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On Continuous Convex or Concave Functions with Respect to the Logarithmic Mean

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Some theorems on the shape of continuous convex or concave functions with respect to the logarithmic mean are presented.

Throughout the paper a, b denote the numbers such that $0 \leq a < b \leq \infty$, I, J stand for the real open intervals such that $J \subset I$, and log denotes the natural logarithm.

Let $M : I \times I \to I$ be a *mean*, i.e.

$$\min\{x,y\} \leq M\{x,y\} \leq \max\{x,y\}, x,y \in I.$$

If, moreover, for all $x,y \in I$, $x \neq y$ these inequalities are sharp then M is said to be a *strict mean*. If $M : I \times I \to I$ is a mean then

$$M(x,x) = x, \; x \in I,$$

and

$$M(J \times J) \subset J, \; J \subset I.$$

The last property allows us to introduce the following

Definition. A function $f : J \to I$ is called:

(i) M – *convex* iff $f(M(x,y)) \leq M(f(x), f(y))$, $x,y \in J$,

(ii) M – *concave* iff $f(M(x,y)) \geq M(f(x), f(y))$, $x,y \in J$,

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(iii) M – *affine* iff $f(M(x,y)) = M(f(x), f(y))$, $x,y \in J$,
(comp. [4], [5]).

We will restrict our considerations to L – *convex* or L – *concave* functions, where $L : (0,\infty)^2 \to (0,\infty)$ denotes the *logarithmic mean* defined by

$$L(x,y) = \frac{x - y}{\log x - \log y}, \ x \neq y, \ x, \ y \in (0,\infty),$$

$$L(x,y) = x, \ x = y, \ x, \ y \in (0,\infty)$$

This mean has the following

Properties ([1],[3]):

1. *L is a strict mean;*
2. *L is a positively homogeneous, i.e. $L(tx,ty) = tL(x,y)$, t, x, $y > 0$;*
3. *L is symmetric, i.e. $L(x,y) = L(y,x)$, x, $y > 0$;*
4. *for every $x > 0$ the function $L(x,\cdot)$ is increasing homeomorphism of $(0, \infty)$ onto itself;*
5. *for all x, $y > 0$ we have*

$$\sqrt{xy} \leq L(x,y) \leq \frac{x + y}{2}; \tag{1}$$

   moreover, the equalities occur iff $x = y$;
6. *L is superadditive, i.e.*

$$L(x_1 + x_2, y_1 + y_2) \geq L(x_1,y_1) + L(x_2,y_2), \ x_1,x_2,y_1,y_2 > 0.$$

The examples of convex or concave functions with respect to the logarithmic mean were given by J. Matkowski and J. Rätz in [4] and [5]. Among others it is known that:

the power function $f(x) = x^p$, $x \in (0,\infty)$ is L – convex iff $p \in \mathbb{R} \setminus (0, 1)$ and it is L – concave iff $p \in [0, 1]$;
the exponential function $f(x) = a^x$, $x \in (0, \infty)$ is L – convex for every $a > 1$ and it is neither L – concave nor L – convex for any $a \in (0, 1)$;
the logarithmic function $f(x) = \log_a x$, $x \in (1, \infty)$ is L – concave for every $a > 1$.

Moreover, J. Matkowski has proved in [3] that *every continuous (at a point) L-affine function $f : (0, \infty) \to (0, \infty)$ is either constant or has the following form $f(x) = kx$, $x > 0$, where k is an arbitrary positive constant.*

We will start our investigations with the intervally monotonic L-convex or L-concave functions. A function $f : (a, b) \to \mathbb{R}$ is called *intervally monotonic* if there exist points x_0, x_1, ..., x_n such that $a = x_0 < x_1 < \ldots < x_n = b$, and the restriction of f to the intervals (x_{i-1}, x_i), $i \in \{1, 2, \ldots, n\}$ are monotonic functions.

Lemma 1. *If $f:(a, b) \to (0, \infty)$ is a monotonic L – convex function then it is continuous.*

Proof. Assume that f is an increasing L-convex function and fix an arbitrary $z \in (a, b)$. Denoting by $f(z^-)$ the lefthand side and by $f(z^+)$ the righthand side limits of f at z we have $f(z^-) \le f(z^+)$. Take an arbitrary sequence $z_n \in (z, b)$ tending to z. By L-convexity of f and on account of (1) we get

$$f(L(z, z_n)) \le L(f(z), f(z_n)) \le \frac{f(z) + f(z_n)}{2}.$$

Letting $n \to \infty$ we obtain

$$f(z^+) \le \frac{f(z) + f(z^+)}{2}.$$

Consequently, $f(z^+) \le f(z)$ and therefore $f(z) = f(z^+)$. Let (z_n) be as above and take the sequence $w_n \in (a, z)$, $\lim_{n\to\infty} w_n = z$ such that $z = L(w_n, z_n)$ for every positive integer n. According to the Lconvexity of f, and (1) we get

$$f(z^+) = f(L(w_n, z_n)) \le L(f(w_n), f(z_n)) \le \frac{f(w_n) + f(z_n)}{2}.$$

Letting $n \to \infty$ and using the equality $f(z) = f(z^+)$ we obtain

$$f(z) \le \frac{f(z^-) + f(z)}{2},$$

which implies that $f(z) = f(z^{-1})$. Thus f is continuous at z. In the same manner our Lemma can be proved in the case of a decreasing function (using now $z_n \in (a, z)$). □

In a similar way (using the inequality $\sqrt{xy} \le L(x, y)$ instead of $L(x, y) \le \frac{x+y}{2}$) the following lemma can be proved.

Lemma 2. *If $f:(a, b) \to (0, \infty)$ is a monotonic L – concave function then it is continuous.* □

Theorem 1. *If $f:(a, b) \to (0, \infty)$ is an intervally monotonic L – convex (L – concave) function, then f is continuous.*

Proof. Let f be an Lconvex intervally monotonic function. By Lemma 1 it is enough to prove that f is continuous at $z \in (a, b)$ in which the monotonicity of f interchanges. Assume that f is decreasing (increasing) in a lefthand neighbourhood (a_1, z) and f is increasing (decreasing) in a righthand neighbourhood (z, b_1) of z. Take a sequence $w_n \in (a_1, z)$ converges to z. Then

$$f(L(w_n, z)) \le L(f(w_n), f(z)) \le \tfrac{f(w_n)+f(z)}{2}.$$

Letting with n to infinity we get $f(z^-) \le f(z)$. Similarly, taking a sequence $z_n \in (z, b_1)$ converges to z we can prove that $f(z^+) \le f(z)$. Now take sequences $w_n \in (a_1, z)$ and $z_n \in (z, b_1)$ converge to z such that $z = L(w_n, z_n)$ for $n \in \mathbb{N}$. By virtue of the L-convexity of f and (1) we have

$$f(z) = f(L(w_n, z_n)) \le \frac{f(w_n) + f(z_n)}{2}.$$

Hence

$$f(z) \le \frac{f(z^-) + f(z^+)}{2}.$$

Therefore it is not true that $f(z) < f(z^-)$ and $f(z) < f(z^+)$ simultaneously. Assume that for instance $f(z^-) \ge f(z)$, i.e. $f(z^-) = f(z)$. Then we have $f(z) \le f(z^+)$ which proves in view of the earlier inequality $f(z^+) \le f(z^-)$ that $f(z^+) = f(z^-)$, too. This ends the proof of continuity of f at point z. Similarly we prove the theorem in the case when f is an L-concave intervally monotonic function. □

The following two theorems refer to the shapes of continuous L-convex and L-concave functions.

Theorem 2. *Let $f : (a, b) \to (0, \infty)$ be a continuous $L-$ convex function. Then there exist c, d, $a \le c \le d \le b$ such that f is strictly decreasing in (a, b), f is constant in (c, d) and f is strictly increasing in (d, b).*

Proof. Let us put

$$m := \inf \{f(x);\ x \in (a, b)\}$$

and

$$T_\alpha := \{x \in (a, b);\ f(x) \le \alpha\}$$

for every $\alpha > m$. Obviously, T_α is closed. We will show that T_α is an interval, for $\alpha > m$. For, take $x_1, x_2 \in T_\alpha$ and assume that there exists an $x \in [x_1, x_2]$ such that $x \notin T_\alpha$. Thus $f(x) > \alpha$. We define C_x and C^x in the following way:

$$C_x := \inf \{s \in [x_1, x];\ f(t) > \alpha,\ t \in (s, x)\};$$
$$C^x := \sup \{s \in [x, x_2];\ f(t) > \alpha,\ t \in (x, s)\}.$$

Note that $x_1 \le C_x < x < C^x \le x_2$. According to the continuity of f we have

$$f(C_x) = f(C^x) = \alpha.$$

Putting

$$C := L(C_x, C^x)$$

we obtain by the L-convexity of f

$$\alpha < f(C) = f(L(C_x, C^x)) \le L(f(C_x), f(C^x)) = \alpha,$$

which is impossible. So, for every $\alpha > m$ the set T_α is an interval. Let

$$T_n = \bigcap_{\alpha > m} T_\alpha.$$

If T_m is nonempty, then $T_m = [c, d] \cap (a, b)$ and evidently f is constant function on (c, d). Assume that $a < c$. We shall show that f is strictly decreasing on (a, c). For indirect proof suppose that there exist x_1 and x_2, $x_1 < x_2$ such that $f(x_1) \leq f(x_2)$. Hence and by the definition of T_α we infer that $x_1, c \in T_{f(x_1)}$. Since $x_2 \in (x_1, c)$ and $T_{f(x_1)}$ is an interval we have $f(x_1) = f(x_2)$. Let us put $x^* = \sup\{x \in [x_2, c];\ f(x) = f(x_1)\}$. Evidently, $f(x^*) = f(x_1)$ and, moreover, $f(u) \leq f(x_1)$ for every $u \in [x_1, x^*]$. Choose a $u \in (x_1, x^*)$ and a $v \in (x^*, c)$ such that $x^* = L(u, v)$. According to the L-convexity of f we get

$$f(x_1) = f(x^*) = f(L(u, v)) \leq L(f(u), f(v)) < f(x_1),$$

a contradiction. Likewise one can show that f is strictly increasing on (d, b). If T_m is empty, then by continuity of f it is sufficient to show in analogous way that f is strictly decreasing or strictly increasing on (a, b). The proof of Theorem 2 is completed. □

Following Robert and Varberg ([6], p. 230) we call a function $f: (a, b) \to \mathbb{R}$ *quasiconvex* if the level sets T_α are convex for every $\alpha \in \mathbb{R}$. Thus we have the following

Corollary 1. *Every continuous $L-$ convex function is quasiconvex.* □

In order to get the analogous (to Theorem 2) result it will be useful to prove the following

Theorem 3. *Let $f: (a, b) \to (0, \infty)$ be an L – concave function. Then the function $h(x) := \frac{1}{f(x)}$, $x \in (a, b)$, is L – convex.*

Proof. By virtue of the definitions of h, L and on account of the L-concavity of f and (1) for all $x, y \in (a, b)$ we get

$$\begin{aligned} L(h,(x), h(y)) &= L\left(\tfrac{1}{f(x)}, \tfrac{1}{f(y)}\right) = \tfrac{1}{f(x)f(y)} L(f(x), f(y)) = \\ &= \tfrac{(L(f(x), f(y)))^2}{f(x)f(y)} \tfrac{1}{L(f(x), f(y))} \geq \\ &\geq \tfrac{(\sqrt{f(x)f(y)})^2}{f(x)f(y)} \tfrac{1}{f(L(x,y))} = h(L(x, y)). \ \square \end{aligned}$$

A simple example of the function $f(x) = \frac{1}{x}$, $x \in (0, \infty)$ shows that the converse theorem does not hold.

Theorem 4. If $f: (a, b) \to (0, \infty)$ *is continuous and $L-$ concave then there exist constants c, d, $a \leq c \leq d \leq b$ such that f is strictly increasing in (a, c), f is constant in (c, d) and f is strictly decreasing in (d, b).*

Proof. According to Theorem 3 the function $\frac{1}{f}$ is L-convex and continuous. Now our assertion follows easily from Theorem 2. □

In [2] we have proved that every bounded L-convex function $f:(0, \infty) \to (0, \infty)$ has to be constant. Theorem 3 also allow us to prove the analogous result for L-concave functions. Namely we have

Theorem 5. *Let $f:(0, \infty) \to (0, \infty)$ be an L – concave function. If there exists a positive number m such that $f(x) \geq m$ for all $x \in (0, \infty)$ then f is constant in $(0, \infty)$.*

Proof. It follows from Theorem 3 and from the below boundedness of f that $\frac{1}{f}$ is bounded L-convex function and therefore constant. Thus f has to be constant, too. □

Immediately from Theorem 5 follows

Corollary 2. *Let $f:(0, \infty) \to (0, \infty)$ be a non-constant increasing function such that* $\lim_{x \to 0^+} f(x) > 0$. Then *$f$ is not L – concave.* □

Corollary 3. *Let p be a positive real number and let $f:(p, \infty) \to (0, \infty)$ be strictly increasing function such that* $\lim_{x \to p^+} f(x) = 0$. *Then f is not L – convex.*

Proof. Suppose that f is a L-convex function. Observe that then the inverse function f^{-1} is L-concave function bounded below by a positive constant. By Theorem 5 f^{-1} is constant, a contradiction. □

All L-convex functions (which we know) are also convex. We do not know whether it is generally true. We will present some partial results of this type.

Theorem 6. *Every decreasing L– convex function $f:(a, b) \to (0, \infty)$ is convex.*

Proof. By monotonicity, L-convexity and (1) we get

$$f\left(\frac{x+y}{2}\right) \leq f(L(x, y)) \leq \frac{f(x) + f(x)}{2}, \quad x, y \in (a, b).$$

This means that f is convex in the Jensen sense and being continuous it is convex. □

In the next statements we use the following

Lemma 3. ([7], p. 13) *Let $f: I \to \mathbb{R}$ be a continuous function. Suppose that for any $t \in I$ and $\delta > 0$ there exist $t_1, t_2 \in I \cap (t - \delta, t + \delta)$ and $\alpha \in [0, 1]$ such that $t = \alpha t_1 + (1 - \alpha)t_2$ and $f(t) \leq \alpha f(t_1) + (1 - \alpha) f(t_2)$. Then f is convex.* □

Theorem 7. *Let $f:(a, b) \to (0, \infty)$ be an L – convex function. If the function $\psi(x) := \frac{f(x)}{x}$, $x \in (a, b)$ is increasing then f is convex.*

Proof. Note that since ψ is increasing then also f is (strictly) increasing in (a, b). It is easy to check that the function $\varphi:(1, \infty) \to \mathbb{R}$ defined by the formula $\varphi(s) := \frac{s - 1 - \log s}{(s-1)\log s}$ is strictly decreasing. For arbitrary $x, y \in (a, b)$, $x < y$ we have

$f(x) < f(y)$. By our assumption on ψ we get $\frac{f(x)}{x} \leq \frac{f(y)}{y}$. Consequently $1 < \frac{y}{x} \leq \frac{f(y)}{f(x)}$ and hence $\varphi\left(\frac{f(y)}{f(x)}\right) \leq \varphi\left(\frac{y}{x}\right)$. Therefore

$$\frac{\frac{f(y)}{f(x)} - 1 - \log\frac{f(y)}{f(x)}}{\left(\frac{f(y)}{f(x)} - 1\right)\log\frac{f(y)}{f(x)}} \leq \frac{\frac{y}{x} - 1 - \log\frac{y}{x}}{\left(\frac{y}{x} - 1\right)\log\frac{y}{x}}$$

or, equivalently

$$L(f(x), f(y)) \leq \frac{f(y) - f(x)}{y - x}(L(x, y) - x) + f(x).$$

According to the L-convexity of f we obtain

$$f(L(x, y)) \leq \frac{f(y) - f(x)}{y - x}(L(x, y) - x) + f(x).$$

This means that the point $(L(x, y), f(L(x, y)))$ lies below the segment joining the points $(x, f(x))$ and $(y, f(y))$. Now our assertion follows from Lemma 3. □

We omit the proof of an analogous theorem for L-concave function.

Theorem 8. *Let $f:(a, b) \to (0, \infty)$ be an increasing L – concave function. If the function $\psi(x) := \frac{f(x)}{x}$, $x \in (a, b)$ is decreasing then f is concave.* □

Example. It follows from Theorem 7 that the function $f:(p, \infty) \to (0, \infty)$ defined by the formula $f(x) = kx - x^{-\alpha}$, where positive constants α, k, p are chosen such that $kp^{\alpha+1} > 1$ is not L-convex in (p, ∞). □

Finally we prove the following

Theorem 9. *Let $f, g:(a, b) \to (0, \infty)$ be functions such that $f(x) < g(x)$, $x \in (a, b)$. If f is L – convex and g is L – concave then the function $h(x) := g(x) - f(x)$, $x \in (a, b)$ is L – concave.*

Proof. Making use of the assumptions and the superadditivity of L we get

$$\begin{aligned} h(L(x, y)) &= g(L(x, y)) - f(L(x, y)) \geq L(g(x), g(y)) - L(f(x), f(y)) = \\ &= L(g(x) - f(x) + f(x), g(y) - f(y) + f(y)) - L(f(x), f(y)) \geq \\ &\geq L(g(x) - f(x), g(y) - f(y)) = L(h(x), h(y)). \quad \square \end{aligned}$$

As a consequence of Theorem 9 we get that the function defined in our example is L-concave in (p, ∞).

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