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On the Locally Fine Construction in Uniform Spaces, Locales and Formal Spaces

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We investigate the connection between the spatiality of locale products and the earlier studies of the author on the locally fine coreflection of the products of uniform spaces. After giving a historical introduction and indicating the connection between spatiality and the locally fine construction, we indicate how the earlier results directly solve the first of the two open problems announced in the thesis of T. Plewe. Finally, we establish a general isomorphism between the covering monoids of the localic product of topological (completely regular) spaces and the locally fine coreflection of the corresponding product of (fine) uniform spaces. Additionally, paper relates the recent studies on formal topology and uniform spaces by showing how the transitivity of covering relations correspond to the locally fine construction.

1. INTRODUCTION

This paper¹ is based on the work carried out on the products of uniform spaces since 1981 by the author and others (see [16] – [24], [25], [26], [27]) using the technique of the locally fine coreflection. We study the connection of these results with the spatiality of localic products of topological spaces and the relationship of the locally fine coreflection with the so-called formal spaces.

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Our work has been motivated by the appearance in 1996 of T. Plewe's Ph.D. thesis [41]² on the spatiality of localic products of topological, especially separable metrizable spaces. Not only did the author discover an analogy between the previous results and those obtained in the thesis in question, but the solution of the first open problem left open (re-solved consistently with ZFC) in [41] was seen to follow from the author's article [23]. Therefore, we decided to write this paper in order to indicate the existence and usefulness of such previous research in an equivalent field. Let us note that the link between the spatiality of the localic products of paracompact spaces and the corresponding product of fine (uniform) spaces was already established by J. Isbell in [32].

On the other hand, the connection between locales and so-called *formal spaces* as clearly indicated in the thesis of I. Sigstam ([46]), led the author to discover the relation of the latter to (pre-)uniform spaces defined by means of filters of covering of a space. The transitivity of the covering relations of formal spaces is obtained by applying the locally fine construction. Hence, we obtain an interesting link between three 'unorthodox' approaches to topology, viz. locales (frames), uniformities and formal spaces. In fact, the locally fine coreflection was first introduced and studied by Ginsburg and Isbell in the beginning of the 1950's as a combinatorial approach to topology (given a monoid or filter of coverings). This project failed as one obtained topology – for metric spaces – only in the case of complete spaces, or more generally, when the spaces considered were paracompact, in the case of so-called *supercomplete* spaces. Their work [12] was published in 1959.

However, the then recently introduced study of 'local lattices' and 'paratopologies' (see J. Benabou [3], D. and S. Papert [39]) was taken up by Isbell resulting in an article on uniform locales [32], in which he characterized [12] as a paper "about objects in a category \mathcal{H} now visible as the hypercomplete uniform locales" (ibid., p. 31). The hypercomplete uniform locales obtained from uniform spaces also have spatial products, provided that the corresponding topological products are paracompact. The study of locales (their opposite objects being named frames) is now a well-established field, both in topology (Johnstone) and uniform spaces, closely related to topos theory (see MacLane and Murdijk). Isbell's student Plewe extended results on spatial products, and also improved our theorem in [27] that Čech-scattered paracompacta have paracompact countable products by establishing that countable localic products of partition-complete ([37], [49]) paracompact spaces are spatial. The connection to the locally fine coreflection was, however, never pointed out.

Formal spaces form a counterpart of the original constructive approaches to topology which considered recursive sequences to define points: One gives a collection of 'pieces' of a space related through a covering relation and studies

² The doctoral dissertation, on which [41] is based, had appeared in 1994.

recursive construction on the basis of these pieces. After being introduced by Fourman and Grayson in 1982 ([9]) and made manifest by Sambin in 1987 ([45]), this approach to a point-free topology has been studied by several authors (see, e.g., Negri and Valentini [38] and Sigstam [46]). As will be seen in the sequel, the transitivity requirement of the covering relation of a formal space is the counterpart of Kuratowski's classical condition that $C^2 = C$ for a closure operation C , and is essentially equivalent to the locally fine construction λ , when the definition is extended from pre-uniformities (covering monoids) to covering relations. This connection will perhaps give a justification for the original attempt by Ginsburg and Isbell to obtain topology combinatorially through λ .

2. The locally fine coreflection

Open covers of topological spaces have an obvious local character in the following sense: If \mathcal{G} is an open cover of a topological space, and for each $G \in \mathcal{G}$, \mathcal{H}_G is an open cover of G , then again the combined family $\cup \{\mathcal{H}_G : G \in \mathcal{G}\}$ is an open cover of the space. This is, however, not valid when 'open' is replaced by 'uniform'. The locally fine operation ℓ can be thought of as an attempt to reach topology from a given filter of coverings through combinatorial localization, i.e., by closing the given filter under the above condition. Let μ be such a filter of coverings (pre-uniformity) on a set X . Assume that $\{U_i\}$ is a member of μ and for each i , we are given a member $\{V_j^i\}$. Then the above condition requires that the 'uniformly locally uniform' cover $\{U_i \cap V_j^i\}$ be again uniform. A pair (X, μ) with this property is called *locally fine*. In case (X, μ) is not locally fine, we may define the closure of μ under this construction as follows.

For generality, let ν be another pre-uniformity on the set X . Then the filter μ/ν is defined to consist of all coverings having a refinement of the form $\{U_i \cap V_j^i\}$, where $U_i \in \mu$ and for each i , $V_j^i \in \nu$. Now let us define by transfinite iteration the consecutive *Ginsburg-Isbell derivatives*³ by setting $\mu^{(0)} = \mu$; $\mu^{(\alpha+1)} = \mu^{(\alpha)}/\mu$, and $\mu^{(\beta)} = \cup \{\mu^{(\alpha)} : \alpha < \beta\}$ when β is a limit ordinal. There will be a least α such that $\mu^{\alpha+1} = \mu^\alpha$; this filter is called the *locally fine coreflection* of μ and denoted $\lambda\mu$. On of the essential results in [12] state that if the filter μ is a uniformity, then so is $\lambda\mu$.

If $\lambda\mu$ sufficient for the topology of the underlying space X , even in the case of a metric uniformity? There is a curious connection with the completeness of

³ Ginsburg and Isbell ([12]) define $\mu^{(\alpha+1)}$ as $\mu^{(\alpha)}/\mu^{(\alpha)}$ which, however, is less suitable for certain inductive purposes. The above 'slowed down' version was introduced by the author in [20]. For products of the form $\mu \times \nu$, we may still slow down the derivation by proceeding one coordinate at a time: Instead of using covers of the form $\mathcal{U} \times \{Y\}$, $\{X\} \times \mathcal{V}$. The corresponding 'coordinatewise refinement condition' can be extended to arbitrary products.

hyperspaces, hence the term ‘hypercomplete’ or ‘supercomplete’. In case the metric uniform space ϱX is complete, then by [12] $\lambda\varrho$ is the fine uniformity of X . (The fine uniformity is the collection of all normal covers of a given completely regular space, or in terms of entourages the filter of all neighbourhoods of the diagonal $\Delta(X)$ in $X \times X$.) In this case every open cover is in $\lambda\varrho$, because (by A. H. Stone) every open cover of a metrizable space is normal. It was previously known that the hyperspace of a complete metric space is again complete. Isbell proved that for a uniform space μX , the uniform hyperspace $H(\mu X)$ is complete if, and only if, the locally fine coreflection $\lambda\mu$ is fine and X is paracompact ([30]). Thus, for such supercomplete spaces, the locally fine construction is sufficient for defining topology (open covers) combinatorially from uniform ones.

3. Locales

For a topological space X , the topology of X , written $T(X)$, is a complete lattice which additionally satisfies the following Heyting axiom:

$$(*) \quad x \wedge \bigvee y_\alpha = \bigvee x \wedge y_\alpha.$$

Given topological spaces X, Y and a continuous mapping $f: X \rightarrow Y$, there is a natural homomorphism $f^*: T(Y) \rightarrow T(X)$ obtained by sending an open set $O \in T(Y)$ to its preimage under f . Complete lattices satisfying the axiom (*) with opposite morphisms are called *local lattices* (Benabou) or simply *locales* (Isbell). Thus, we have a contravariant functor $T: \mathbf{Top} \rightarrow \mathbf{Loc}$. Very general ‘local structures’ satisfying (*) were considered by Ehresmann in [7], who also defined the notion of ‘paratopology’, studied in the papers [39], [3]. The category of *frames* is the opposite \mathbf{Loc}^{op} of the category of locales, i.e., they are complete right distributive lattices with morphisms in the standard direction.

The other way goes from locales to spaces. Recall that the *points* of a locale L may be considered homomorphism $\phi: L \rightarrow \mathbb{2}$, where $\mathbb{2}$ denotes the lattice with two elements 0, 1 such that $0 < 1$. One defines the topological space $\text{Pt}(L)$ of points by choosing for the subbasis the sets $x^* = \{\phi \in \text{Hom}(L, \mathbb{2}) : \phi(x) = 1\}$. A question arises concerning the relations of $X, T(X)$ and $\text{Pt}(T(X))$. One says that L is *spatial* (‘has enough points’) if distinct elements of L can be distinguished by points, i.e., if for $x, y \in L, x \neq y$, there is a point $\phi: L \rightarrow \mathbb{2}$ such that $\phi(x) \neq \phi(y)$. Then if L has enough points, L is isomorphic to $T(X)$ for some space X . Isbell has proved e.g. that quasi-compact regular (more generally ‘subfit’) locales are spatial ([32], 2.1). On the other hand, spatiality is hard to preserve in the products of locales. We postpone the constructive definition of localic products to the last section where we prove our main result. We simply note here that the category of locales has products, denoted here with the symbol Π_{loc} . (Similarly, the category of frames has co-products.) One always has the equality (topological homeomorphism).

$$\text{Pt}(\prod_{\text{loc}} L_i) \cong \prod \text{Pt}(L_i).$$

Even assuming that the L_i come from spaces, i.e., $L_i = T(X_i)$ and the spaces X_i are 'sober' so that $\text{Pt}(L_i)$ is homeomorphic to X_i , this is still not enough. (A topological space is called *sober* if its every irreducible non-empty closed subset is the closure of a unique singleton subset.) We can only infer that

$$\text{Pt}(\prod_{\text{loc}} T(X_i)) \cong \prod X_i,$$

and this does not say anything about the *pointless part* of the localic product. We need to establish that the spaces X_i are preserved under the localic product, i.e., $\prod_{\text{loc}} T(X_i) \cong T(\prod X_i)$. We often replace such an isomorphism with an equality.

The following section seeks to show the relation of the studies (mainly in the 1980's) on the locally fine coreflection of product uniformities to the thesis of T. Plewe.

4. Product theorems

Detailed studies on the behaviour of λ on complete metric and other uniform spaces were carried out by the author and J. Pelant in the 1980's. Let us first consider supercomplete spaces. The questions considered were mainly related to product spaces, a topic directly connected with (and in the metrizable case equivalent to) the spatiality of localic products (Plewe's thesis [41]). If the topological spaces X, Y are paracompact (and Hausdorff), then the fine uniform spaces $\mathcal{F}(X), \mathcal{F}(Y)$ are supercomplete. The condition for the product to be supercomplete is that it is (topologically) paracompact and the equation

$$\lambda(\mathcal{F}(X) \times \mathcal{F}(Y)) = \mathcal{F}(X \times Y)$$

holds. In the infinite case, the corresponding equation has the form $\lambda \prod \mathcal{F}(X_i) = \mathcal{F}(\prod X_i)$. The main results proved in the series *On Supercomplete Spaces I - V* are listed below. A space X is called \mathcal{K} -scattered with respect to a class \mathcal{K} of spaces if every non-empty closed subspace of X contains a point with a closed neighbourhood which belongs to \mathcal{K} . In the sequel, \mathcal{C} (resp. $\check{\mathcal{C}}$) denotes the class of compact (resp. Čech-complete) spaces.

Theorem 1. ([21, [24]): *A binary product $\mathcal{F}(X) \times \mathcal{F}(Y)$ is supercomplete for every paracompact space Y if, and only if, X is paracompact and \mathcal{C} -scattered.*

This characterization was partially obtained in [21], and completed in [24]. The paper [21] ([17]) contained the result that $\mathcal{F}(X) \times \mathcal{F}(Y)$ is supercomplete for every paracompact Y whenever X is \mathcal{C} -scattered and paracompact. Furthermore, a partial converse obtained stated that if X is a paracompact p -space (of Arhangel'skii), then $\mathcal{F}(X) \times \mathcal{F}(S)$ is supercomplete for every separable metrizable S iff X is \mathcal{C} -scattered. Therefore, a metrizable space X is a *multiplier* in the

class of supercomplete (topologically) metrizable spaces iff it is C -scattered. This result implies one of the results obtained in Plewe [41], namely that for metrizable spaces, the locale $X \times_{\text{loc}} Y$ is spatial for all Y if and only if X is completely metrizable and does not contain a closed copy of the irrationals. (See below for a discussion on the spatiality of metrizable products.) Indeed, for metrizable spaces, the properties of 1) being C -scattered and 2) being completely metrizable and not containing a closed copy of the irrationals are equivalent. (If X is C -scattered, then by [48], Theorem 1.7, it is an absolute G_δ space, and hence completely metrizable. The space X cannot contain a closed copy of the irrationals \mathbb{J} , because \mathbb{J} is nowhere locally compact. On the other hand, if X is not C -scattered, then X contains a closed subspace F which is nowhere locally compact. Then by [15], p. 157 (or ([41], 2.1), if X is completely metrizable, F contains a closed copy of \mathbb{J} .)

We note here that Isbell ([33], Th. 4) proved that in the class of *completely regular spaces*, $X \times_{\text{loc}} Y$ is spatial for all Y if, and only if, X is C -scattered.

Theorem 2. ([26]): *If X is paracompact and \mathcal{C} -scattered, then the countable power $\mathcal{F}(X)^\mathbb{N}$ is supercomplete.*

Indeed, if X is assumed to be merely a countable union of closed \mathcal{C} -scattered subspaces, then the so-called *metric-fine* coreflection $m(\mathcal{F}(X)^\mathbb{N})$ is supercomplete. This gave a new topological corollary for paracompact spaces, made possible by the Noetherian tree technique (see below) related to the operation λ . Well-known topological cases known previously (the locally compact paracompact case in Arhangel'skii [2] and Frolík [11], \mathcal{C} -scattered Lindelöf spaces (Alster, [1]), scattered paracompact spaces (Rudin, Watson [44]) and paracompact \mathcal{C} -scattered spaces (Friedler, Martin, Williams [10])) all follow from the uniform case by suitably choosing the uniformity considered. The result of [26] was extended to $\check{\mathcal{C}}$ -scattered paracompact spaces by the author and Yun Ziqiu in [27], first announced in 1990 (conferences in Dubrovnik and Tsukuba). This time the topological corollaries were new: The product theorem holds for \mathcal{C} -scattered Lindelöf, paracompact, and ultraparacompact spaces.

Theorem 3. ([27]): *If X is a $\check{\mathcal{C}}$ -scattered paracompact space, then the countable power $\mathcal{F}(X)^\mathbb{N}$ is supercomplete.*

On the other hand, the autor proved 'omitting' theorems for supercompleteness in products. The method was based on the notion of n -cardinality, due to T. Przymusiński and van Douwen, who gave similar applications to topological spaces (cf. [43]).

Theorem 4. ([22]): *For each $n \in \{1, 2, 3, \dots\}$ there is a subset $X \subset \mathbb{R}$ such that $\mathcal{F}(X)^k$ is supercomplete for $k = 1, \dots, n$ but $\mathcal{F}(X)^{n+1}$ is not, in other words $\lambda(\mathcal{F}(X)^n) = \mathcal{F}(X)^n$ but $\lambda(\mathcal{F}(X)^{n+1}) \neq \mathcal{F}(X)^{n+1}$.*

In the same paper, it was also established that the set X can be chosen so that all finite powers of $\mathcal{F}(X)$ are supercomplete, while the countable power is not. As mentioned in [22], the sets X were constructed as Bernstein sets. One of the corollaries in Plewe's thesis ([41]) is a result that follows from the above theorem, and is in fact directly equivalent to it:

Theorem 4'. (Corollary 5.6 in [41]): *For each $n \in \{2, 3, \dots, \omega\}$ there exist Bernstein sets whose m th localic power is spatial for each $m < n$, while the n th localic power is not.*

Indeed, the equivalence of spatiality and supercompleteness was pointed out already by Isbell in 1972 ([32], Theorem 3.12): The product locale of supercomplete spaces X_i is the locale underlying the hypercompletion (as a locale) of their product space. Thus, the product locale is the locale derived from the product space (and hence the product is spatial) if, and only if, the product space itself is hypercomplete. As countable products of metrizable spaces are always paracompact, the product of at most countably many metrizable X_i is spatial (as locale) if, and only if, $\lambda(\prod \mathcal{F}(X_i)) = \mathcal{F}(\prod X_i)$.

This equivalence is not, however, the end of the story. In his thesis Plewe listed two unanswered questions, of which the first is directly related to Theorems 4 – 4': *Do there exist non-complete spaces with spatial countable localic powers?* He proved that the question has a positive answer in case one assumes a set-theoretical hypothesis consistent with the ZFC, namely that $|\mathbb{R}| \geq \omega_2$ and the unions of ω_1 first category subsets is again of the first category (this is implied by Martin's Axiom). However, the author had extended the technique of n -cardinality and published a solution to the equivalent problem for uniform spaces in 1988:

Theorem 5. ([23], 3.2): *There is a non-analytic subset $X \subset [0, 1]$ such that $\lambda(\mathcal{F}(X)^\omega) = \mathcal{F}(X^\omega)$.*

As a corollary ([23], 3.3), one directly obtains from Gleason's factorization theorem (cf. [32], p. 130) that the equality in the above theorem is valid for any power. For definiteness, let us give here the corollary to Theorem 5 for spatiality:

Theorem 5': *There is a non-analytic and hence non-complete subset $X \subset [0, 1]$ such that the countable localic power of $T(X)$ is spatial.*

The original notion of n -cardinality was used to extend the validity of the CH for Borel sets (Alexandroff) to the n -cardinality version of the CH for subsets of finite products. Let X be an arbitrary set and let $A \subset X^n$. Consider finding a set $Y \subset X$, as small as possible, such that the codimension 1 'hyperplanes' $\pi_i^{-1}(y)$, $y \in Y$ cover the set A . Accordingly, we define the n -cardinality of A , written $|A|_n$ as the minimum cardinality of a subset $Y \subset X$ such that

$$A \subset Y \times X^{n-1} \cup X \times Y \times X^{n-2} \cup \dots \cup X^{n-1} \times Y,$$

or equivalently, $A \subset \cup \{\pi_i^{-1}[Y] : 1 \leq i \leq n\}$. The result proved by Przymusiński in [43] states that if X is a Polish space and $A \subset X^n$ is an analytic subset with $|A|_n > \omega$, then the n -cardinality equals 2^ω .

For dealing with infinite powers, the author defined in [23] the notion of *relative* ω -cardinality: The ω -cardinality of a subset $A \subset X^\omega$ with respect to a subset $S \subset X$, written $|A, S|_\omega$, is the minimum cardinality of a subset $Y \subset S$ (if such a set exists) such that

$$A \subset \cup \{\pi_i^{-1}[Y] : i \in \omega\}.$$

In case there is no such $Y \subset S$, we define $|A, S|_\omega = |X|$. Due to the relativity condition, this is a non-trivial extension of the notion of n -cardinality. (On the other hand, a similar notion of relative n -cardinality permits a simple proof of the basic result: For a Polish space X , and an analytic subspace $A \subset X^n$, $|A, S|_n > \omega$ implies $|A, S|_n = 2^\omega$ for any $S \subset X$; for details, see [23], 2.1.) The main principle in the inductive proof of Theorem 5 was the result that if X is a Polish space, $S \subset X$ is arbitrary, and $A \subset X^\omega$ is analytic, then $|A, S|_\omega > \omega$ implies $|A, S|_\omega = 2^\omega$.

Remark 1: A basic example of a non-spatial product is given by $\mathbb{Q} \times \mathbb{Q}$, where \mathbb{Q} denotes the rationals as the subspace of the real line. This example was explicitly handled by Johnstone in his book ([34], II 2.14), which appeared in 1982. Coincidentally, in the same year, the author had shown as a particular corollary to his results on supercompleteness that $\lambda(\mathcal{F}\mathbb{Q} \times \mathcal{F}\mathbb{Q}) \neq \mathcal{F}(\mathbb{Q} \times \mathbb{Q})$. This followed from the following result: Given Tychonoff spaces X, Y such that $X \times Y$ is Lindelöf, then $\lambda(\mathcal{F}(X) \times \mathcal{F}(Y)) = \mathcal{F}(X \times Y)$ if, and only if, for each compact $K \subset (\beta X \times \beta Y) \setminus (X \times Y)$ there are Čech-complete paracompact subspaces M, N of $\beta X, \beta Y$, respectively, such that $X \subset M, Y \subset N$ and $K \cap (M \times N) = \emptyset$. (See [21], 3.5). For a subset X of the unit interval I , there is an easier way of paraphrasing this result: $\mathcal{F}X \times \mathcal{F}X$ is supercomplete if, and only if, for each compact $K \subset I^2 \setminus X^2$ there is a first category subset $A \subset I \setminus X$ such that

$$K \subset (A \times I) \cup (I \times A).$$

However, there is an entire geometric circle $C \subset I^2 \setminus \mathbb{Q}^2$, and it cannot be covered by the projection pre-images $\pi_i^{-1}[A]$, $i = 1, 2$ of any first category set A .

It was also shown that $\mathcal{F}(\mathbb{J}) \times \mathcal{F}(\mathbb{Q})$, where \mathbb{J} denotes the irrationals, is not supercomplete. Thus, it follows that whenever a Tychonoff space X contains a closed copy of the irrationals, then $\mathcal{F}(X) \times \mathcal{F}(\mathbb{Q})$ is not supercomplete, and the localic product $X \times_{\text{loc}} \mathbb{Q}$ is not spatial. \square

5. Noetherian trees.

The method used in proving the positive countable product theorems 2 – 3 was based on trees with only finite branches. The application of the locally fine

condition in the successive constructions of the covers in the derivatives μ^α leads to such ‘Noetherian’ covering trees. In such a tree, the immediate successors of an element form its uniform cover, and the collection $\text{End}(T)$ of all maximal elements of the tree T cover the space. This technique was first used in Pelant’s proof ([40])⁴ of Isbell’s conjecture that every locally fine space is ‘subfine’ (a subspace of a fine space). Pelant showed that the ‘ λ equation’ considered above, $\lambda(\Pi\mathcal{F}(M_i)) = \mathcal{F}(\Pi M_i)$ holds for *any* collection of completely metrizable spaces M_i . (See below for a current extension of this result.) Noetherian trees were used to represent the recursive construction of covers in the consecutive derivatives $\mu^{(\alpha)}$. The essential lemma used by Pelant states (in our formulation) that $\mathcal{U} \in \lambda\mu$ if, and only if, there is a Noetherian tree T of subsets of the underlying space such that 1) T satisfies the *uniform covering condition* with respect to μ (i.e., the immediate successors of a non-maximal element form its uniform cover); 2) the maximal elements $\text{End}(T)$ form a cover which refines \mathcal{U} and 3) T has X as its root. Each cover $\mathcal{G} \in \mu^{(\alpha)}$ can be refined by such a Noetherian tree and vice versa. This enables one to replace the consecutive derivatives and transfinite induction by arguments based on well-foundedness. It should be noted that general (localic) products of completely metrizable spaces (not being paracompact) are not spatial; the equation $\lambda(\Pi\mathcal{F}(M_i)) = \mathcal{F}(\Pi M_i)$ is not sufficient alone but must be complemented with the condition that each open cover of the product is normal. We will give a more general result in the last section.

Remark 2: Noetherian trees have well-defined *ranks*, and complete metric spaces of a finite or countable rank were studied by the author in [20]. (We say the rank of a complete metric space qX is the least α such that $q^{(\alpha)} = \mathcal{F}(X)$, the existence of which is guaranteed by [12], 4.2.) Among other results, it was proved that for a finite or countable α , the rank of qX equals α if, and only if, X has a compact set K such that outside of any neighbourhood of K , X is uniformly locally of a strictly lesser rank. This naturally led the author to recursively constructed decompositions of such spaces into Noetherian trees of closed subspaces in which the maximal elements are compact. The extended results obtained in [25] by the author and Pelant have to be bypassed here.

6. A game-theoretical characterization.

Noetherian trees can be used to give a direct motivation to a game-theoretical characterization of supercompleteness introduced – but not studied – by the author

⁴ Z. Frolík had an interesting interpretation of this result: The fine spaces (i. e., Tychonoff topologies) form the smallest coreflective class such that all subspaces are locally fine. Thus, Tychonoff spaces are obtained from the locally fine spaces through a purely categorical construction.

in 1983 [19]. There are two players I and II. For each game we choose an open cover \mathcal{V} of the given uniform space μX . Player I begins by choosing a uniform cover $\mathcal{U}_0 \in \mu$. If possible, Player II responds by selecting an element $U_0 \in \mathcal{U}_0$ such that $U_0 \subset V$ for no $V \in \mathcal{V}$. Then Player I continues by choosing a uniform cover \mathcal{U}_1 of U_0 . Player II again selects – if possible – an element $U_1 \in \mathcal{U}_1$ such that no member of \mathcal{V} contains this U_1 . Inductively, after the choice U_n by Player II, Player I chooses a uniform cover \mathcal{U}_{n+1} of U_n and Player II selects, whenever possible, an element $U_{n+1} \in \mathcal{U}_{n+1}$ such that $U_{n+1} \subset V$ for no $V \in \mathcal{V}$. Otherwise, the play stops at U_n . If this play of the game $G(\mu X, \mathcal{V})$ has infinitely many moves, then Player II wins, otherwise Player I wins. Then we may state the following characterization of supercompleteness in terms of the games $G(\mu X, \mathcal{V})$:

Theorem 6. ([19], Theorem 5') *A uniform space μX is supercomplete if and only if, for any open cover \mathcal{V} of X , Player I has a winning strategy in the game $G(\mu X, \mathcal{V})$.*

Proof. If μX is supercomplete, then $\mathcal{V} \in \lambda \mu$, and there is a Noetherian tree T with $\text{Root}(T) = X$, T satisfies the uniform covering condition and $\text{End}(T) < \mathcal{V}$. By proceeding along the branches of T , and using the uniform covering condition, Player I has a (stationary) winning strategy in the game $G(\mu X, \mathcal{V})$.

On the other hand, suppose that Player I always has such a winning strategy. Given an open cover \mathcal{V} of X , it is enough to produce a Noetherian tree T as in the preceding paragraph. As Player I has a winning strategy in $G(\mu X, \mathcal{V})$, one is able to find a uniform cover \mathcal{U} of X such that Player I knows how to win every play following Player II selecting elements $U \in \mathcal{U}$. The construction of T stops at every $U \in \mathcal{U}$ which is contained in some member of \mathcal{V} . (Those are choices that Player II cannot make). On the other hand, we will continue with all other members of \mathcal{U} . Player I chooses, for each such member a uniform cover, and the definition of T is inductively continued. By taking the union of all the inductive steps, we get a Noetherian tree T , because each branch corresponds to a play of $G(\mu X, \mathcal{V})$ in which Player I wins. By the construction of T , we have $\text{End}(T) < \mathcal{V}$, as required. \square

Thus, the winning strategy in the game $G(\mu X, \mathcal{V})$ is directly obtained from the Noetherian tree associated with any refinement of \mathcal{V} in $\lambda \mu$. Each particular play can be won by Player I by following a particular branch of such a tree. A game-theoretical characterization of spatiality in localic products $X \times_{\text{loc}} Y$ was given by Plewe in [41], likewise related to trees ([41], p. 647).

By applying the above theorem to products of uniform spaces, we immediately obtain a characterization of their supercompleteness as follows. It is enough to consider the case in which the factors X, Y are fine paracompact spaces. In the game $G(X, Y, \mathcal{W})$, we are given an open cover \mathcal{W} of $X \times Y$. We may assume – if necessary – that \mathcal{W} consists of open rectangles $W_1 \times W_2$. Player I chooses open

covers $\mathcal{U}_0, \mathcal{V}_0$ of X and Y , respectively, claiming that the rectangular cover $\mathcal{U}_0 \times \mathcal{V}_0$ refines \mathcal{W} . Player II selects, if possible, a rectangle $U_0 \times V_0$, $U_0 \in \mathcal{U}_0$, $V_0 \in \mathcal{V}_0$, such that $U \times V$ is not contained in any member of \mathcal{W} . Then Player I chooses open covers $\mathcal{U}_1, \mathcal{V}_1$ of U_0 and V_0 , respectively, obtained by restricting open covers of X and Y , and claims that $\mathcal{U}_1 \times \mathcal{V}_1$ refines the restriction of \mathcal{W} to $U_0 \times V_0$. The rest of the play is defined inductively, and Player I wins, if it only involves finitely many moves; otherwise, Player II wins. Again, the product is supercomplete if Player I has a winning strategy in $G(X, Y, \mathcal{V})$ for each open cover \mathcal{V} .

It is not directly possible to change the rules of the games $G(X, Y, \mathcal{V})$ so that Player I would choose simple rectangles $U_i \times V_i$, instead of choosing covers. The crux of the rules is to guarantee that the choices of Player I are ‘rectangular’ in the sense that once an open set $U_i \subset X$ is selected, all choices $V_i \subset Y$ would then have to be combined into products $U_i \times V_i$, and similarly for the other factor. In Plewe’s game ([41], p. 645) (we switch the players to follow our original notation) this is obtained by letting the other player choose points $x_i \in X$, $y_i \in Y$ in *alternative* steps. In our situation, Player I would choose, in alternative steps, open sets $U_i \subset X$, $V_i \subset Y$ with $x_i \in U_i$, $y_i \in V_i$. Consider a set of choices x_i by Player II large enough so that the corresponding sets U_{i,x_i} , selected by using a winning strategy, form a cover. Then for each such U_{i,x_i} , consider a similarly formed cover by sets of the form V_{i,x_i,y_i} . The cover consisting of all rectangles of the form $U_{i,x_i} \times V_{i,x_i,y_i}$ is in the first derivative of the product uniformity $\mathcal{F}(X) \times \mathcal{F}(Y)$. Thus, the corresponding game $G'(X, Y, \mathcal{V})$ is related to $G(X, Y, \mathcal{V})$ in the sense that while Player I chooses rectangular uniform covers in the latter, the covers chosen in the former are uniformly locally uniform.

In Plewe’s game, the players start from an open cover of an open *rectangle* of the product. However, as noted in his article ([41], p. 646), for regular spaces this is tantamount to taking the entire product as the initial rectangle. Therefore, it is now easy to see that for uniform spaces, his game is *equivalent to ours with respect to supercompleteness*. Thus, for paracompact factors we may state the following characterization of the spatiality of the localic product:

Theorem 6’: *Let X, Y be paracompact spaces. Then the localic product $T(X) \times_{\text{loc}} T(Y)$ is spatial if, and only if, Player I has a winning strategy in the game $G(X, Y, \mathcal{V})$ for each open cover \mathcal{V} of $X \times Y$. \square*

However, it is to be noted that the game in [41] is more general than the ones described above, because they are not restricted to uniform spaces or paracompact products, which always are completely regular. Nevertheless, our characterization can be extended to products of general regular spaces by using the main result of this paper to be given in the last section. We obtain a deeper connection between spatiality and the locally fine operation by moving to ‘covering monoids’ of spaces.

7. Formal spaces

Motivated by locales, Fourman and Grayson [9] introduced in 1981 a ‘formal space’ of a theory, based on four conditions of an *entailment relation* in a propositional language, a pre-ordered set. (A pre-ordered set is defined here as a set with a binary relation \leq that is reflexive and transitive.) These conditions were taken up by Sambin in 1987 (cf. [45]) who developed a theory of formal spaces from a ‘pure’ standpoint in the spirit of the intuitionistic type theory of Martin-Löf. Accordingly, intuitionistic versions of classical theorems for topological spaces have been proved by several authors (see, e.g., Tychonoff’s Theorem in Negri and Valentini [38] and Coquand’s version of van der Waerden’s theorem on arithmetic progressions [4].) Formal spaces were used in 1990 by Sigstam to give an effective theory of spaces in her thesis [46]. The approach is opposite (‘top-down’) to the traditional (‘bottom-up’) constructive approaches to say, real numbers: While the same recursive constructions are used, one applies them to given parts of a space, rather than to an assumed collection of (computable) points.

Definition 7.1: *Given a pre-ordered set (P, \leq) , a covering relation is a subset $\text{Cov} \subseteq P \times 2^P$ satisfying the following axioms:*

- C1) if $a \in U$, then $\text{Cov}(a, U)$.
- C2) if $a \leq b$, then $\text{Cov}(a, \{b\})$.
- C3) if $\text{Cov}(a, U)$ and $\text{Cov}(a, V)$, then $\text{Cov}(a, U \wedge V)$. Here $U \wedge V$ denotes the set of elements bounded by both U and V .
- C4) if $\text{Cov}(a, U)$ and $\text{Cov}(u, V)$ for all $u \in U$, then $\text{Cov}(a, V)$.

It is the last axiom⁵ which is directly connected with our discussion. It corresponds to the Heyting axiom of right distributivity (characterizing locales) and also to the locally fine condition. Indeed, for a pre-uniformity μ (recall that in this paper, a pre-uniformity is a filter of coverings of a set) on a set X , we define a relation $R \subseteq P(X) \times P(P(X))$ by setting $(A, \mathcal{U}) \in R$ if there is a cover $\mathcal{V} \in \mu$ such that the restriction of \mathcal{V} to A refines \mathcal{U} . Then R satisfies the above conditions C1) – C3). Indeed, to see this, C1) is obvious because if A is a member of \mathcal{U} , then we may take the ‘trivial’ cover $\{X\} \in \mu$ as \mathcal{U} . Condition C2) is similar, and C3) follows from the requirement that μ be closed under finite meets.

On the other hand, the *transitivity* condition C4) (called that of *composition* in [9]) is satisfied if, and only if, the pre-uniformity μ is locally fine, i.e., $\lambda\mu = \mu$.

⁵ In addition to the circle of notions represented by 1) the locally fine operation, 2) locales and 3) formal spaces, we may add 4) Grothendieck topologies, because the covering relation gives the conditions for a Grothendieck topology on a pre-ordered set. This may be followed by 5) modal logics (see [13]), closing the circle with the equivalence between the modal system S4 and the closure operation in topology, well known since the 1930’s (see [36]).

To see this, suppose that R satisfies Condition C4). Let $\{U_i\} \in \mu$, and for each i , let $\{V_j^i\} \in \mu$. Thus, $(X, \{U_i\}) \in R$, and for each i , we have $(U_i, \{U_i \cap V_j^i\}) \in R$, by the definition of R . By the condition under consideration, we obtain that $(X, \{U_i \cap V_j^i\}) \in R$. Thus, there is a member $\mathcal{V} \in \mu$ which refines $\{U_i \cap V_j^i\}$, whence the latter is a member of μ as well. Conversely, assume that μ is locally fine, and suppose $(A, \mathcal{U}) \in R$, and for all $U \in \mathcal{U}$, let $(U, \mathcal{V}) \in R$. There is $\mathcal{U}' \in \mu$ such that $\mathcal{U}' \upharpoonright A < \mathcal{U}$ and for each $U \in \mathcal{U}'$, there is $\mathcal{V}_U \in \mu$ such that $\mathcal{V}_U \upharpoonright (U \cap A) < \mathcal{V}$. The cover $\mathcal{W} = \cup \{\mathcal{V}_U \upharpoonright U : U \in \mathcal{U}'\}$ is in $\mu^{(1)} = \mu$, and is easily seen that $\mathcal{W} \upharpoonright A < \mathcal{V}$. Therefore, $(A, \mathcal{V}) \in R$, as desired. \square

The reader should note the Condition 4) above (transitivity) is the characteristic ‘topological condition’, expressed in locales by the Heyting axiom and in classical topology by the *idempotency* of the Kuratowski closure operator (or by the *transitivity* of the corresponding relation between sets). In this sense, λ corresponds to topology.

Given only a *set of generators* $G \subset P \times 2^P$, the associated covering relation Cov_G is obtained by closing G under the conditions C1) – C4). This means forming all Noetherian trees T such that for each element x of T , the immediate successors are derived by using one of the four conditions. This corresponds to the idea of using Noetherian trees to construct ‘recursively defined’ refinements of open covers of uniform spaces, in particular in the products of paracompact spaces. Such constructions start from the basis of uniform covers, which is a commutative monoid under the operation of meet, and closes the collection under the condition of transitivity, which we have seen to be equivalent to the locally fine condition. By the same token, formal spaces are often described by giving a ‘formal base’, a commutative monoid $(S, \cdot, 1)$ with unit, and the corresponding *rules of inference* equivalent to the above condition C1) – C4). For example, they could be given as the rules

$$1) \frac{a \in U}{a \vDash U} \quad 2) a \cdot b \vDash a \quad 3) \frac{a \vDash U \quad a \vDash V}{a \vDash U \cdot V} \quad 4) \frac{a \vDash U \quad U \vDash V}{a \vDash V}.$$

We will call a pair (P, Cov) a *covering monoid*, if P is a pre-ordered set with a unique maximal element 1 and $\text{Cov} \subset P \times 2^P$ is a relation closed under the conditions C1 – C3. A homomorphism between covering monoids $(P, \text{Cov}), (Q, \text{Cov}')$ is a map $f: P \rightarrow Q$ such that $(a, U) \in \text{Cov}$ implies $(f(a), f(U)) \in \text{Cov}'$. With a covering monoid (P, Cov) , we may associate a monoid (P, μ_{Cov}) of *covers of P under Cov* , i.e., μ_{Cov} consists of all $U \subset P$ such that $(1, U) \in \text{Cov}$. If f – as given above – preserves the maximal element, then f ‘restricts’ to a homomorphism $(P, \mu_{\text{Cov}}) \rightarrow (Q, \mu_{\text{Cov}'})$. We denote the closure of a relation $R \subset P \times 2^P$ under C4 by λR . This closure can be obtained by applying the following version of Ginsburg-Isbell derivation on Cov : Let $\text{Cov}^{(0)} = \text{Cov}$, and given $\text{Cov}^{(\alpha)}$, let $\text{Cov}^{(\alpha+1)}$ be the collection of all pairs (a, V) for which there is $(a, U) \in \text{Cov}$ such that for all $u \in U$, $(u, V) \in \text{Cov}^{(\alpha)}$. For limit ordinals β , define $\text{Cov}^{(\beta)} = \cup \{\text{Cov}^{(\alpha)} : \alpha < \beta\}$. The

first stable derivative is then λCov . This closure may also be described in terms of Noetherian trees: $(a, V) \in \lambda\text{Cov}$ if, and only if, there is a Noetherian tree T such that 1) the root of T is a ; 2) for each element p of T , the immediate successors of p form a set $U \subset 2^p$ such that $(p, U) \in \text{Cov}$ and 3) $V = \text{End}(T)$.

As seen above, any pre-uniformity μ on a set X is associated with a covering monoid $(P(X), \text{Cov}_\mu)$ in a natural way. Motivated by this relation, we will call pre-uniformities *monoids of covers* to emphasize their formal independence of actual pre-uniform spaces. Uniform spaces will correspond to *normal monoids* of covers μ , i.e., in which for each $u \in \mu$ there is $v \in \mu$ with $v^2 \leq u$. Corresponding to the fine uniformity (the filter of all normal covers of a Tychonoff space), we have the *fine monoid of covers* of a space X , written $\mathcal{O}(X)^*$, consisting of all covers of X with an open refinement. This should be contrasted with the *fine covering monoid* $\mathcal{O}(X)$ of X consisting of all pairs (U, \mathcal{G}) where U is an open subspace of X and \mathcal{G} is a cover of U with an open refinement. We will call a monoid of covers on a space X , written (X, μ) , (super)complete if $\lambda\mu$ is fine. In the next section, we will obtain a product theorem which implies a far-reaching equivalence of locales, formal spaces and covering monoids (and extends our previous results on supercompleteness to non-paracompact spaces). Let us first give two essential lemmas on products of covering monoids.

We note that the product of a family (P_i, Cov_i) of covering monoids is a pair (P, Cov) , where P is the weak direct product of the P_i consisting of all elements a of $\prod P_i$ with $a_i = 1_i$ for almost all i , and where $(a, U) \in \text{Cov}$ if, and only if, there is for each $a_i \neq 1_i$ a pair $(a_i, U_i) \in \text{Cov}_i$ such that $\bigwedge \{(U_i)_i : a_i \neq 1_i\}$ refines U , where $(U_i)_i$ denotes the set of all $u \in \prod P_j$ such that $u_i \in U_i$ and $u_j = 1_j$ for $j \neq i$. By considering only pairs of the form $(1, U)$, this restricts to the usual product of pre-uniform spaces. Indeed, in the special situation in which the elements of P_i are subsets of a set X_i , we take the subbasis of $\prod \text{Cov}_i$ to consist of pullbacks $\pi_i^{-1}(a, U) = (\pi_i^{-1}[a], \pi_i^{-1}[U])$, where $\pi_i : \prod X_j \rightarrow X_i$ is a projection. In the general situation, we consider instead 'insertions' $q_i : P_i \rightarrow \prod P_j$ given by $q_i(a) = (x_j)$, where $x_i = a$ and $x_j = 1_j$ for $j \neq i$. However, in the following three lemmas we consider the (set-theoretical) situation of topological spaces.

The following Observation is obvious.

Observation 7.2: *Let X be a topological space. Then $\mathcal{O}(F) = \mathcal{O}(X) \upharpoonright F$ for each closed subspace $F \subset X$. \square*

Lemma 7.3: *Let (X_i) be a family of topological spaces, and let $(T(X_i), \text{Cov}_i)$ be a corresponding family of covering monoids. Then $\lambda \prod \text{Cov}_i$ has a basis consisting of pairs (a, U) , where U is a collection of basic open rectangles.*

Proof. An inductive proof can be obtained by using the consecutive derivatives $\text{Cov}^{(\alpha)}$, where $\text{Cov} = \prod \text{Cov}_i$. The claim is clearly valid for $\alpha = 0$. Thus, suppose it is valid for α and let $(a, U) \in \text{Cov}(\alpha + 1)$. Then there is a cover V of a such that $(a, V) \in \text{Cov}$, and for each $v \in V$ a cover W_v such that $(v, W_v) \in \text{Cov}^{(\alpha)}$. But V is

refined by a cover V' consisting of open basic rectangles, and for each $v \in V$, there is such a refinement W'_v of W_v . It is clear that the elements $v' \wedge w'$, $v' \in V'$, $w' \in W'_v$ form a refinement U' of U , the elements of which are open basic rectangles, and $(a, U') \in \lambda\text{Cov}$. The case of limit ordinals is obvious. \square

Theorem 7.4: *Let (X_i) be a family of regular topological spaces. Then $(1, U) \in \lambda\Pi\mathcal{O}(X_i)$ if, and only if, $U \in \lambda\Pi\mathcal{O}(X_i)^*$.*

Proof. We will again proceed by induction. By the definition of the direct product of covering monoids, $(1, U) \in \mu = \Pi\mathcal{O}(X_i)$ if and only if $U \in \nu = \Pi\mathcal{O}(X_i)^*$. So, suppose $(1, U) \in \mu^{(\alpha)}$ iff $U \in \nu^{(\alpha)}$ (taking μ, ν with respect to arbitrary regular spaces). To show that this is valid for α replaced with $\alpha + 1$, it is sufficient to consider the right implication. Thus, let $(1, U) \in \mu^{(\alpha+1)}$. Thus, there is $(1, V) \in \mu$ such that for each $v \in V$, we have $(v, U) \in \mu^{(\alpha)}$. By the assumption of regularity, there is a cover W of ΠX_i by closures of basic open rectangles in μ and hence in ν which refines V . For each $w \in W$, there is an extension of U to a cover U_w of ΠX_i the restriction of which to w refines U .

We may assume that $U_w \in \nu^{(\alpha)}$. Indeed, w is a (topological) product of regular spaces, and we may use the inductive hypothesis. Write $w = \bigwedge \{\pi_i^{-1}[w_i] : i \in E\}$, where E is finite and w_i is the closure of an open subset of X_i . Set $X'_i = w_i$ for $i \in E$ and let $X'_i = X_i$ otherwise. Then consider the products $\mu' = \Pi\mathcal{O}(X'_i)$, $\nu' = \Pi\mathcal{O}(X'_i)^*$. The restriction U' of U to w satisfies $(w, U') \in (\mu')^{(\alpha)}$, and hence by the inductive hypothesis $U' \in (\nu')^{(\alpha)}$. By using the equations $\mathcal{O}(X'_i)^* = \mathcal{O}(X_i)^* \upharpoonright w_i$ for $i \in E$, and recalling that the Ginsburg-Isbell derivatives preserve substructures (i.e., $(\xi \upharpoonright A)^{(\alpha)} = (\xi)^{(\alpha)} \upharpoonright A$) it easily follows there is cover $U_w \in \nu^{(\alpha)}$ the restriction of which to w refines U' , as desired.

But then the elements $w \wedge u_w$, $u_w \in U_w$, form a cover U'' such that $U'' \in \nu^{(\alpha+1)}$ and U'' refines U , implying $U \in \nu^{(\alpha+1)}$. As above, the limit ordinal case is obvious. \square

8. A general product theorem

It can be seen from the previous section that the theory of formal spaces corresponds to that of locally fine covering monoids. In this section, we will use notions and lemmas developed above to link supercompleteness in products to spatiality in a general fashion. We extend the characterization of supercompleteness in a paracompact product $\mathcal{F}X \times \mathcal{F}Y$ by the equation

$$\lambda(\mathcal{F}X \times \mathcal{F}Y) = \mathcal{F}(X \times Y)$$

to a similar one (8.6) characterizing spatiality, even without paracompactness.

We will first describe the locale product simply as the locally fine (or λ -) product. The product theorem given in this section grew out of the author's attempt

to understand the proof given by Dowker and Strauss ([6]) for their product theorem. The following definitions are well-known, see, e.g., [46].

Let Cov be a covering relation on a pre-ordered set P . For subsets $U, V \subset P$, define $U \leq V$ if for all $u \in U$, we have $\text{Cov}(u, V)$. Then define an equivalence relation \sim by setting $U \sim V$ if $U \leq V$ and $V \leq U$. Denote the corresponding equivalence classes by $[U]$. The locale associated with the covering relation Cov is the set

$$\mathcal{L}_{\text{Cov}} = \{[U] : U \subseteq P\}$$

equipped with the lattice operations (recall the definition of $U \wedge V$)

$$[U] \wedge [V] = [U \wedge V], \quad \bigvee_{i \in I} [U_i] = [\bigcup \{U_i : i \in I\}].$$

We say that the covering relation Cov (or more exactly the pair (P, Cov)) *generates* L . We extend this definition to covering monoids by stipulating that the locale associated with a covering monoid (P, μ) is the one generated by $\lambda\mu$.

On the other hand, given a locale L , define a canonical covering relation Cov_L by setting $\text{Cov}_L(a, U) \Leftrightarrow a \leq \bigvee U$. Then $\mathcal{L}_{\text{Cov}_L} \cong L$. Thus, every locale has a canonical generating covering relation, and it follows from the right distributivity of the locale that this relation is locally fine, i.e., defines a formal space. If (P, Cov) generates L , then there is a canonical embedding (of covering monoids) $(P, \text{Cov}) \hookrightarrow (2^P, \text{Cov}_L)$ given by $a \mapsto [a]$; we will consider the generating monoid a submonoid of $(2^P, \text{Cov}_L)$.

One says a subset $U \subseteq L$ is a *cover* of a locale L if $\bigvee U = 1$. A subset V is a *refinement* of U if for each $v \in V$ there is $u \in U$ such that $u \leq v$. We denote the monoid of all covers of L by $\text{Cov}(L)$. Thus, $\text{Cov}(L)$ is the collection of all $U \subseteq L$ such that $\text{Cov}_L(L, U)$, and by transitivity, $\text{Cov}(L)$ is locally fine. (Note that since $1 \in L$, $\text{Cov}_L(L, U)$ implies $\bigvee U = 1$.)

We will construct the co-product $\prod L_i$ of given frames L_i . (We remind the reader that the difference with locales is that morphisms go in the opposite direction. With the product of locales, we have projections $\pi_j : \prod L_i \rightarrow L_j$, whereas with the co-product of frames, we have ‘insertions’ $q_j : L_j \rightarrow \prod L_i$.)

Let (L_i) be a family of frames, and consider the Cartesian product frame $\prod L_i$. (This is a frame, but not the *product* of the L_i in the category of locales!) Take a subframe $B \subset \prod L_i$ which consists of all $b = (b_i)$ such $b_i = 1_i$ except for finitely many i (the direct product of monoids). We define a covering relation by first choosing a set $G \subset (B, 2^B)$ of generators to consist of all (a, U) , where for some i , U has the form: For $j \neq i$, $\pi_j[U] = \{a_j\}$, and $(a_i, \pi_i[U]) \in \text{Cov}_{L_i}$. Thus, U has been obtained from a by ‘splitting’ it along exactly one coordinate direction with respect to the corresponding covering relation. (Notice that this condition corresponds to the ‘coordinatewise derivation condition’ from Section 2.) The associated covering relation Cov_G is obtained by closing G under the conditions

C1) – C4) of covering relations. The frame $L = \mathcal{L}_{\text{Cov}_G}$ will be our co-product. Recall that the elements of L are equivalence classes $[U]$ of subset $U \subset B$ under the equivalence relation: $U \sim V$ iff $U \leq V$ and $V \leq U$. It follows that U is a cover of L , i.e., $\bigvee U = 1_L$, if $(1, U) \in \text{Cov}_G$, where 1 denotes the maximal element of B . We will show that for each such U there is $\mathcal{U} \in \lambda \Pi \mu_i$ with $\phi(\mathcal{U}) < U$, where $\mu_i = \text{Cov}(L_i)$ and $\phi : \text{Cov}(\lambda \Pi \mu_i) \rightarrow \text{Cov}(L)$ is an embedding of covering monoids.

The product of the μ_i has a subsbasis defined by the insertions $q_i : L_i \rightarrow B$ by setting $q_i(x) = (a_j)$, where $a_i = x$ and $a_j = 1_j$ for $j \neq i$. It is obvious that $(a, U) \in \text{Cov}_L$, implies $(q_i(a), q_i(U)) \in G$, by the definition of G . By taking finite meets, it turns out that $\Pi \mu_i$ has a basis \mathcal{B} contained in Cov_G . We note that $\Pi \mu_i$ is obtained from \mathcal{B} by applying the rules C1 – C3.

How are the elements $(a, U) \in \text{Cov}_G$ obtained? By the definition, (a, U) belongs to Cov_G if, and only if, there is Noetherian tree T such that 1) the root of T is a ; 2) the immediate successors of an element $p \in T$ are obtained from p by applying G or one of the conditions C1 – C4 and 3) $U = \text{End}(T)$. It is clear that $\lambda \Pi \mu_i$ is closed under these conditions. Thus, it is contained in Cov_G . The opposite inclusion is clear, too, and hence Cov_G and $\Pi \mu_i$ are the same covering relation.

Let \mathcal{E} be a cover of L . Thus, $\bigvee \mathcal{E} = 1_L$. Hence, there are sets $E_i \subset B$ such that $\mathcal{E} = \{[E_i] : i \in I\}$, and therefore

$$1_L = \bigvee_{i \in I} [E_i] = [\bigcup_{i \in I} E_i]$$

It follows that $(1, \bigcup_{i \in I} E_i) \in \text{Cov}_G$, where 1 denotes the maximal element of B . Thus, $\mathcal{U} = \bigcup \{E_i : i \in I\}$ is an element of $\lambda \Pi R_i$ such that $[\mathcal{U}] = \{[u] : u \in \mathcal{U}\}$ refines \mathcal{E} . Denote the covering monoid associated with Cov_G by $\text{Cov}(G)$. The mapping $u \mapsto [u]$ defines a natural homomorphism $\phi : \text{Cov}(\Pi \mu_i) \rightarrow \text{Cov}(\Pi L_i)$ of covering monoids. However, the factors L_i are frames and hence partially ordered and so is the weak direct product B . It follows that $u \mapsto [u]$ yields an embedding $B \rightarrow L$, which extend to covers. Thus, we have proved the following result:

Theorem 8.1: *Let (L_i) be a family of frames. Then there is an embedding ϕ of covering monoids*

$$(*) \quad \lambda(\Pi \text{Cov}_{L_i}) \rightarrow^\phi \text{Cov}_{\Pi L_i}$$

where Cov_{L_i} is the canonical covering relation on L_i and the left-hand side of $(*)$ is a locally fine generating covering monoid for ΠL_i . Moreover, the mapping ϕ is induced by the embedding $u \mapsto [u]$, and for each cover \mathcal{V} of ΠL_i there is $\mathcal{U} \in \lambda \Pi \text{Cov}_{L_i}$, such that $\phi(\mathcal{U}) < \mathcal{V}$. \square

For pre-uniform spaces (X_i, μ_i) , the direct product is a pair $(\Pi X_i, \Pi \mu_i)$, where $\Pi \mu_i$ is generated by the basis of all finite meets of single pullbacks $\pi_i^{-1}[\mathcal{U}]$, $\mathcal{U} \in \mu_i$

(‘basic rectangular covers’). Moreover, $\lambda\Pi\mu_i$ is generated by covers consisting of basic rectangles, which form a monoid. Lacking better notation, we denote this monoid of rectangles by $[\lambda\Pi\mu_i]_{\mathcal{R}}$. Its covering relation is induced by the pre-uniform structure of the product: A collection of rectangles cover a rectangle if, and only if, they cover the latter as a (pre-)uniform cover.

This is special case of the product of covering monoids (P_i, μ_i) , in which the basic rectangular covers are finite meets of pullbacks of the form $\pi_i^{-1}(a, U) = (\pi_i^{-1}[a], \pi_i^{-1}[U])$, where $\pi_i^{-1}[a]$ is a basic open rectangle covered by the cover $\pi_i^{-1}[U]$ consisting of basic rectangles. 8.1 implies the following result.

Corollary 8.2: *Let (X_i) be a family of topological spaces. For each X_i , let $\mathcal{O}_i(X_i)$ denote the fine covering monoid induced by open-refinable covers. Then there is an embedding ϕ*

$$[\lambda\Pi\mathcal{O}_i(X_i)]_{\mathcal{R}} \cong \lambda\Pi \text{Cov}_{T(X_i)} \hookrightarrow^{\phi} \text{Cov}_{\text{IIT}(X_i)},$$

where $T(X_i)$ is the topology of X_i . Moreover, for any cover \mathcal{V} of $\text{IIT}(X_i)$, there is a rectangular cover \mathcal{U} in $[\lambda\Pi\mathcal{O}_i(X_i)]_{\mathcal{R}}$ such that $\phi(\mathcal{U}) < \mathcal{V}$. \square

Since $\lambda\Pi \text{Cov}_{T(X_i)}$ generates the localic product of the $T(X_i)$, we (ab)use the above corollary to say that $[\lambda\Pi\mathcal{O}_i(X_i)]_{\mathcal{R}}$ generates it, too.

In [32], Isbell showed that the product of paracompact locales is paracompact. Dowker and Strauss [6] extended this result to include the cases of metacompact and Lindelöf (regular) locales. These results (and an unlimited number of others) follow from Theorem 8.2.

Indeed, for a topological space X , let $\mathcal{L}\mathcal{F}(X)$ be the monoid of all covers which have an open, locally finite refinement. Let us call $\mathcal{L}\mathcal{F}(X)$ the locally finite monoid of covers on X . Then $\mathcal{L}\mathcal{F}(X)$ is locally fine, and X is paracompact if $\mathcal{L}\mathcal{F}(X)$ is fine, i.e., contains (and thus equals) $\mathcal{O}(X)$. It is easy to see that arbitrary direct products of locally finite monoids of covers is again locally finite. (We call μ on a space X locally finite if it contains $\mathcal{L}\mathcal{F}(X)$.) This follows from the easy observation that any binary, and more generally finite, product of locally finite covers is again locally finite. Finally, λ preserves local finiteness, so that $\lambda\Pi\mathcal{L}\mathcal{F}(X_i)$ is locally finite. The same is true of point-finite, locally countable, point-countable, Lindelöf, and compact monoids of covers (call μ compact if every cover has a finite open refinement in μ). These considerations are valid for general covering monoids. Therefore, we obtain the following corollary:

Corollary 8.3: *If the members of a family (L_i) of locales are compact (resp. paracompact, metacompact, Lindelöf, para-Lindelöf, meta-Lindelöf), then so is $\Pi_{\text{loc}}L_i$. \square*

In fact, we may use 8.2 to establish a relation between the spatiality of localic products and the locally fine condition. To this end, we might first give a game-theoretical characterization for the λ of the product of fine monoids to be fine, and show its equivalence with Plewe’s game-theoretical characterization of

spatiality in products. However, we will proceed directly. Let (X_i) be a family of sober spaces, i.e., $\text{Pt}(T(X_i)) \cong X_i$. We will first show that the localic product $\Pi_{\text{loc}}T(X_i)$ is spatial, $\Pi_{\text{loc}}T(X_i) = T(\Pi X_i)$ if, and only if, $\lambda\Pi\mathcal{O}(X_i)$ is the fine monoid $\mathcal{O}(\Pi X_i)$.

Theorem 8.4: *The localic product of a family (X_i) of sober topological spaces is spatial if, and only if, $\lambda\Pi\mathcal{O}(X_i) = \mathcal{O}(\Pi X_i)$.*

Proof. Suppose that $\lambda\Pi\mathcal{O}(X_i) = \mathcal{O}(\Pi X_i)$. The locales $T(X_i)$ are generated by the fine covering monoids $\mathcal{O}(X_i)$. Hence, their localic product $\Pi_{\text{loc}}T(X_i)$ is generated by $\lambda\Pi\mathcal{O}(X_i)$, which is by assumption the fine monoid of the topological product, and hence generates $T(\Pi X_i)$, as desired. On the other hand, suppose that $\Pi_{\text{loc}}T(X_i)$ is spatial.

Given an open cover U of a basic open rectangle a in ΠX_i , we may consider U a cover of a in $L = \Pi_{\text{loc}}T(X_i)$. But L is generated by $\lambda\Pi\mathcal{O}(X_i)$, and hence there is, for each $u \in U$, collection V_u of open sets (basic rectangles) such that $u = [V_u]$ and hence $(a, \mathcal{V}) \in \lambda\Pi\mathcal{O}(X_i)$, where $\mathcal{V} = \cup\{V_u : u \in U\}$. Therefore, \mathcal{V} is a refinement of U in the locally fine closure of the product of the $\mathcal{O}(X_i)$, which consequently refines the fine monoid of the topological product, i.e., it is itself fine. \square

Notice in particular that we have not assumed the factors to be regular. However, this result cannot be directly applied to spaces (via spatiality) along the lines of 8.3, because the fine covering monoids $\mathcal{O}(X)$ carry – within their structure – all the open subspaces. As a consequence, after taking the locally fine coreflection, the corresponding products $\Pi\mathcal{O}(X_i)$ produce in general monoids of covers finer than the ones obtained from products of monoids $\mathcal{O}(X_i)^*$ of covers (as generalized pre-uniform spaces). In order to bridge the gap, we need to assume regularity. The following lemma provides a link between spatiality and λ -covers.

Lemma 8.5: *Let (X_i) be a family of topological spaces, and let \mathcal{U} be a collection of basic open rectangles in ΠX_i such that $\mathcal{U}' = \{[u] : u \in \mathcal{U}\}$ covers the points of $\Pi_{\text{loc}}T(X_i)$. If \mathcal{U} belongs to $\lambda\Pi\mathcal{O}(X_i)^*$, then \mathcal{U}' covers $\Pi_{\text{loc}}T(X_i)$.*

Proof. This follows immediately from the result that $\lambda\Pi\mathcal{O}(X_i)$ generates $\Pi_{\text{loc}}T(X_i)$. \square

The condition that $\lambda\Pi\mathcal{O}(X_i)^* = \mathcal{O}(\Pi X_i)^*$ is analogous to the condition – studied by the author – that $\lambda(\Pi\mathcal{F}(X_i))$ contain all normal covers of ΠX_i . (In [40] this was shown to be true whenever the X_i are completely metrizable spaces; in [28], this result has been extended to paracompact spaces which are countable unions of closed, partition-complete subspaces.)

Theorem 8.6: *The localic product of a family (X_i) of regular topological spaces is spatial if, and only if, $\lambda(\Pi\mathcal{O}(X_i)^*) = \mathcal{O}(\Pi X_i)^*$.*

Proof. Suppose that $\Pi_{\text{loc}}T(X_i)$ is spatial. Then by 8.4, $\lambda\Pi\mathcal{O}(X_i) = \mathcal{O}(\Pi X_i)$, and hence by 7.4 we have $\lambda(\Pi\mathcal{O}(X_i)^*) = \mathcal{O}(\Pi X_i)^*$, as required.

On the other hand, suppose that this condition holds. We recall that a regular locale is spatial iff it does contain a non-empty, closed pointless sublocale. If $\prod_{\text{loc}} T(X_i)$ were not spatial, then there would exist such a sublocale F , and the collection \mathcal{U} of all basic open rectangles u of $\prod X_i$ such that $[u] \wedge F = 0$ would form an open cover $\prod X_i$ for which $\mathcal{U}' = \{[u] : u \in \mathcal{U}\}$ covers the points of the localic product. But by the assumption $\mathcal{U} \in \lambda\Pi\mathcal{O}(X_i)^*$, and hence (by the preceding lemma) \mathcal{U}' would cover the product locale, which is impossible. Hence, the product in question is spatial. \square

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