \int \left( \int h^{A_\lambda} \, dU \right) \left( \int h^{A_\psi} \, dU \right) \, d\mu = \int h^{A_\lambda} h^{A_\psi} \, dF_U = \int h^{2A_\lambda} \, dF_U \]

\[ = \int \left( \int h^{A_\lambda} \, dU \right)^2 \, d\mu = \int f_\lambda^2 \, d\mu. \]

Thus, (5) is satisfied.

2. Let (5) be fulfilled.

Accordingly to Lemma 1, without any loss of generality we may suppose a closed set \( \Lambda \subset [0, +\infty) \), \( 0 \in \Lambda \) and \( \int f_\lambda^2 \, d\mu = \lambda \) for all \( \lambda \in \Lambda \).

Then, we define \( U : [0, +\infty) \to L_2(\mu) \) by the formula

\[ \forall t \in [0, +\infty) \quad U_t = f_\lambda \quad \text{where} \quad \lambda_t = \max \{ \lambda \in \Lambda : \lambda \leq t \}. \quad (10) \]

Accordingly to Lemma 1, \( U \) is an o.i.-process and, evidently,

\[ f_\lambda = U_\lambda = U_\lambda - U_0 = \int h^{[0,\lambda]} \, dU \quad \text{whenever} \quad \lambda \in \Lambda . \]

Hence, we set \( h \equiv 1 \) and \( A_\lambda = (0, \lambda] \) for all \( \lambda \in \Lambda \) to show (9).

Q.E.D.

3. A solution for the original problem

The previous section gives a solution of the original task when (1) is fulfilling (2).

**Theorem 2.** Indexed functions (1) fulfill (2) with \( V \) being a Wiener process iff there are an o.i.-process \( U \) in \( L_2(\mu) \), a function \( h : [0, +\infty) \to \mathbb{R} \) and a collection of sets \( A_\lambda \subset (0, +\infty), \lambda \in \Lambda \) such that

\[ h^{A_\lambda} \in L_2(F_U) \quad \forall \lambda \in \Lambda , \quad (11) \]

\[ \xi(\lambda) = \int h^{2A_\lambda} \, dF_U \quad \forall \lambda \in \Lambda , \quad (12) \]

\[ A_\lambda \subset A_\psi \quad \text{whenever} \quad \xi(\lambda) \leq \xi(\psi) \quad \forall \lambda, \psi \in \Lambda , \quad (13) \]

\[ a_\lambda = \int h^{A_\lambda} \, dU \quad m-a.e. \quad \forall \lambda \in \Lambda . \quad (14) \]

**Proof.** The theorem is a particular case of Theorem 1 with \( \mu = m \).

Q.E.D

**Theorem 3.** Indexed functions (1) fulfill (2) with \( V \) being a Brownian bridge iff there are an o.i.-process \( U \) in \( L_2(\mu) \), a function \( h : [0, +\infty) \to \mathbb{R} \) and a collection of sets \( A_\lambda \subset (0, +\infty), \lambda \in \Lambda \) such that

\[ h^{A_\lambda} \in L_2(F_U) \quad \forall \lambda \in \Lambda , \quad (15) \]
\[
\xi(\lambda) = \int h^2 \|_{A_\lambda} \, dF_U \leq 1 \quad \forall \lambda \in \Lambda, \quad (16)
\]

\[
A_\lambda \subset A_\psi \quad \text{whenever} \quad \xi(\lambda) \leq \xi(\psi) \quad \forall \lambda, \psi \in \Lambda, \quad (17)
\]

\[
a_\lambda = \int h(\|_{A_\lambda} - \xi(\lambda) \|_{A_\lambda}) \, dU \quad m-a.e. \quad \forall \lambda \in \Lambda. \quad (18)
\]

**Proof.** Theorem follows immediately Theorem 2 because the transformation \((W(t) - tW(1), t \in [0, 1])\) transforms a Wiener process to a Brownian bridge and the transformation \((B(t) + tN, t \in [0, 1])\), where \(N\) is a standard Gaussian r.v. independent with \(B\), reverses a Brownian bridge to a Wiener process.

Q.E.D.

**Theorem 4.** Indexed functions (1) fulfill (2) with \(V\) being an Ornstein–Uhlenbeck process iff there are an o.i.-process \(U\) in \(L^2(m)\), a function \(h : [0, +\infty) \rightarrow \mathbb{R}\) and a collection of sets \(A_\lambda \subset (0, +\infty), \lambda \in \Lambda\) such that

\[
h_{\|_{A_\lambda}} \in L^2(F_U) \quad \forall \lambda \in \Lambda, \quad (19)
\]

\[
\xi(\lambda) = \frac{1}{2} \log \left( \int h^2 \|_{A_\lambda} \, dF_U \right) \quad \forall \lambda \in \Lambda, \quad (20)
\]

\[
A_\lambda \subset A_\psi \quad \text{whenever} \quad \xi(\lambda) \leq \xi(\psi) \quad \forall \lambda, \psi \in \Lambda, \quad (21)
\]

\[
a_\lambda = e^{-\xi(\lambda)} \int h_{\|_{A_\lambda}} \, dU \quad m-a.e. \quad \forall \lambda \in \Lambda. \quad (22)
\]

**Proof.** Theorem follows immediately Theorem 2 because the transformation \((e^{-t}W(e^2t), t \in \mathbb{R})\) transforms a Wiener process to an Ornstein–Uhlenbeck process and the transformation \((\sqrt{t}U \left( \frac{1}{2} \log (t) \right), t \in \mathbb{R}_+)\) alters an Ornstein–Uhlenbeck process to a Wiener process.

Q.E.D.

**References**

